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Large deflections of variable-arc-length beams under uniform self weight: Analytical and experimental

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Abstract. This paper presents the solution of large static deflection due to uniformly distributed self weight and the critical or maximum applied uniform loading that a simply supported beam with variable-arc-length can resist. Two analytical approaches are presented and validated experimentally. The first approach is a finite-element discretization of the span length based on the variational formulation, which gives the solution of large static sag deflections for the stable equilibrium case. The second approach is the shooting method based on an elastica theory formulation. This method gives the results of the stable and unstable equilibrium configurations, and the critical uniform loading. Experimental studies were conducted to complement the analytical results for the stable equilibrium case. The measured large static configurations are found to be in good agreement with the two analytical approaches, and the critical uniform self weight obtained experimentally also shows good correlation with the shooting method.

Key words: large sag deflection; variable-arc-length beams; uniformly distributed self weight; finite-element solution; shooting method; experimental studies.

1. Introduction

Recently, the simply-supported Variable-Arc-Length (VAL) beam has been investigated using various loading conditions and solution methods. The variability in arc-length of this beam arises from one end being pinned and the other end being supported by a frictionless roller at a fixed

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distance from the pinned end. The distance between the two supports is specified while the total arc-length of the beam is unknown and must be determined. Since the beam span length is very long, the change in beam arc-length due to sag is not negligible. Subjected to either self weight or applied forces, the beam can experience large static sag deflection in which sag ratio is significantly larger than that obtained from Linear Beam Theory (LBT). The sag ratio is defined as the ratio of mid-span static deflection due to a given loading to the beam span length, $y_{s_{max}}/L$.

The flexible elastic horizontally sagging pipelines or very long span suspended elastica pipes, which are frequently found in the fields of offshore engineering and petroleum industries, may be considered as VAL beams. Interesting features of these beams are that they may have at least two equilibrium configurations under given loading conditions and that there exists a critical or maximum loading. Therefore, it is essential for an engineer designing a VAL beam to have a good understanding of static behavior.

Chucheepsakul *et al*. (1995, 1996, 1997a, b, 1999) solved the large deflection of VAL beams subjected end moment and point loading by using the finite element method, the shooting-optimization method, and the elliptic integral. The different methods provide independent verification of each solution. Studies have addressed additional static problems such as beam deflection using intrinsic coordinates (Golley 1997), large deflections of beams under point loads (Wang *et al.* 1997), and beam static behavior under follower force (Hartono 2000).

However, the static solution of the VAL beam configuration subjected to uniformly distributed loading or beam self weight has not yet been found. Therefore, the present study continues in this line of research by providing the large static (sag) deflection solution of beam under a given uniform self weight magnitude. The maximum or critical value of applied uniform beam self weight that the beam can resist is also highlighted. The measured results of static sag deflection and the critical applied loading are conducted to validate the analytical results.

Two analytical approaches are presented to solve the problem. The finite-element approach is based on the variational formulation, in which the energy functional involving strain energy due to bending, the potential energy from uniform beam self weight, and the additional strain energy due to axial force is formulated. Because of the unknown arc-length, the finite-element discretization of span length is employed. The nonlinear equation is solved to obtain the stable static configurations using the Newton-Raphson iterative process. The second approach is the shooting method based on the elastica theory formulations in which the set of governing differential equations is numerically integrated using the Runge-Kutta-Felhberg algorithm. This method gives the results which included both stable and unstable equilibrium configurations, and the critical value of applied uniform loading is also obtained.

In association with analytical formulations, static tests in precisely controlled experiments were conducted. The range of tested beam specimens were built to behave as analytical models and measured to obtain the large stable static beam configurations at the specific point along beam arc-length as well as the critical uniform self weight. The experimental results, for both the stable static configuration due to a given uniform self weight and for the critical loading, are in very good agreement with analytical results.

2. Analytical solutions

The VAL beam is pinned at end *A* and supported on the frictionless roller at end *B* located at a

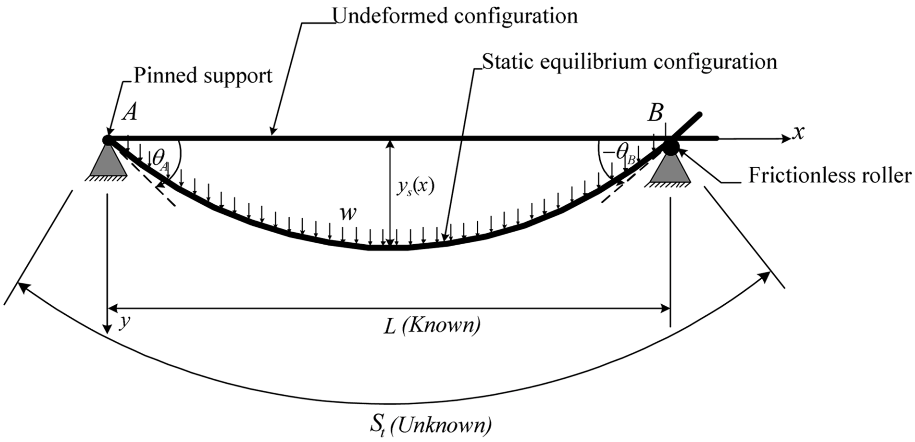

Fig. 1 Undeformed and deformed configuration of VAL beams

fixed distance L from A as shown in Fig. 1. The total arc-length of beam, S_t, is varied due to the "sag" of the beam from its self weight given as w (force per unit arc-length). The VAL beam is made from an elastic material. The beam's section dimension is very small in comparison with its length making it a "slender" beam. The overhung part of the beam is short in comparison with its span-length and therefore its effects are neglected. Since axial movement is unrestrained at the roller, the effects of axial deformation are not included. Shear deformation is small and therefore neglected as usual for slender beams.

2.1 Finite element solution

To obtain the large static equilibrium deformation, $y_s(x)$, the energy functional of the beam system is established and minimized using the finite element method. The beam formulations are derived based on the function of x, the projection of beam on the known span length, L, instead of the beam arc-length, S. Because the total arc-length of VAL beam is an initial unknown, the use of beam arc-length, as the independent variable as commonly used for conventional beam elements, may not be convenient for establishing the beam boundary conditions. The boundary conditions can be conveniently established by using the horizontal known span length as the independent variable.

Considering the case of large displacement at a position, $0 \le x \le L$, the element energy functional due to bending under beam self weight and axial force is expressed in rectangular coordinates as

$$\pi_k = \int_0^{l_k} \frac{1}{2} \frac{EIy_s''^2}{(1 + y_s'^2)^{5/2}} dx - \int_0^{l_k} wy_s \sqrt{1 + y_s'^2}\, dx + \int_0^{l_k} (N + \lambda) \sqrt{1 + y_s'^2}\, dx \tag{1}$$

The terms $y_s(x)$, $y_s'(x)$, and $y_s''(x)$ are the large static displacements and its derivatives with respect to the variable x, where

x is measured along the undeformed median line of beam,
l_k is the x projection of the kth element,
EI is the transverse bending stiffness of the beam, and
dx is the x projection of length ds of the beam.

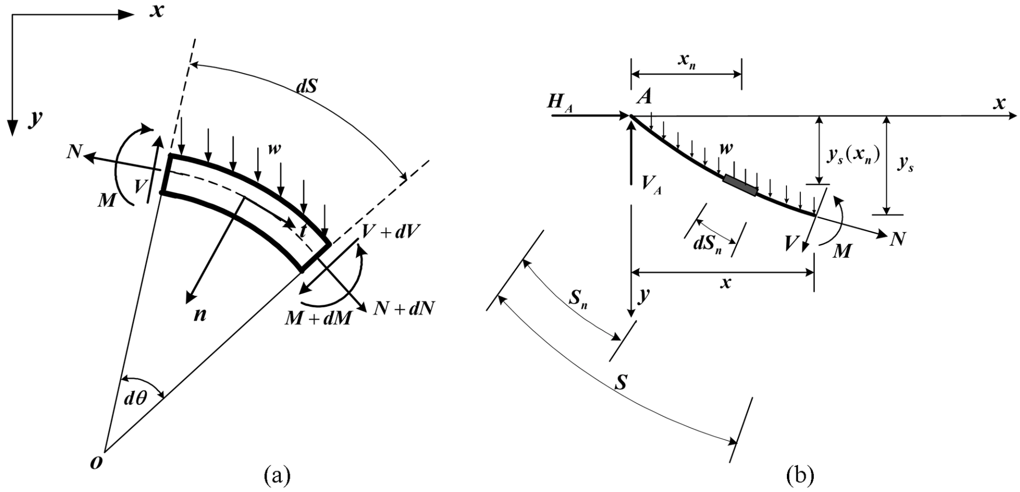

Fig. 2 Equilibrium of forces on beam segment: (a) normal and tangential coordinate, (b) rectangular coordinate

The first term in Eq. (1) is bending strain energy. The second term is the potential energy of uniformly distributed load. The multiplier N of the last term is identified as axial force, while λ represents a multiplier to account for the arc-length being unvaried due to bending at equilibrium position. The multiplier N can be obtained by considering the static equilibrium of forces on the segment of beam in the normal and tangential directions (Malvern 1969) as shown in Fig. 2(a). This multiplier is written as

$$N = -\frac{EI}{2}\left(\frac{d\theta}{dS}\right)^2\Bigg|_0^S - \int_0^s w\sin\theta dS = -\frac{EI}{2}\frac{y_s''^2}{(1+y_s'^2)^3}\Bigg|^x - wy_s\Big|^x \tag{2}$$

The multiplier λ is also given in Huang and Chucheepsakul (1985) as

$$\lambda = \frac{EI}{2}\frac{y_s''^2}{(1+y_s'^2)^3} \tag{3}$$

The finite element procedure based on the span length discretization (Chucheepsakul *et al.* 1995, 1997a) is used to solve the stationary condition, $\delta\pi_k = 0$. In the procedure, the span length is divided into a number of elements. The large static displacement of the span element is approximated by

$$y_s = [N]\{q\} \tag{4}$$

in which $[N]$ = row of fifth-order polynomial shape functions; and $\{q\}$ is a nodal displacement vector of y_s and its first and second derivatives at both ends of the element.

By using the fifth-order polynomial shape function, the beam's displacements, y_s, and its derivatives up to second order are directly computed at the nodes. In addition, a better result is obtained for a fifth-order polynomial rather than a cubic polynomial. A thorough explanation of use of fifth-order polynomial shape functions is given in Chucheepsakul and Huang (1997b).

Substituting the approximated displacement y_s and its derivative in the element energy, the energy functional of the kth element, π_k, can be expressed in terms of its local degrees-of-freedom. The equilibrium condition, $\delta\pi_k = 0$, leads to the highly nonlinear equilibrium equation

$$\left\{\frac{\partial\pi_k}{\partial q_i}\right\} = \int_0^{l_k}\left\{\begin{array}{l}[N'']^T\dfrac{EIy_s''}{(1+y_s'^2)^{5/2}} - [N']^T\dfrac{5EIy_s''^2y_s'}{2(1+y_s'^2)^{7/2}} - [N]^T w\sqrt{1+y_s'^2} \\ + [N']^T\dfrac{(N+\lambda)y_s'}{\sqrt{1+y_s'^2}}\end{array}\right\}dx \tag{5}$$

in which $y_s' = [N']\{q\}$, and $y_s'' = [N'']\{q\}$. The multipliers N and λ as well as uniform loading w are unvaried during the process of performing the variation $\delta\pi_k$.

By assembling the total contribution, one obtains the global equilibrium equation ($\delta\pi = 0$) in terms of the global degrees of freedom $\{Q_i\}$ as

$$\left\{\frac{\partial\pi}{\partial Q_i}\right\} = \{0\} \tag{6}$$

The system of Eq. (6) with the boundary conditions, $y_s(0) = 0$ and $y_s(L) = 0$ is solved by the Newton-Raphson iterative procedure. With this procedure, the incremental equation is

$$\left[\frac{\partial^2\pi}{\partial Q_i\partial Q_j}\right]\{\Delta Q\} = -\left\{\frac{\partial\pi}{\partial Q_i}\right\} \tag{7}$$

where the matrix on the left side of Eq. (7) is the global incremental stiffness matrix.

2.2 Shooting method

The beam is subjected to a uniformly distributed self weight per unit arc-length, w, within its span length. By considering a free body diagram at static configuration of beam segment as shown in Fig. 2(b), the bending moment, M, is given as

$$M = \frac{wS_t}{2}x + \frac{wS_t}{2}\tan(-\theta_B)y_s - \int_0^s w(x-x_n)dS_n \tag{8}$$

The constitutive relation and the geometric relations are given by

$$M = -EI\frac{d\theta}{dS} \tag{9}$$

In view of Eqs. (8) and (9), the governing differential equations and the boundary conditions can be written as

$$EI\frac{d\theta}{dS} = \frac{wS_t}{2}[-x - y_s\tan(-\theta_B)] + \int_0^s w(x-x_n)dS_n \tag{10a}$$

$$\left.\frac{d\theta}{dS}\right|_{S=0} = 0 \quad \text{and} \quad \left.\frac{d\theta}{dS}\right|_{S=S_t} = 0 \tag{10b}$$

$$\frac{dx}{dS} = \cos\theta, \quad x(0) = 0, \quad x(S_t) = L, \text{ and} \tag{11a-c}$$

$$\frac{dy_s}{dS} = \sin\theta, \quad y_s(0) = 0, \quad y_s(L) = 0 \tag{12a-c}$$

Introducing the following non-dimensional parameters

$$\bar{x} = \frac{x}{L}, \quad \bar{x}_n = \frac{x_n}{L}, \quad \bar{y}_s = \frac{y_s}{L}, \quad S^* = \frac{S}{S_t}, \quad \bar{S}_t = \frac{S_t}{L}, \quad \bar{w} = \frac{wL^3}{EI}, \quad dS^* = \frac{dS}{S_t}, \quad dS_n^* = \frac{dS_n}{S_t} \tag{13a-h}$$

and substituting these non-dimensional parameters into Eqs. (10) to Eqs. (12) yields

$$\frac{d\theta}{dS^*} = \frac{\bar{w}\bar{S}_t^2}{2}[-\bar{x} - \bar{y}_s\tan(-\theta_B)] + \bar{w}S_t^2\int_0^{S^*}(\bar{x} - \bar{x}_n)dS_n^* \tag{14a}$$

$$\left.\frac{d\theta}{dS^*}\right|_{S^* = 0} = 0, \qquad \left.\frac{d\theta}{dS^*}\right|_{S^* = 1} = 0 \tag{14b,c}$$

$$\frac{d\bar{x}}{dS^*} = \bar{S}_t\cos\theta, \quad \bar{x}(0) = 0, \quad \bar{x}(\bar{S}_t) = 1, \text{ and} \tag{15a-c}$$

$$\frac{d\bar{y}_s}{dS^*} = \bar{S}_t\sin\theta, \quad \bar{y}_s(0) = 0, \quad \bar{y}_s(\bar{S}_t) = 0 \tag{16a-c}$$

Thus, for a given value of $\bar{w}$, there are four unknowns (θ, $\bar{x}$, $\bar{y}_s$ and $\bar{S}_t$) to be evaluated from three first order differential equations together with four end conditions ($\bar{x}(0) = 0$; $\bar{x}(1) = 0$; $\bar{y}_s(0) = 0$; and $\bar{y}_s(1) = 0$). However if the given value, $\bar{w}$, is greater than the critical or maximum value, the solution does not exist. Instead of assigning $\bar{w}$, the value of θ had been assigned and unknowns $\bar{x}$, $\bar{y}_s$, $\bar{w}$ and $\bar{S}_t$ are solved. Thus, given any θ_A or θ_B, one guesses $\bar{w}$ and $\bar{S}_t$ at the first iteration from the linear small deflection theory. Eqs. (14) to Eq. (16) are numerically integrated using the fifth order Cash-Karp Runge Kutta with adaptive step size adjustment during integration following Felhberg method (Press *et al.* 1992).

In the calculation, the error is minimized by the simplex method (Nelder and Meade 1965) in which the objective function for the minimization is

$$\underset{\bar{w},\bar{S}_t}{\text{minimize}}\,\phi = |\bar{x}(0)| + |\bar{y}_s(0)| \tag{17}$$

In computation, the desired value of ϕ is zero for a solution.

3. Measurement procedure

Three laboratory-scale beam specimens made from tempered spring steel and designed to behave as VAL beams, defined as in Fig. 1, were built with dimensions and material properties as presented in Table 1. The beam specimens were mounted on an isolation table for self-adjusting the level of the testing system. A micrometer was used to measure the vertical static sag deflections of beam configuration at specific points along the beam arc-length. This micrometer can measure the

Table 1 Beam specimen information

Beam No.	Weight per unit volume (MN/m^3) ρ	Modulus of elasticity (MN/m^2) E	Depths (m) d	Widths (m) b	Span length (m) L	Non-dimensional self weight $\overline{w}$
1	7.5824×10^{-2}	1.9933×10^{5}	1.00×10^{-3}	2.54×10^{-2}	0.980	4.2962
2	7.5824×10^{-2}	1.9933×10^{5}	0.80×10^{-3}	2.54×10^{-2}	0.940	5.9240
3	7.5824×10^{-2}	1.9933×10^{5}	0.80×10^{-3}	2.54×10^{-2}	1.030	7.8173

Specimen material: Tempered spring steel (SK5)
$\overline{w} = wL^3/EI$ = Non-dimensional beam self weight
$w = \rho A = \rho(bd)$ = Uniform self weight of beam (forces per unit arc-length)

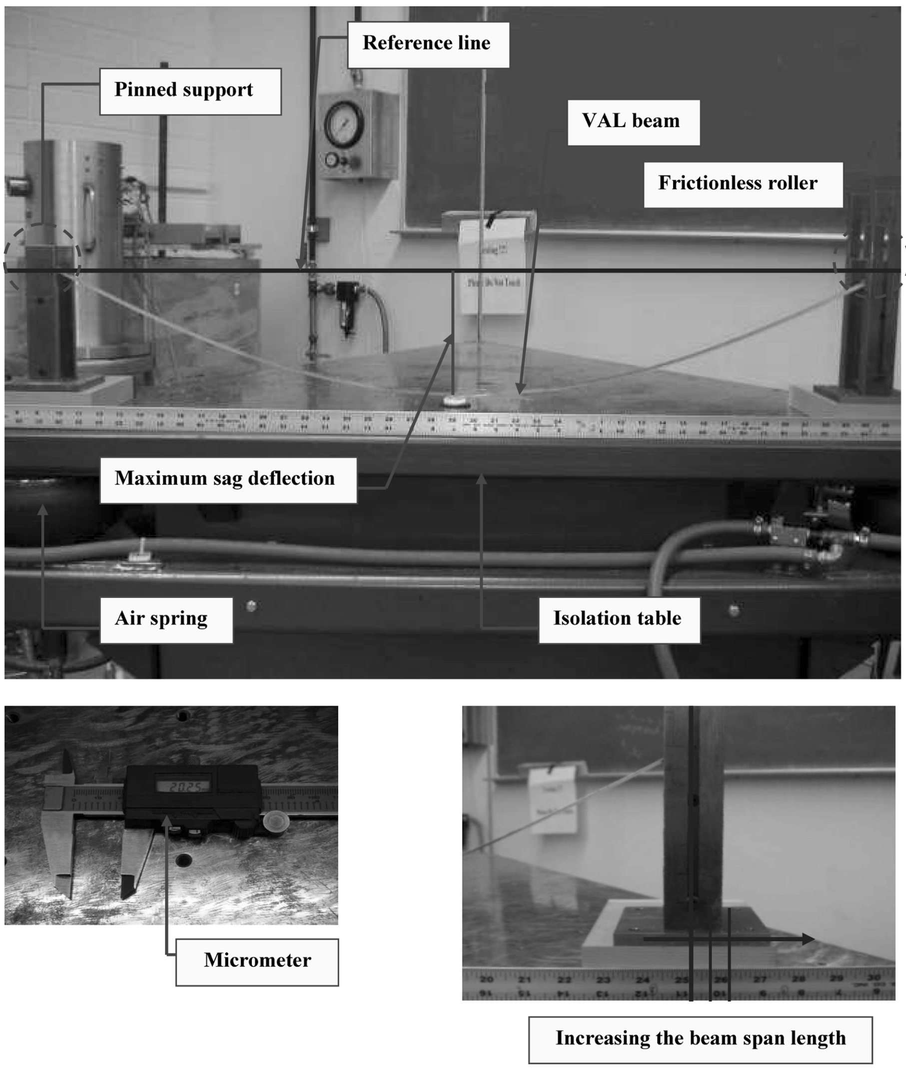

Fig. 3 Typical set-up of testing of beam No. 3, $\overline{w}$ = 7.8173 or sag ratio, $y_{s_{max}}/L$ = 0.14754

displacement for range of measurement range (300 mm.) with accuracy better than 0.01 mm. The typical set-up is shown in Fig. 3. Due to initial lack of straightness, the beam was tested, turned over, and tested again. The testing results were averaged.

The testing to determine the maximum or critical uniform beam self weight was performed by increasing the applied non-dimensional uniform beam self weight, $\bar{w} = wL^3/EI$, until the beam became unstable. The applied non-dimensional uniform beam self weight or $\bar{w}$ was increased step-by-step by holding the beam cross-section constant in which the beam weight and bending stiffness were also unchanged, and adjusting the span length with a small incremental step of 2.5 mm. The span-length of the beam specimen was increased step-by-step until the beam began slipping rapidly at the frictionless roller support. This behavior was the unstable state. The beam span length of the previous step was used to calculate the critical self weight or $\bar{w}_{cr}$ obtained from experiment. The accuracy of the experimental critical loading depends on the size of incremental step. The smaller incremental step yields the better results.

4. Results and comments

The results of large stable static deflection are calculated for various values of non-dimensional beam self weights, $\bar{w}$, by using the non-linear finite element method (FEM) and shooting method (SM) and are used to compare with those based on Linear Beam Theory (LBT). The stable equilibrium solution from FEM and the SM are in very close agreement. The LBT results agree quite closely with FEM and SM results for the low $\bar{w}$ values or little sag ($\bar{w}$ less than 2.00 or $y_{s_{max}}/L$ less than 0.0264) but begin to diverge for higher $\bar{w}$ values. Fig. 4 illustrates these results.

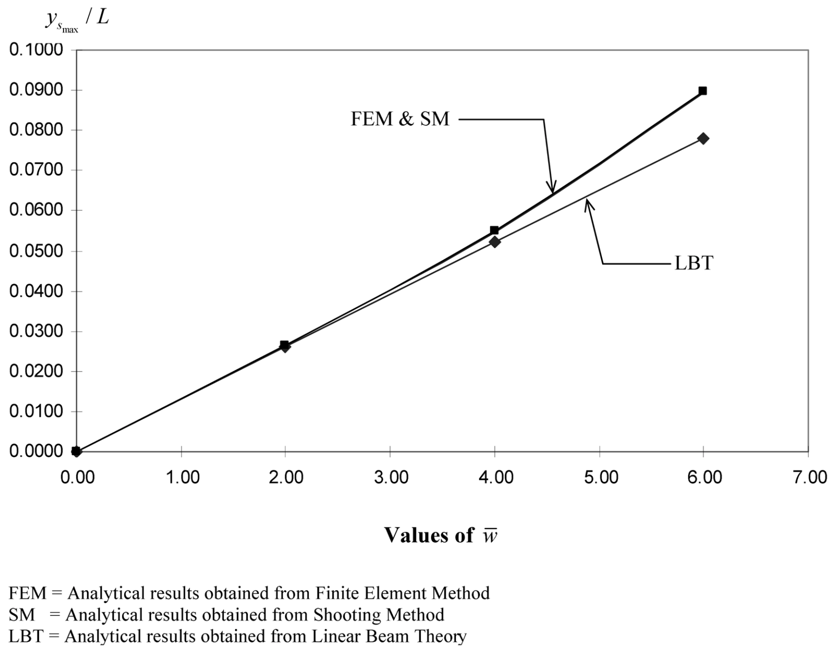

Fig. 4 Variation of maximum static displacement, $y_{s_{max}}/L$, with various values of $\bar{w}$

These results indicate that VAL beams with large static displacements are more accurately modeled using the presented FEM or SM formulations than by simply using LBT.

Fig. 5 shows the plots of $\overline{w}$, $y_{s_{\max}}/L$, and S_t/L against end rotations in the range of $0 \le \theta \le \pi/2$ obtained from SM. The maximum or critical values that the beam can resist, $\overline{w}_{cr}$, is 8.2461 at θ_A (or θ_B) = 0.5627 rad. The beams have two possible static configurations for any value of $\overline{w}$ which is less than the critical loading, $\overline{w}_{cr}$. One with smaller rotation is stable, while the other with a larger rotation is unstable. Plots of the stable and unstable equilibrium configuration for $\overline{w}$ = 6.00

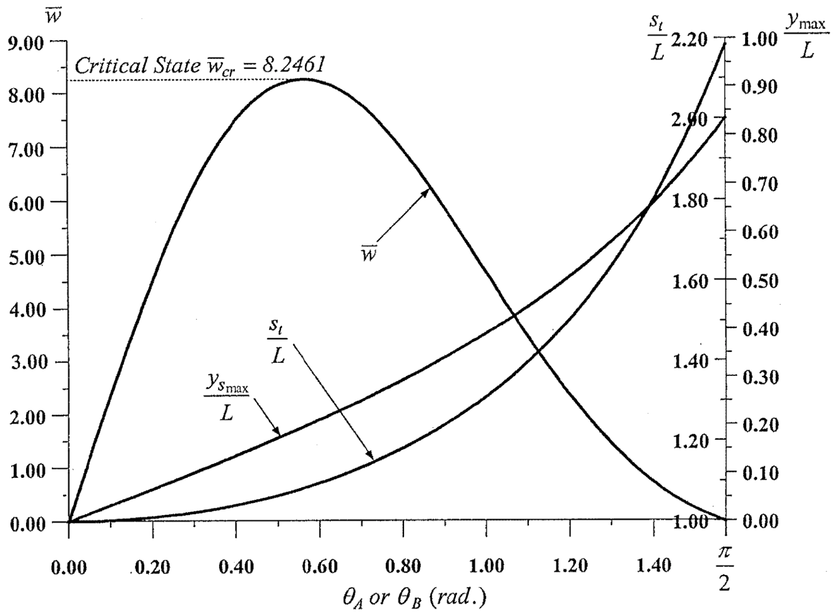

Fig. 5 Plot of $\overline{w}$, $y_{s_{\max}}/L$, and S_t/L versus θ_A or θ_B

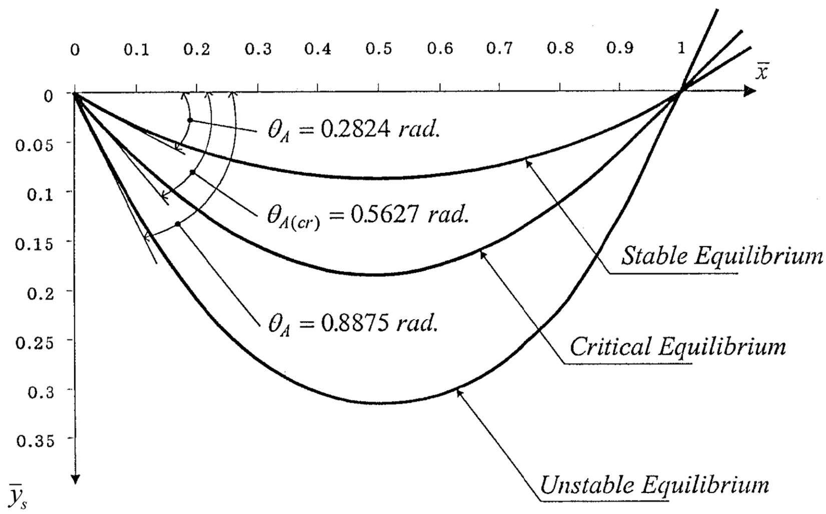

Fig. 6 Stable and unstable equilibrium configurations of $\overline{w}$ = 6.00

as well as critical equilibrium configuration are shown in Fig. 6. In Fig. 5, the results also indicate that the maximum $y_{s_{max}}/L$ and S_t/L are found to be 0.8347 and 2.1884, respectively at the end rotation equal to $\pi/2$. These values are independent of loading conditions (Chucheepsakul *et al*. 1995, 1996, 1997b).

The large static sag configuration of VAL beams for the stable equilibrium case under a given uniform self weight were measured at specific locations along the beam arc-length and compared with those obtained from FEM and SM. Because of the symmetry of beam configuration, the comparisons are presented for a half of beam arc-length as shown in Table 2. The average measured static displacements are 2.50% larger than those obtained analytically (FEM and SM) for all beam specimens.

The maximum or critical loading value due to uniform self weight, $\bar{w}_{cr}$, obtained experimentally is 8.0726. The deviation is within 2.10% lower than those obtained from SM. The deviation of critical uniform loading may come from the effect of initial deflection of the tested beam specimens. Theoretically, the undeformed configuration of VAL beams is assumed to be straight but in the testing procedure there exists a small amount of initial deflection prior to testing because of beam's self weight.

From these results, the comparisons between experimental and two analytical results show that very good agreement was obtained in this investigation for large static sag deformation due to beam self weight as well as for critical uniform self weight.

Table 2 Analytical and experimental comparison of large static sag deflection of VAL beam due to self weight

$\bar{w}$	Static sag ratio, $y_{s_{max}}/L$						
	Location, x/L	0.0833	0.1666	0.2500	0.3333	0.4167	0.500
4.2962	FEM	0.01570	0.03017	0.04241	0.05166	0.05741	0.05936
	SM	0.01570	0.03016	0.04241	0.05165	0.05741	0.05936
	Average	0.01570	0.03017	0.04241	0.05166	0.05741	0.05936
	Experiment	0.01595	0.03061	0.04255	0.05173	0.05755	0.05918
	%Deviation	–1.5673	–1.4374	–0.3290	–0.1353	–0.2432	+0.3041
5.9240	FEM	0.02364	0.04536	0.06364	0.07738	0.08589	0.08877
	SM	0.02364	0.04535	0.06363	0.07716	0.08542	0.08877
	Average	0.02364	0.04536	0.06364	0.07727	0.08565	0.08877
	Experiment	0.02417	0.04672	0.06521	0.07930	0.08790	0.09070
	%Deviation	–2.1928	–2.9109	–2.4076	–2.5598	–2.5397	–2.1278
7.8173	FEM	0.03844	0.07345	0.10250	0.12403	0.13719	0.14163
	SM	0.03843	0.07342	0.10249	0.12402	0.13724	0.14162
	Average	0.03843	0.07344	0.10250	0.12403	0.13721	0.14162
	Experiment	0.03944	0.07523	0.10605	0.12805	0.14221	0.14655
	%Deviation	–2.5645	–2.3793	–3.3474	–3.1394	–3.5159	–3.3604

FEM = Stable static displacement obtained analytically by using Finite Element Method
SM = Stable static displacement obtained analytically by using Shooting Method
Average = Average static displacement from two analytical solutions
% Deviation = Difference between average analytical and experimental results.

5. Conclusions

Two analytical approaches are presented for the solution of VAL beams subjected to a uniformly distributed self weight and validated experimentally. The finite-element method gives the stable equilibrium configuration and shows good agreement with shooting method. The results of shooting method yield the beam configurations for both stable and unstable equilibrium cases. Analytical results reveal that static sag deflections due to self weight of VAL beams are larger than linear beam theory (LBT). This is especially true for beams with $\overline{w}$ higher than 2.0.

The experimental studies were conducted for the stable equilibrium case. The static beam configurations due to a given uniform loading determined experimentally closely matched the analytical results, where average deviations were less than 2.50%. Good agreement of critical uniform loading, $\overline{w}_{cr}$, was determined to be 8.2461 obtained analytically (SM), and 8.0726 obtained experimentally.

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References

Chucheepsakul, S., Buncharoen, S. and Huang, T. (1995), "Elastica of simple variable-arc-length beam subjected to end moment", *J. Engrg. Mech*., **121**(7), 767-772.

Chucheepsakul, S., Theppitak, G. and Wang, C.M. (1996), "Large deflection of simple variable-arc-length beams subjected to a point load", *Struct. Engrg. Mech.*, **4**(1), 49-59.

Chucheepsakul, S., Theppitak, G. and Wang, C.M. (1997a), "Exact solution of variable-arc-length elastica under moment gradient", *Struct. Engrg. Mech.*, **5**(5), 529-539.

Chucheepsakul, S. and Huang, T. (1997b), "Finite element solution of variable-arc-length beam under a point load", *J. Struct. Engrg.*, **123**(7), 968-970.

Chucheepsakul, S., Wang, C.M., He, X.Q. and Monprapussorn, T. (1999), "Double curvature bending of variable-arc-length elasticas", *J. Appl. Mech.*, **66**, 87-94.

Golley, B.W. (1997), "The solution of open and closed elasticas using intrinsic coordinate finite elements", *J. Comp. Meth. Appl. Mech. Engrg.*, **146**, 127-134.

Hartono, W. (2000), "Behavior of variable-arc-length elastica with frictionless support under follower force", *Mech. Res. Comm.*, **27**(6), 653-658.

Huang, T. and Chucheepsakul, S. (1985), "Large displacement analysis of a marine riser", *J. Energy Resources Tech.*, **107**(3), 54-59.

Malvern, L.E. (1969), *Introduction to the Mechanics of Continuous Media*, Prentice-Hall, Inc.

Nelder, J.A. and Meade, R. (1965), "A simplex method for the function minimization", *Comp. J*., **7**, 308-313.

Press, W.H., Teukolsky, S.A., Vettering, W.T. and Flannery, B.P. (1992), *Numerical Recipes in Fortran*, 2nd ed., Cambridge University Press.

Wang, C.M., Lam, K.Y., He, X.Q. and Chucheepsakul, S. (1997), "Large deflections of an end supported beam subjected to a point load", *Int. J. Nonl. Mech.*, **32**(1), 63-72.