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Static analysis of shear-deformable shells of revolution via G.D.Q. method

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Abstract. This paper deals with a novel application of the Generalized Differential Quadrature (G.D.Q.) method to the linear elastic static analysis of isotropic rotational shells. The governing equations of equilibrium, in terms of stress resultants and couples, are those from Reissner-Mindlin shear deformation shell theory. These equations, written in terms of internal-resultants circular harmonic amplitudes, are first put into generalized displacements form, by use of the strain-displacements relationships and the constitutive equations. The resulting systems are solved by means of the G.D.Q. technique with favourable precision, leading to accurate stress patterns.

Key words: shell of revolution; generalized differential quadrature; static analysis; numerical method.

1. Introduction

Shells of revolution are common structural elements and can be found in many fields of engineering technology. Their use spans over different branches of engineering such as pressure vessels, cooling towers, water tanks, tires etc. As it is well known, for the various shell models presented in the literature, only a few cases of limited applicative importance have a closed-form solution (Reissner 1946, Kunieda 1984) and often this needs some further simplification to be obtained. In the last three decades numerical approaches to shell analysis have become more and more important, in a way that nowadays approximate solutions are the only ones to be studied, making shell analysis a rather important area of computational mechanics (Yang *et al.* 2000). Amongst different numerical techniques both for the statics and dynamics of shells, surely the most widely used one is the finite element method. We recall the displacement-based version (Luah and Fan 1990), the mixed version (Prato 1969, Gould and Sen 1971, Reddy 1984) and the recent hybrid mixed version (Kim and Kim 2000). Other methods developed for the problem in argument are the strip method (Mizusawa 1988), the boundary element method (Liu 1998), the element-free method (Krysl and Belytschko 1996) and the differential quadrature method in the so-called δ-point version (Mirfakhraei and Redekop 1998, Redekop and Xu 1999, Jiang and Redekop 2002).

The differential quadrature was first introduced (Bellman and Casti 1971, Bellman *et al.* 1972) as a rapid tool for solving linear and nonlinear PDE systems. In the recent past further developments

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of this numerical approach have brought a quantity of research results in structural mechanics and in computational mechanics (Bert and Malik 1996) and even in other fields of applied sciences (Shu and Richards 1992, Bellomo 1997), due to the great versatility and applicability of the differential quadrature method.

The scope of the present paper is to enlighten an efficient and accurate innovative application of a version of the differential quadrature approach, called G.D.Q. method, for the solution of the static problem of doubly curved shells of revolution. The present work is based on a first order shear deformation theory (FSDT) for thin shells and hence the solution does not need the δ-point technique (Bert and Malik 1996) which by itself introduces further simplification to the model.

The governing equations of static equilibrium, for the shell structure, are a set of five bi-dimensional partial differential equations with variable coefficients. They are obtained by means of the principle of minimum total potential energy for rotational thin shell structures, provided the strain-displacements relationships for points lying on the reference surface are known. These equations are initially expressed in terms of stress resultants and couples per unit length of parametric lines of the middle surface of the shell.

By introducing the constitutive equations and the kinematic relationships between strain measures and displacements, the equilibrium equations can be written in terms of generalized displacement components only. The expansion of all variables of the problem into Fourier series' with respect to the circumferential coordinate θ permits separation of the independent variables and the initial two-dimensional problem is reduced to a series of simple one-dimensional problems; in this way, axisymmetric and nonsymmetric loading cases can be studied easily and the resulting governing equations can be solved separately for each circular harmonic component.

Precisely, they can now be discretized and solved with the aim of the Generalized Differential Quadrature technique so to give a series of sets of linear equations. The solution obtained so far is obviously in terms of middle surface translations and rotations. A simple interpolation rule (Lagrange interpolation) along meridional coordinate direction is sufficient to obtain an accurate pattern for the complete kinematic assessment of the shell. Applying the strain displacements relations to the approximated displacement field and the constitutive equations to the strain measures so obtained allows to get to the stress resultants and couples for the shell under consideration. Several examples treated in the present paper show how this simple and quick procedure produces reliable and accurate results and put the basis for a further development of the research to the field of more complex shaped rotational shells. The G. D. Q. solution shows good convergence characteristics and appears to be accurate when tested by comparison to F. E. M. analyses or analytical solutions available from the scientific literature. To the very best of the authors knowledge, such an application of the generalized differential quadrature method in combination with a partial Fourier series decoupling of problem variables, has never been tried before to rotational shell structural elements elasticity problems.

2. Shell geometry and fundamentals

The typical rotational shell with its reference (middle) surface is presented if Fig. 1. The shell geometry can be completely assigned in different ways: here the parallel radius scalar equation $R = R(z)$ and the thickness variation $t = t(z)$ are given. The coordinates on the middle surface are the circumferential angle θ and the vertical ascissa z. The position of a point lying out of the middle

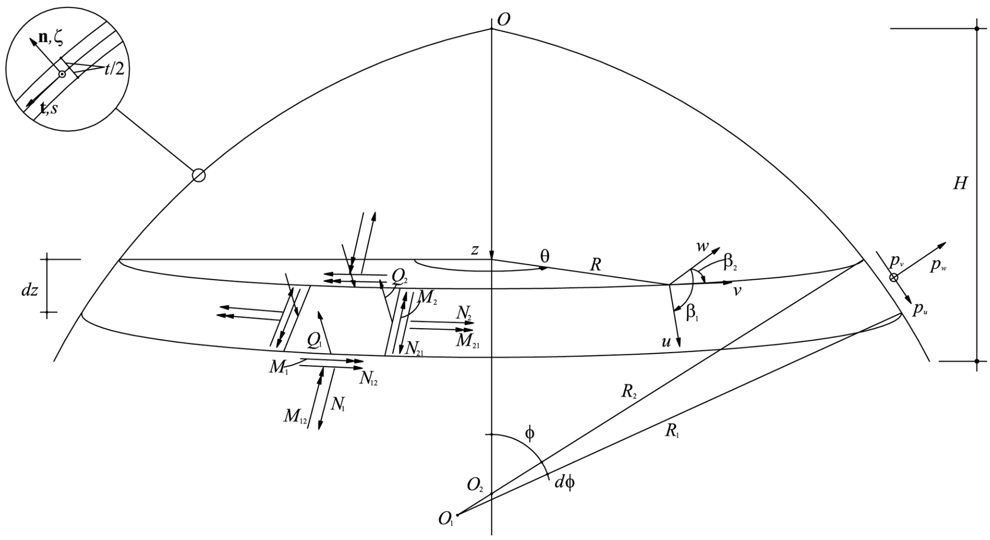

Fig. 1 Shell geometry and coordinate system on the reference surface

surface is given by the distance ζ, measured along the outward normal to the middle surface.

For further analytic developments it is convenient to introduce the following geometric parameters of the surface:

$$A_1 = A_1(z) = [1 + (R,_z)^2]^{1/2} \tag{1a}$$

$$A_2 = A_2(z) = R(z) \tag{1b}$$

which represent the fundamental Lamè parameters along the coordinate lines (meridians and parallels), and:

$$\sin\phi = \sin\phi(z) = [1 + (R,_z)^2]^{-1/2} \tag{2b}$$

$$\cos\phi = \cos\phi(z) = R,_z[1 + (R,_z)^2]^{-1/2} \tag{2b}$$

where ϕ is the colatitudinal oriented angle (Fig. 1) and where $(\ )_{,z}$ represents derivation with respect to z.

The generalized displacement components for an arbitrary point at location $(z, \theta, 0)$ are the translations u, v, w, along meridian, circumferential, and normal directions, respectively, and normal-to-mid-surface rotations β_1 and β_2. Assuming the shell to be sufficiently thin, the displacement field can be represented in this form:

$$U(z, \theta, \zeta) = u + \zeta\beta_1, \quad u = U(z, \theta, 0) \tag{3a}$$

$$V(z, \theta, \zeta) = v + \zeta\beta_2, \quad v = V(z, \theta, 0) \tag{3b}$$

$$W(z, \theta, \zeta) = w, \quad w = W(z, \theta, 0) \tag{3c}$$

It is evident how meridional and circumferential translations U and V vary linearly through the thickness, while the transverse component W remains constant with respect to ζ. The stress resultants and couples, obtained with equilibrium arguments (Gould 1999) are represented in Fig. 1 with their positive sense and acting on shell element positive-normal faces. It is assumed that twisting moments and in-plane shear resultants acting on adjacent sides of the element are equal with reasonable approximation. Also, distributed load components p_u, p_v and p_w are assigned, per unit mid-surface area.

3. Statement of the problem and variational formulation of equilibrium

For points lying on the reference middle surface the following strain-displacement relationships can be obtained from (3):

$$\varepsilon_1 = (1/A_1)u,_z + w/R_1 \tag{4a}$$

$$\varepsilon_2 = (1/R)(v,_\theta + R,_z/A_1 u + \sin\phi w) \tag{4b}$$

$$\varepsilon_{12} = (1/A_1)(v,_z) + (1/R)(u,_\theta - R,_z/A_1 v) \tag{4c}$$

$$\kappa_1 = (1/A_1)(\beta_{1,z}) \tag{4d}$$

$$\kappa_2 = (1/R)[\beta_{2,\theta} + (R,_z/A_1)\beta_1] \tag{4e}$$

$$\kappa_{12} = 1/2[(1/A_1)\beta_{2,z} + (1/R)\beta_{1,\theta} - (R,_z/RA_1)\beta_2] \tag{4f}$$

$$\gamma_1 = \beta_1 + w,_z/A_1 - u/R_1 \tag{4g}$$

$$\gamma_2 = \beta_2 + (1/R)(w,_\theta - \sin\phi v) \tag{4h}$$

in which ε_1, ε_2, $\varepsilon_{12} = \varepsilon_{21}$ are in-plane meridional, circumferential and shearing strains; κ_1, κ_2, $\kappa_{12} = \kappa_{21}$ are meridional, circumferential curvature changes and twisting strain, γ_1 and γ_2 are meridional and circumferential transverse shearing strains, which are fully taken into account in this framework. Again, symmetric mixed components are retained as approximately equal in the present treatment.

The total potential energy minimum principle, for a rotational thin shell, reads as follows (Reddy 1984):

$$\delta\Pi = \delta\{\int_0^{2\pi}[\int_0^H LA_1 dz + B]A_2 d\theta\} = 0 \tag{5}$$

where

$$L = \mathbf{f}_1^T\delta\boldsymbol{\varepsilon}_1 + \mathbf{f}_2^T\delta\boldsymbol{\varepsilon}_2 - \mathbf{p}^T\delta\mathbf{u} \tag{6}$$

represents the energy density of the system, while B is the potential of the edge loads, assigned on boundary parallel circles. Now, the vectors in (6) are written in terms of a partial harmonic Fourier expansion with respect to the circumferential coordinate θ. This leads to:

$$\mathbf{f}_1 = \begin{bmatrix} N_1 \\ N_{12} \\ Q_1 \\ M_1 \\ M_{12} \end{bmatrix} = \sum_{n=0}^{\infty} \begin{bmatrix} N_1^n \cos n\theta \\ N_{12}^n \sin n\theta \\ Q_1^n \cos n\theta \\ M_1^n \cos n\theta \\ M_{12}^n \sin n\theta \end{bmatrix}; \quad \boldsymbol{\varepsilon}_1 = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_{12} \\ \gamma_1 \\ \kappa_1 \\ \kappa_{12} \end{bmatrix} = \sum_{n=0}^{\infty} \begin{bmatrix} \varepsilon_1^n \cos n\theta \\ \varepsilon_{12}^n \sin n\theta \\ \gamma_1^n \cos n\theta \\ \kappa_1^n \cos n\theta \\ \kappa_{12}^n \sin n\theta \end{bmatrix} \tag{7a}$$

$$\mathbf{f}_2 = \begin{bmatrix} N_2 \\ M_2 \\ Q_2 \end{bmatrix} = \sum_{n=0}^{\infty} \begin{bmatrix} N_2^n \cos n\theta \\ M_2^n \cos n\theta \\ Q_2^n \sin n\theta \end{bmatrix}; \quad \boldsymbol{\varepsilon}_2 = \begin{bmatrix} \varepsilon_2 \\ \kappa_2 \\ \gamma_2 \end{bmatrix} = \sum_{n=0}^{\infty} \begin{bmatrix} \varepsilon_2^n \cos n\theta \\ \kappa_2^n \cos n\theta \\ \gamma_2^n \sin n\theta \end{bmatrix} \tag{7b}$$

$$\mathbf{p} = \begin{bmatrix} p_u \\ p_v \\ p_w \\ 0 \\ 0 \end{bmatrix} = \sum_{n=0}^{\infty} \begin{bmatrix} p_u^n \cos n\theta \\ p_v^n \sin n\theta \\ p_w^n \cos n\theta \\ 0 \\ 0 \end{bmatrix}; \quad \mathbf{u} = \begin{bmatrix} u \\ v \\ w \\ \beta_1 \\ \beta_2 \end{bmatrix} = \sum_{n=0}^{\infty} \begin{bmatrix} u^n \cos n\theta \\ v^n \sin n\theta \\ w^n \cos n\theta \\ \beta_1^n \cos n\theta \\ \beta_2^n \sin n\theta \end{bmatrix} \tag{7c}$$

where the superscripted quantities in (7a-c) are the Fourier coefficients of the nth circumferential harmonic. By substituting (7a-c) into (6) and integrating $\delta\Pi$ over θ, one may write the final form of the variational principle

$$\delta\Pi^n = \delta\left(\int_0^H L^n A_1 dz + B^n\right) = 0 \tag{8}$$

where

$$L^n = \mathbf{f}_1^{nT} \delta\boldsymbol{\varepsilon}_1^n + \mathbf{f}_2^{nT} \delta\boldsymbol{\varepsilon}_2^n - \mathbf{p}^{nT} \delta\mathbf{u}^n \tag{9}$$

is the modified expanded energy density for the nth harmonic circular number and the integration extrema 0 and H represent the limits for the z vertical coordinate (see Fig. 1).

Omitting, for brevitiy's sake, the complete analytical derivations in (8), one obtains:

$$[(RN_\phi)_{,z} + (A_1 N_{\phi\theta})_{,\theta} - R_{,z} N_\theta] + A_1 Q_\phi (R/R_\phi) + p_u R A_1 = 0 \tag{10a}$$

$$[(RN_{\phi\theta})_{,z} + (A_1 N_\theta)_{,\theta} + R_{,z} N_{\phi\theta}] + Q_\theta (A_1 \sin\phi) + p_v R A_1 = 0 \tag{10b}$$

$$[(RQ_\phi)_{,z} + (A_1 Q_\theta)_{,\theta}] - A_1 N_\phi (R/R_\phi) - N_\theta (A_1 \sin\phi) + p_w R A_1 = 0 \tag{10c}$$

$$-(RM_{\phi\theta})_{,z} - (A_1 M_\theta)_{,\theta} - R_{,z} M_{\phi\theta} + A_{1,2} M_\phi + R A_1 Q_\theta = 0 \tag{10d}$$

$$(RM_\phi)_{,z} + (A_1 M_{\phi\theta})_{,\theta} - R_{,z} M_\theta - R A_1 Q_\phi = 0 \tag{10e}$$

which represent the static equilibrium conditions for each harmonic number considered.

Last, a proper set of boundary conditions depending on how the B term is written (i.e., how boundary loads are set) comes from the stationarity of (6) and one gets a fully well posed b.v. problem. The following are the types of boundary assignments presented in the numerical examples studied in the following:

$$\text{free edge: } N_1^n = N_{12}^n = Q_1^n = M_1^n = M_{12}^n = 0 \tag{11a}$$

$$\text{simply supported edge: } u^n = v^n = w^n = M_1^n = \beta_2^n = 0 \tag{11b}$$

$$\text{clamped edge: } u^n = v^n = w^n = \beta_1^n = \beta_2^n = 0 \tag{11c}$$

which do not need necessarily to be homogeneus.

It should be noted that reported in Reddy (1984), one can deduce the correct equations for specialized geometry directly from Eq. (10), by setting correct values to the geometric parameters and not needing to restart from the variational formulation. For instance (one is refereed to Figs. 1, 2, 3):

- Circular plate: $R_\phi \to \infty$
- Circular cylindrical shell: $R = const., \quad R_\phi \to \infty$
- Spherical shell: $R_\phi = const.$

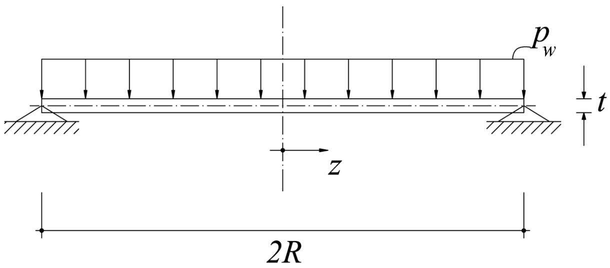

Fig. 2 Simply supported circular plates subjected to a uniform pressure

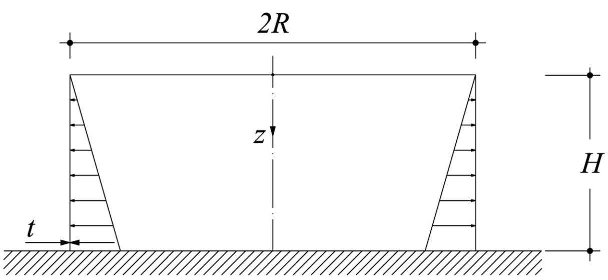

Fig. 3 Circular cylindrical tank with hydrostatic loading

4. Fundamental system

The shells examined in this paper are made of linearly elastic isotropic material and the corresponding constitutive equations can be put in harmonic form too:

$$\begin{bmatrix} N_1^n \\ N_2^n \\ N_{12}^n \\ M_1^n \\ M_2^n \\ M_{12}^n \\ Q_1^n \\ Q_2^n \end{bmatrix} = \begin{bmatrix} E_1 & E_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ E_2 & E_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & E_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & E_4 & E_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & E_5 & E_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & E_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & E_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & E_7 \end{bmatrix} \begin{bmatrix} \varepsilon_1^n \\ \varepsilon_2^n \\ \varepsilon_{12}^n \\ \kappa_1^n \\ \kappa_2^n \\ \kappa_{12}^n \\ \gamma_1^n \\ \gamma_2^n \end{bmatrix} \tag{12}$$

where E is Young modulus, ν is the Poisson ratio and $E_1 = Et/2(1-\nu^2)$, $E_2 = \nu E_1$, $E_3 = (1-\nu)E_1/2$, $E_4 = Et^3/[12(1-\nu^2)]$, $E_5 = \nu E_4$, $E_6 = (1-\nu)E_4/2$, $E_7 = \lambda E_3$. λ is a transverse shearing factor such that $\lambda = 5/6$. Introducing the kinematic equations into the above relations and then substituting the modified constitutive equations into the set of equilibrium equations leads to the following sets of harmonically-uncoupled fundamental equations, expressed in terms of harmonic components of generalized displacements:

$$K_{11}^n u^n + K_{12}^n v^n + K_{13}^n w^n + K_{14}^n \beta_1^n + K_{15}^n \beta_2^n = p_u^n R \tag{13a}$$

$$K_{21}^n u^n + K_{22}^n v^n + K_{23}^n w^n + K_{24}^n \beta_1^n + K_{25}^n \beta_2^n = p_v^n R \tag{13b}$$

$$K_{31}^n u^n + K_{32}^n v^n + K_{33}^n w^n + K_{34}^n \beta_1^n + K_{35}^n \beta_2^n = p_w^n R \tag{13c}$$

$$K_{41}^n u^n + K_{42}^n v^n + K_{43}^n w^n + K_{44}^n \beta_1^n + K_{45}^n \beta_2^n = 0 \tag{13d}$$

$$K_{51}^n u^n + K_{52}^n v^n + K_{53}^n w^n + K_{54}^n \beta_1^n + K_{55}^n \beta_2^n = 0 \tag{13e}$$

The coefficients K_{ij} are listed in Viola *et al.* (2003). The G.D.Q. technique is applied to the systems (13a-e), keeping in mind that the resulting equations contain derivatives with respect to the longitudinal coordinate z only.

5. The G.D.Q. method implementation and solution

The basis of the G.D.Q. method, when applied to linear time-invariant PDE systems is resumable by the following sequence (Bellomo 1997):

- Collocation of the space variables into a finite number of grid points

- Approximation of the dependet variable(s) of the problem by means of a proper linear interpolation rule, through collocation points values.
- The space derivatives are approximated themselves using the aforementioned interpolation.
- The differential b.v. problem is transformed into a set of algebraic problems, each imposed at a grid point. Boundary conditions are similarly assigned at those points corresponding to the boundary of the domain of the independent variables.
- The solution, in terms of local values of dependent variables, is obtained by solving a linear algebraic set of equations. These values are subsequently interpolated along the domain, with the previous method.

In this way, the representation of the derivatives of a z-dependent variable, say $f(z)$, defined in the closed interval $[0, H]$, is given as:

$$\left.\frac{\partial^r f(z)}{\partial z^r}\right|_{z = z_i} = \sum_{j=1}^{m} A_{ij}^{(r)} f(z_j) \tag{14}$$

which is a weighted sum of function values at every grid point along the domain. Here A_{ij} are the weighting coefficients of the rth order derivative at the ith sampling point along the domain, and m is the number of sampling points in the collocation $\{0 = z_1, z_2, \ldots, z_i, \ldots, z_m = H\}$.

The weighting coefficients (w.c.) can be determined, depending on the chosen interpolation rule. For the cases treated in the present paper, Lagrange polynomial functions have been adopted; therefore some simple recursive formulas are available in finding weighting coefficients (Shu and Richards 1992):

$$A_{ij}^{(1)} = \frac{\pi(z_i)}{(z_i - z_j)\pi(z_j)}; \quad i, j = 1, 2, \ldots, m; \quad i \neq j \tag{15}$$

where

$$\pi(z_i) = \prod_{j=1}^{m} (z_i - z_j); \quad i, j = 1, 2, \ldots, m; \quad i \neq j \tag{16}$$

for the first order derivative while, for higher order derivatives, one gets iteratively:

$$A_{ij}^{(r)} = r\left[A_{ij}^{(r-1)} A_{ik}^{(1)} - \frac{A_{ij}^{(r-1)}}{(z_i - z_k)}\right]; \quad (i \neq j), \quad 2 \leq r \leq m-1 \tag{17}$$

$$A_{ii}^{(r)} = -\sum_{\substack{k=1 \\ k \neq i}}^{m} A_{ik}^{(r)} \quad i = 1, 2, \ldots, m, \quad 1 \leq r \leq m-1 \tag{18}$$

As will be seen later, in the present study, only one-dimensional dependent variables will be involved and the Chebyshev polynomials roots will be used for numerical reasons (Bellomo 1997), as collocation points along the domain:

$$z_i = \frac{1-\cos[(i-1)\pi/(m-1)]}{2}H, \quad i = 1, 2, \ldots, m \tag{19}$$

Moreover, the boundary conditions in terms of generalized displacements harmonic amplitudes can be cast directly within the approximated equations. Applying the quadrature rule, in order to discretize the systems (13) and the boundary conditions, enables to impose the approximate equilibrium equations at every collocation point, for each harmonic number. After a proper subdivision of nodal generalized displacements, the solving sets of equilibrium harmonic equations take the following form:

$$\begin{bmatrix} \mathbf{K}_{bb}^n & \mathbf{K}_{bd}^n \\ \mathbf{K}_{db}^n & \mathbf{K}_{dd}^n \end{bmatrix} \begin{bmatrix} \mathbf{u}_{bb}^n \\ \mathbf{u}_{dd}^n \end{bmatrix} = \begin{bmatrix} \mathbf{p}_{bb}^n \\ \mathbf{p}_{dd}^n \end{bmatrix} \tag{20}$$

where the subscripts b and d stand for *boundary* and *domain*, respectively, in a way that b-equations represent the discretized boundary conditions, which are valid only for the points lying on constrained edges of the shell or at the apex; while d-equations are proper equilibrium equations, imposed on points of the interior of the domain.

The solution procedure takes advance, now, of the static condensation, in order to obtain a system of equations in terms of *domain* d.o.f. only:

$$\overline{\mathbf{K}}_{dd}^n \mathbf{u}_{dd}^n = \overline{\mathbf{p}}_{dd}^n \tag{21}$$

where

$$\overline{\mathbf{K}}_{dd}^n = \mathbf{K}_{dd}^n - \mathbf{K}_{db}^n (\mathbf{K}_{bb}^n)^{-1} \mathbf{K}_{bd}^n \tag{22}$$

and

$$\overline{\mathbf{p}}_{dd}^n = \mathbf{p}_{dd}^n - \mathbf{K}_{db}^n (\mathbf{K}_{bb}^n)^{-1} \mathbf{p}_{bb}^n \tag{23}$$

When the global condensed system (21) is solved for the internal degrees of freedom, by simply recalling that

$$\mathbf{u}_{bb}^n = (\mathbf{K}_{bb}^n)^{-1} (\mathbf{p}_{bb}^n - \mathbf{K}_{bd}^n \mathbf{u}_{dd}^n) \tag{24}$$

gives the boundary values for displacements variables. Finally, applying the steps listed in section 5 and using both the strain-displacement and constitutive relations, one obtains the complete approximate state of stress and deformation of the shell structure.

6. Numerical results and discussion

6.1 Simply supported circular plate subjected to uniform pressure

To show the accuracy and precision of the present method, we first consider a circular plate under

Table 1 Comparison between present results and Kim and Kim hybrid mixed finite element solution

	z (m)	Theor.	Kim & Kim	G.D.Q.
$w \times 10^{-3}$ (m)	0	0.938	0.939	0.938
	0.25	0.868	0.868	0.868
	0.50	0.668	0.668	0.668
	0.75	0.364	0.363	0.364
	1.00	0	0	0
$\beta_1 \times 10^{-3}$	0	0	0	0
	0.25	0.551	0.551	0.551
	0.50	1.031	1.031	1.031
	0.75	1.371	1.371	1.371
	1.00	1.500	1.500	1.500
M_1 (kNm/m)	0	0.188	0.188	0.188
	0.25	0.176	0.175	0.176
	0.50	0.141	0.141	0.141
	0.75	0.082	0.082	0.082
	1.00	0.000	0.000	5×10^{-17}

a uniform pressure (Fig. 2). The boundary conditions impose closed apex continuity at the centre point and null transverse displacement along the extreme parallel; it is remarkable how the point continuity conditions at the centre can be exactly imposed with the aim of the G.D.Q. technique (see Viola and Artioli 2004). The material data (as taken from Kim and Kim 2000) are $E = 10^6$ kN/m^2, $\nu = 0.0$, $R = 1$ m, $t = 0.01$ m, $p = 1$ kN/m^2. In Table 1, the present results are plotted, and those available from Kim and Kim (2000), resulting from a recent hybrid-mixed rotational finite element and those from elementary plate theory (Reddy 1984). For this analysis, only 5 sampling points are sufficient to yield good accuracy both for kinematic and static quantities. With regards to this example, the computational efficiency of the present method, when compared to the reference solution, is recognizable in the less computational cost involved in obtaining the global linear system which for both methods has the same dimension.

6.2 A clamped-free cylindrical tank with idrostatic load

Fig. 3 shows the geometry of a liquid-filled circular cylindrical shell with one clamped edge, the other one being free; as for the preceding example the only circular harmonic involved is $n = 0$, due to axial symmetry of the system. The geometry and material data of the shell are: $E = 3 \times 10^6$ kN/m^2, $\nu = 0.25$, $R = 360$ in., $t = 14$ in., $p = 0.03616 \times z$ lb/in^2. Results from a G.D.Q. modelling with 11 grid points along meridian are shown in Fig. 4 and Fig. 5. The accuracy of the solution in terms of internal stress resultants and moments is achieved and a good agreement between the present solution and the results from Gould and Sen (1971) is evident. The reference solution was obtained using a mixed rotational finite element and required a discretization with more that 25 nodes along meridional curve. Fig. 6 shows that to obtain a sufficient accuracy for the free edge translations less nodes were required in the analysis.

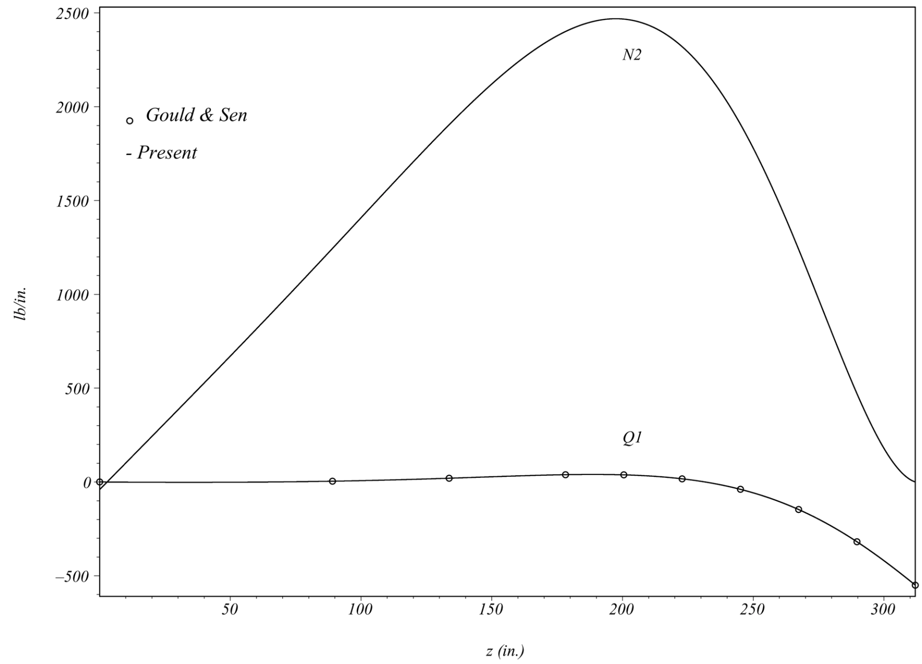

Fig. 4 Circumferential stress resultant N_2 and transverse meridional shear Q_1 for tank shell in Fig. 3. Comparison with results from Gould and Sen

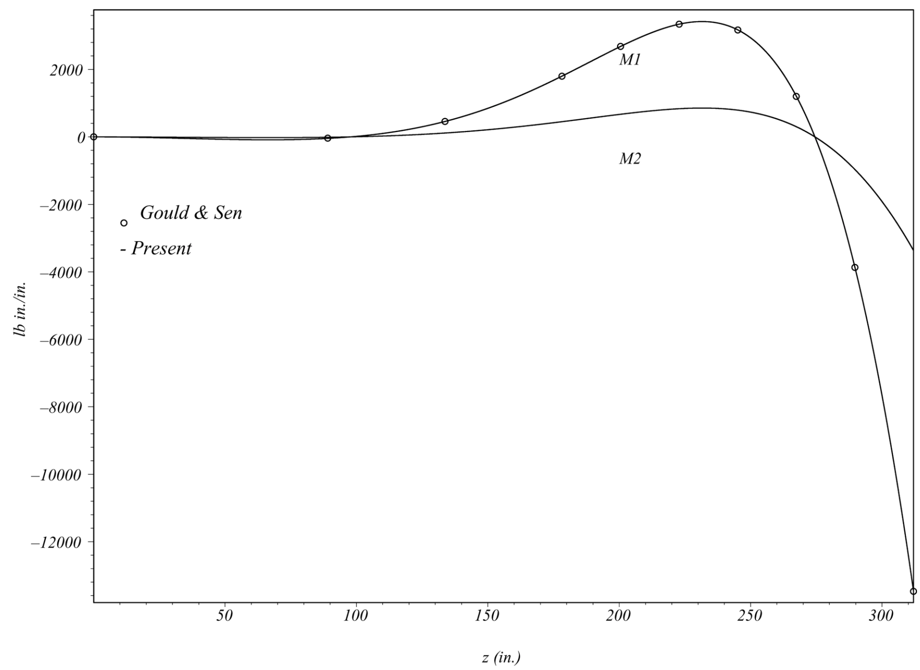

Fig. 5 Stress couples M_1 and M_2 for tank shell in Fig. 3. Comparison with results from Gould and Sen

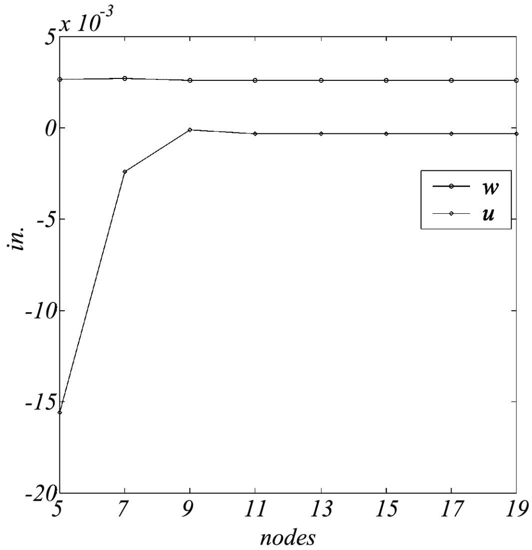

Fig. 6 Convergence pattern for the meridional and transverse displacements at top of cylindrical tank with hydrostatic load if Fig. 3

6.3 Hyperboloid of revolution subjected to wind pressure

A rather classic example in the static analysis of non-symmetrically loaded shells of revolution is shown in Fig. 7. The clamped hyperboloidal shell with uniform thickness shown in Fig. 7(a) is subjected to a wind pressure p whose pattern around the parallel has been reported in Fig. 4(b), while it is assumed that no variation occurs along the vertical coordinate z. The external normal load has been developed in Fourier series accounting for the first ten harmonic components (Kim

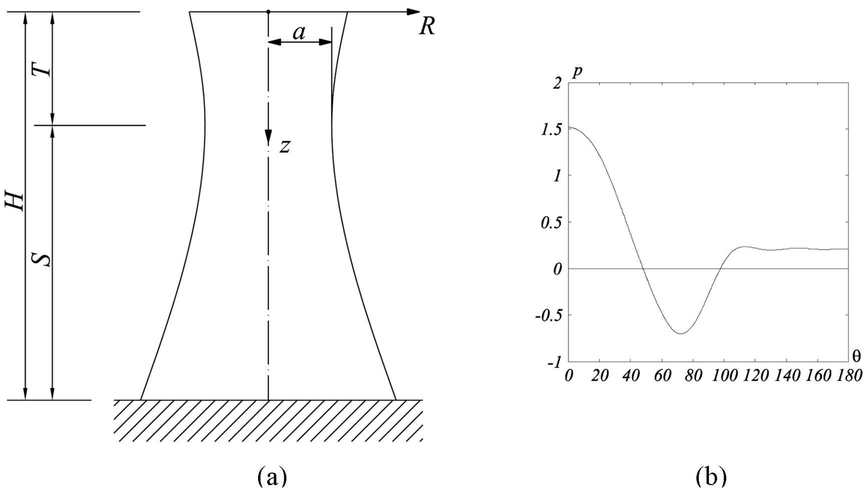

Fig. 7 (a) Clamped-free hyperboloid of revolution subjected to unsymmetrical wind pressure (b) Wind pressure pattern and Kim 2000). The assumed shell meridian equation is the following:

$$\frac{z^2}{a^2} - \frac{R^2}{b^2} = 1$$

with the geometric parameters a = 84 ft and b = 209.66 ft. The elastic parameters are $E = 432 \times 10^6$ lb/ft^2, $\nu = 0.15$ and t = 7 in. and the total height of the tower is $H = S + T = 270 + 60 = 330$ ft. The present solution has first been compared with that obtained by Kim and Kim (2000) with a brand-newly developed hybrid-mixed harmonic rotational element and then with that obtained by Luah and Fan (1990) using a spline finite element method. The present analysis made use of 19 grid points along the domain, producing a global system of size 85. The analysis by the hybrid-mixed element of Kim and Kim requires a total amount of 15 unequally spaced elements with a grand total of almost 230 condensed d.o.f.. Fig. 8 again shows a quite good agreement between the present analysis and the scaled values extracted by the paper of Kim and Kim, for what concerns the transverse displacement along centre meridian $\theta = 0$. In Fig. 9 and Fig. 10 a comparison is made between present results computed with the G.D.Q. method and those from a rotational spline finite element (Luah and Fan 1990). This excellent agreement also for membrane and couple stress resultants is achieved with a significantly lower number of nodes (19) than those used in the reference paper. It is to be noted also that, differing from the F.E. method, within the present approach no integration prior to the global system assembly is necessary and that this results in a further reduction of computational cost.

As mentioned earlier, (Fig. 11), even a fewer number of nodal points are necessary for the good prediction of displacements.

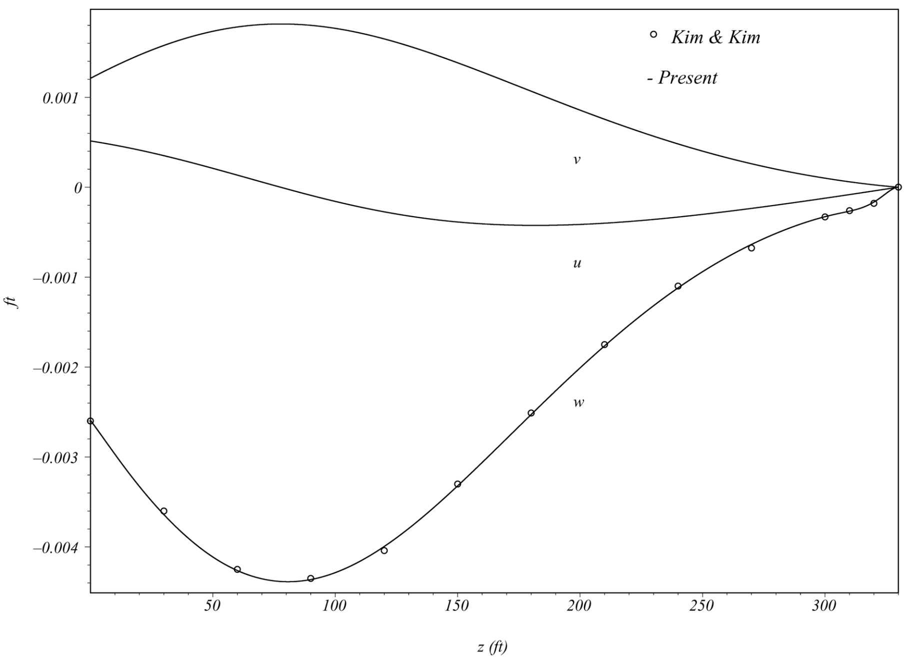

Fig. 8 Translational displacement u, v and w for the wind-loaded hyperbolic cooling tower in Fig. 7(a). Comparison with results from Kim and Kim

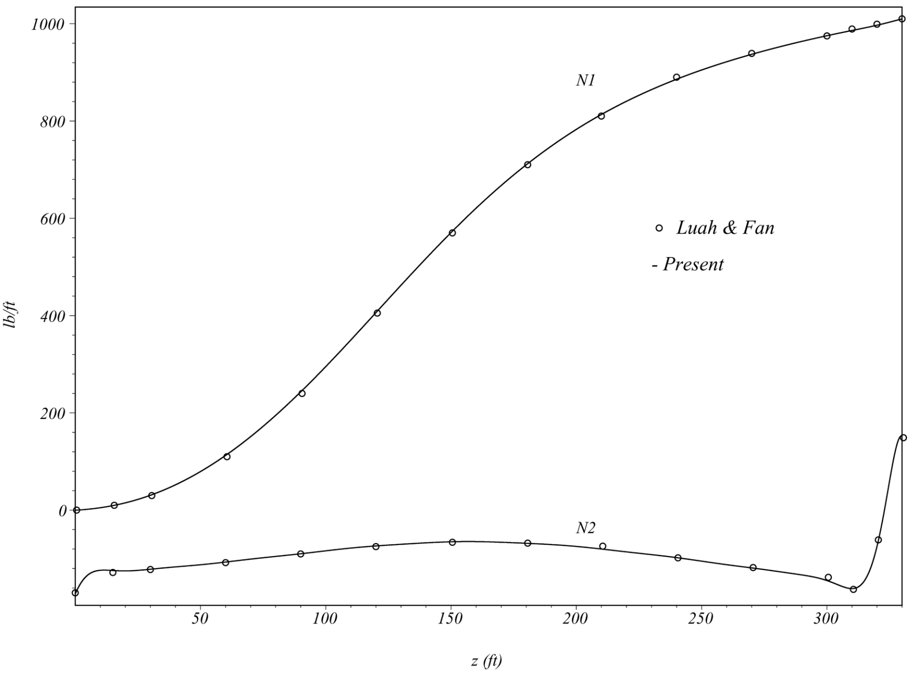

Fig. 9 Internal stress resultants N_1 and N_2 for the hyperbolic rotational shell subjected to wind pressure in Fig. 7(a). Comparison with results available from Luah and Fan

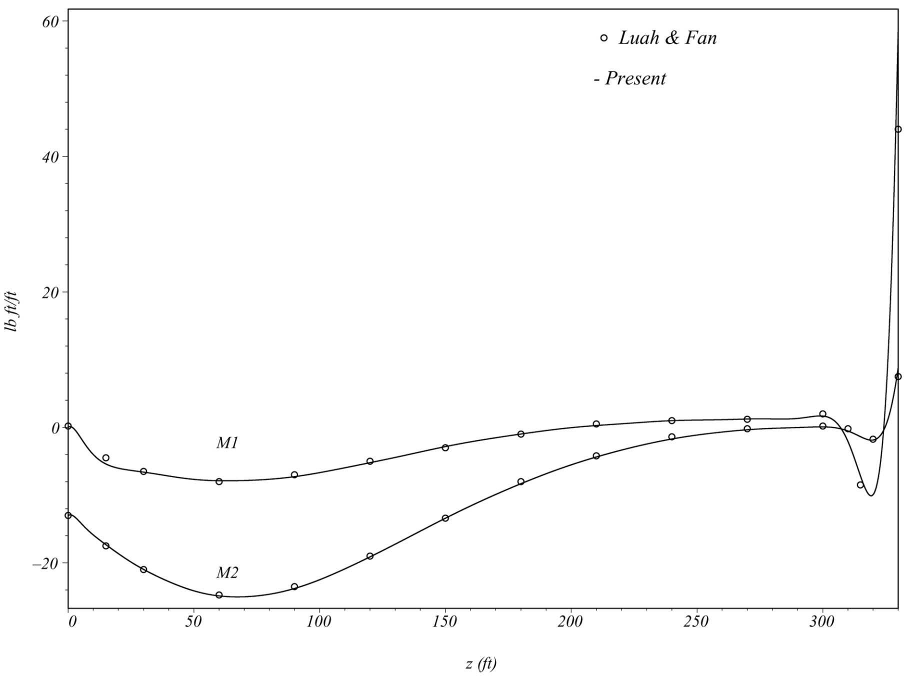

Fig. 10 Internal stress couples M_1 and M_2 for the hyperbolic rotational shell subjected to wind pressure in Fig. 7(a). Comparison with results available from Luah and Fan

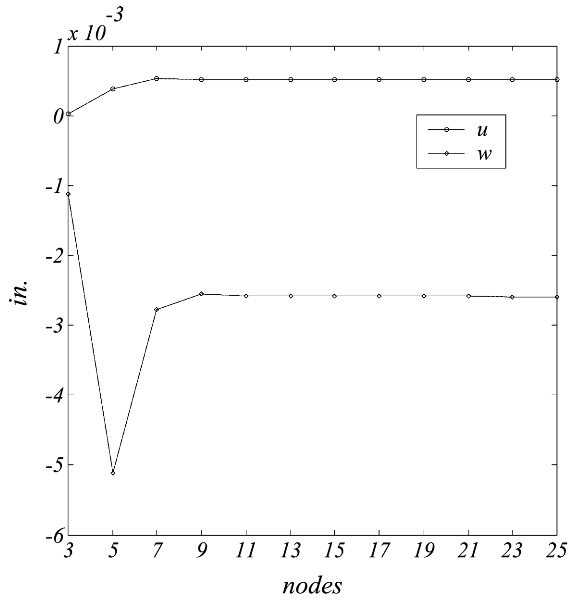

Fig. 11 Convergence characteristics for top edge displacements of the wind-loaded hyperbolic cooling tower in Fig. 7

7. Conclusions

A G.D.Q. procedure for the static analysis of rotational shells has been presented. Starting from general shell equations a proper set of equilibrium equations in terms of circular harmonic components of generalized displacements and a set of boundary conditions has been obtained. These partial differential equations in the axial coordinate only are easily discretized by means of the technique in argument, to yield a set of common linear systems each for a circular harmonic number. Some comparisons with available results confirm how this simple numerical method provides accurate and computationally low-cost results.

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References

Bellman, R. and Casti, J. (1971), “Differential quadrature and long-term integration”, *J. Mathematics Analytic Applications,* **34**, 235-238.

Bellman, R., Kashef, B. and Casti, J. (1972), “Differential quadrature: a technique for the rapid solution of nonlinear partial differential equations”, *J. of Computational Physics,* **10**(1), 40-52.

Bellomo, N. (1997), “Nonlinear models and problems in applied sciences from differential quadrature to generalized collocation methods”, *Mathematical and Computer Modelling*, **26**(4), 13-34.

Bert, C.W. and Malik, M. (1996), “Differential quadrature method in computational mechanics: a review”, *Appl. Mech. Rev.,* **49**, 1-27.

Bert, C.W. and Malik, M. (1996), “Free vibration analysis of thin cylindrical shells by the differential quadrature method”, *J. of Pressure Vessel Technology,* **118**, 1-12.

Gould, P.L. (1999), *Analysis of Shells and Plates*, Upper Saddle River, Prentice Hall.

Gould, P.L. and Sen, S.K. (1971), “Refined mixed method finite elements for shells of revolution”, *Proc. Air Force third Conference on matrix methods in structural mechanics*, AFB Ohio.

Jiang, W. and Redekop, D. (2002), “Polar axisymmetric vibration of a hollow toroid using the differential quadrature method”, *J. Sound Vib.*, **251**(4), 761-765.

Kim, J.G. and Kim, Y.Y. (2000), “A higher-order hybrid-mixed harmonic shell-of-revolution element”, *Comput. Meth. Appl. Mech. Eng.,* **182**, 1-16.

Krysl, P. and Belytschko, T. (1996), “Analysis of thin shells by the element-free Galerkin method”, *Int. J. Solids Struct.,* **33**(20-22), 3057-3080.

Kunieda, H. (1984), “Flexural axisymmetric free vibrations of a spherical dome: exact results and approximate solutions”, *J. Sound Vib.*, **92**(1), 1-10.

Liu, Y. (1998), “Analysis of shell-like structures by the boundary element method based on 3-D elasticity: Formulation and verification”, *Int. J. Numer. Meth. Eng.,* **41**, 541-558.

Luah, M.H. and Fan, S.C. (1990), “New spline finite element for analysis of shells of revolution”, *J. Eng. Mech.,* ASCE, **116**(3), 709-725.

Mirfakhraei, P. and Redekop, D. (1998), “Buckling of circular cylindrical shells by the Differential Quadrature Method”, *Int. J. of Pressure Vessels and Piping,* **75**, 347-353.

Mizusawa, T. (1988), “Application of spline strip method to analyse vibration of open cylindrical shells”, *Int. J. Numer. Meth. Eng.,* **26**, 663-676.

Prato, C. (1969), “Shell finite element method via Reissner’s principle”, *Int. J. Solids Struct.,* **5**, 1119-1133.

Reddy, J.N. (1984), *Energy and Variational Methods in Applied Mechanics*, John Wiley & Sons.

Redekop, D. and Xu, B. (1999), “Vibration analysis of toroidal panels using the differential quadrature method”, *Thin Walled Structures*, **34**, 217-231.

Reissner, E. (1946), “Stresses and small displacements of shallow spherical shells – Part II”, *J. Mathematics Physics,* **25**, 80-85.

Shu, C. and Richards, B.E. (1992), “Application of generalized differential quadrature to solve two-dimensional incompressible Navier-Stokes equations”, *Int. J. Numer. Meth. Eng.*, **15**, 791-798.

Viola, E. and Artioli, E. (2004), “The G.D.Q. method for the harmonic dynamic analysis of rotational shell structural elements”, *Structural Engineering and Mechanics*, **17**(6), 789-817.

Viola, E., Sgallari, F. and Artioli, E. (2003), “Computational efficiency of the Generalized Differential Quadrature method in the elastic analysis of shells of revolution”, Technical Report, DISTART, University of Bologna, Italy.

Wu, C.P. and Lee, C.Y. (2001), “Differential quadrature solution for the free vibration analysis of laminated conical shells with variable stiffness”, *Int. J. Mech. Sci.,* **43**, 1853-1869.

Yang, H.T.Y., Saigal, S., Masud, A. and Kapania, R.K. (2000), “A survey of recent shell finite elements”, *Int. J. Numer. Meth. Eng.,* **47**, 101-127.

Notation

z, θ	: vertical and colatitudinal curvilinear coordinates on the reference surface
ζ	: normal coordinate along shell thickness
$R(z)$	: parallel circle normal radius

$R_\phi(z)$	: meridian radius
A_1, A_2	: Lamè parameters of the reference surface
U, V, W	: displacement components of points lying within shell thickness
u, v, w	: displacement components of points lying on shell midsurface
β_1, β_2	: midsurface normal rotations
$\mathbf{f}_1$, $\mathbf{f}_2$	: internal stress resultants and couples
$\boldsymbol{\varepsilon}_1$, $\boldsymbol{\varepsilon}_2$	: midsurface strain measures
$\mathbf{p}$	: midsurface distributed load intensities
$\mathbf{u}$	: midsurface distributed load intensities
λ	: shearing correction factor
$A_{ik}^{(r)}$	: weighting coefficients for the G. D. Q. approximations of derivatives
n	: circular harmonic number
$\mathbf{K}_{ij}^n$, $i, j = b, d$	: stiffness submatrices of discretized structure (boundary and domain)
$\mathbf{u}_{ij}^n$, $i, j = b, d$	: discretized structure nodal degrees of freedom (boundary and domain)
$\underline{\mathbf{p}}_{ij}^n$, $i, j = b, d$	: discretized structure nodal load vectors (boundary and domain)
$\overline{\mathbf{K}}_{dd}^n$	: condensed stiffness matrix of discretized structure
$\overline{\mathbf{p}}_{dd}^n$	: condensed load vector of discretized structure