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Numerically integrated modified virtual crack closure integral technique for 2-D crack problems

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Abstract. Modified virtual crack closure integral (MVCCI) technique has become very popular for computation of strain energy release rate (SERR) and stress intensity factor (SIF) for 2-D crack problems. The objective of this paper is to propose a numerical integration procedure for MVCCI so as to generalize the technique and make its application much wider. This new procedure called as numerically integrated MVCCI (NI-MVCCI) will remove the dependence of MVCCI equations on the type of finite element employed in the basic stress analysis. Numerical studies on fracture analysis of 2-D crack (mode I and II) problems have been conducted by employing 4-noded, 8-noded (regular & quarter-point), 9-noded and 12-noded finite elements. For non-singular (regular) elements at crack tip, NI-MVCCI technique generates the same results as MVCCI, but the advantage for higher order regular and singular elements is that complex equations for MVCCI need not be derived. Gauss numerical integration rule to be employed for 8-noded singular (quarter-point) element for accurate computation of SERR and SIF has been recommended based on the numerical studies.

Key words: fracture mechanics; finite element method; stress intensity factor; strain energy release rate; numerical integration.

1. Introduction

The fracture behaviour of structural components under fatigue loading or during static overload can be estimated through linear elastic fracture mechanics (LEFM) principles, and SIF is the influencing design parameter. A detailed review of fatigue and fracture behaviour of structural components has been presented by Cotterell (2002) and Schijve (2003). Using the finite element

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method (FEM) for basic stress analysis (Zienkiewicz and Taylor 2000), SIF can be computed through post-processing of finite element analysis (FEA) results (Liebowitz and Moyer 1989). The techniques based on displacement extrapolation, strain energy release rate, virtual crack extension, modified virtual crack closure integral, equivalent domain integral and J-integral are generally preferred (Owen and Fawkes 1982), for computing SIF through post-processing of FEA results. The major disadvantage in the extrapolation methods is that the accuracy in evaluating SIF depends on the accuracy of displacement and stress distribution in the vicinity of crack tips. As such, these methods are not suitable with conventional finite elements and generally require stress analysis using singular elements only. The strain energy release rate and the virtual crack extension techniques require two runs of analysis for evaluating SERR. Considering the merits and demerits of these techniques, it is observed that for LEFM problems, MVCCI technique in combination with FEM is an efficient tool for evaluating SERR from which SIF can be calculated. One of the popular post-processing techniques is MVCCI developed by Rybicki and Kanninen (1977) based on Irwin's crack closure integral (CCI) technique (Irwin 1958) with appropriate modifications for computation of SERR and SIF. The advantage of MVCCI technique is its simplicity and also the ease with which individual mode SERR/SIF can be estimated in mixed-mode problems.

Rybicki and Kanninen (1977) expressed Irwin's CCI technique in a form consistent with the finite element (FE) formulation and evaluated SERR for mode I and II (G_I and G_{II}) in terms of nodal forces and displacements. Further, these computations can be carried out from a single FEA, as against from two analyses with crack lengths differing by an infinitesimally small crack length as conceived originally. Buchholz (1984) realized the element dependence of MVCCI equations and presented appropriate equations for 8-noded quadrilateral elements, but did not establish a formal procedure for deriving them. Badari Narayana and Dattaguru (1996) and Badari Narayana *et al.* (1990) presented the generalised MVCCI equations for conventional and singular quadrilateral elements for 2-D problems with cracks. Raju (1986) also derived MVCCI equations for 6-noded and 8-noded quarter-point singular elements. Young and Sun (1993) demonstrated the application of MVCCI technique to plate bending problems. Buchholz *et al*. (2001) and Dhondt *et al*. (2001) conducted fracture analysis to study the 3-D and mode coupling effects by employing MVCCI method.

For the successful application of MVCCI technique, it is essential (Badari Narayana 1991) to derive element dependent MVCCI equations for computation of SERR. The derivation of MVCCI equations involves evaluation of constants used in the polynomial assumed to represent displacement and stress variation and evaluation of many integrals. In view of these, the derivation of MVCCI equations becomes a tedious exercise for higher order and singular 2-D and 3-D finite elements. Therefore, a need is felt to develop a generalized MVCCI technique involving numerical integration for computation of the required constants and to evaluate the associated integrals. Towards this, NI-MVCCI technique has been proposed in this paper for computation of SERR and SIF for 2-D crack problems. NI-MVCCI is a generalized technique and removes the dependence on the type of finite elements employed. NI-MVCCI technique has been demonstrated for 4-noded bilinear, 8-noded Serendipity (regular & quarter-point), 9-noded Lagrangian and 12-noded cubic isoparametric finite elements. Numerical studies on fracture analysis of 2-D crack (mode I and II) problems have been conducted. Gauss numerical integration rule to be employed for 8-noded singular (quarter-point) element for accurate computation of SERR and SIF has been recommended based on the numerical studies. It may be noted that no results have been reported in the literature using MVCCI technique for 9-noded and 12-noded elements. In this paper, NI-MVCCI technique has been used for the first time for this purpose.

2. Formulation of NI-MVCCI technique

Irwin (1958) proposed CCI technique for evaluation of SERR. CCI was derived using a fundamental concept that when crack extension takes place, the energy required to close this part of crack in a solid is same as energy released during crack extension. The rate of change of this energy with crack extension is SERR, which is generally denoted as G. Fig. 1 shows a crack tip in an infinite isotropic media subjected to remote tensile loading causing mode I crack deformation.

The normal stress distribution ahead of the crack tip is σ_{yy}. Let the crack of length, 'a' be extended by a small virtual increment of 'Δa'. The crack opening displacement (COD) behind the new crack tip is U_y (half of the total COD). The energy required to close the extended crack 'Δa' can be estimated as the work done by forces corresponding to the stress distribution, σ_{yy} on COD, U_y. This can be expressed as

$$W = \frac{1}{2}\int_0^{\Delta a} \sigma_{yy} U_y dx \tag{1}$$

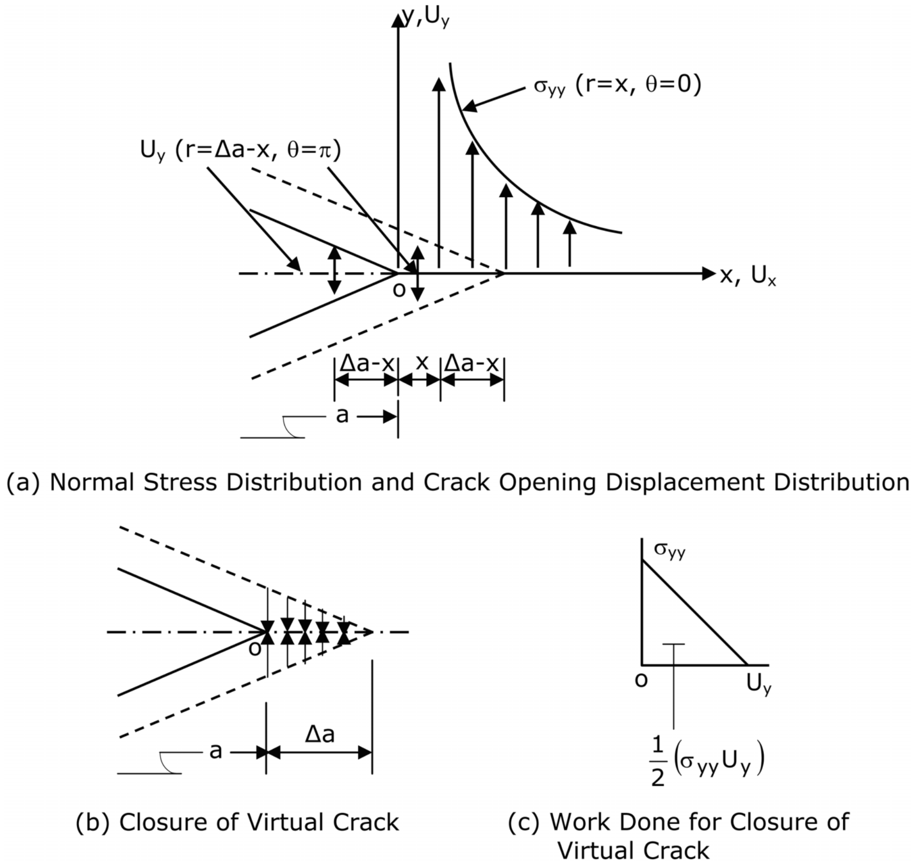


Fig. 1 Schematic of virtual crack extension (Mode I)

The above CCI can be used to compute SERR, as

$$G = \underset{\Delta a \to 0}{Lt} \frac{1}{2\Delta a} \int_0^{\Delta a} \sigma_{yy} U_y dx \tag{2}$$

Taking polar coordinate system (r, θ) with the origin at the crack tip in a 2-D domain and using Eq. (2), SERR for mode I and II cracks (G_I and G_{II}) can be expressed as

$$G_I = \underset{\Delta a \to 0}{Lt} \frac{1}{2\Delta a} \int_0^{\Delta a} \sigma_{yy}(r = x, \theta = 0) U_y(r = \Delta a - x, \theta = \pi) dr \tag{3}$$

$$G_{II} = \underset{\Delta a \to 0}{Lt} \frac{1}{2\Delta a} \int_0^{\Delta a} \sigma_{xy}(r = x, \theta = 0) U_x(r = \Delta a - x, \theta = \pi) dr \tag{4}$$

where σ_{yy} ($r = x$, $\theta = 0$) and σ_{xy} ($r = x$, $\theta = 0$) are distribution of stresses ahead of the crack tip. U_x ($r = \Delta a - x$, $\theta = \pi$) and U_y ($r = \Delta a - x$, $\theta = \pi$) are the relative sliding and opening displacements between the crack faces and Δa is the virtual crack increment.

The problem is basically to evaluate SERR, represented as $G = (\partial U/\partial a)$. If one could do two stages FEA, SERR can be obtained from the difference in strain energies for the configuration corresponding to crack sizes 'a' and '$a + \Delta a$'. However, if 'Δa' is kept very small, one could use the stress distribution ahead of crack tip and COD behind crack tip derived from single FEA to evaluate MVCCI using Eqs. (3) and (4). The evaluation of CCI by using the results obtained from single FEA as a post-processing approach is known as MVCCI technique. The derivation of the element dependent MVCCI equations for computing G_I and G_{II} involves evaluation of constants used in the polynomial assumed to represent displacement and stress variation and evaluation of many integrals. In view of these, the derivation of MVCCI equations becomes a tedious exercise for higher order and singular 2-D and 3-D finite elements. NI-MVCCI technique proposed in this paper involves numerical integration for computation of the constants and to evaluate the crack closure integrals for G_I and G_{II} as given by Eqs. (3) and (4). The numerical integration has to be carried out in two stages: one for evaluating constants representing the stress distribution ahead of crack tip in terms of nodal forces and the second to evaluate SERR itself.

Consider a typical FE mesh at the crack tip as shown in Fig. 2. The mesh shown consists of quadrilaterals with n number of nodes on edge OA. For mode I crack, G_I can be evaluated by multiplying the stress distribution along OA (ahead of crack tip) with the corresponding displacement distribution along OB (behind crack tip) and integrating this product over Δa. For evaluation of G_I the stress distribution along OA is expressed in terms of the nodal forces $F_{y,\,j}$, $F_{y,\,j+1}$, etc. acting at the nodes j, $j + 1$, etc. respectively. The COD distribution along OB is expressed in terms of the nodal displacements at j, $j - 1$, $(j - 1)'$, etc. G_I is derived by evaluating the energy required to close the crack over a length 'Δa' in terms of these nodal forces and displacements. The shape functions for elements ① and ② along OB can be obtained by substituting $\eta = -1$, in the respective element shape functions. Let these shape functions be N_i and the general formulation for any value of n would be as follows.

The COD distribution along OB can be expressed in terms of nodal displacements $\{(U_y)_i\}$ as

$$U_y = [N_i]\{(U_y)_i\} \quad i = 1, \ldots, n \tag{5}$$

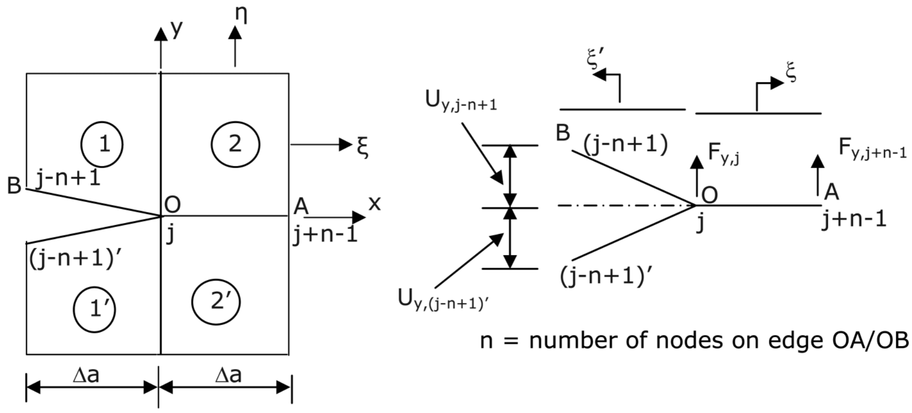


Fig. 2 Typical FE mesh of crack tip region

where n is the number of nodes on edge OA or OB of the respective element. Consistent with the isoparametric formulation, coordinate of any point $X(x, y)$ is given by

$$X = [N_i]\{(X)_i\} \quad i = 1, \ldots, n \tag{6}$$

where $\{(X)_i\}$ are nodal coordinates. The transformation between the global and natural coordinate system for the respective element can be obtained by using Eq. (6). Consistent with the element shape functions, the displacement variation along OB can be expressed as function of ξ' for non-singular elements as

$$U_y(\xi') = a_0 + a_1\xi' + \ldots + a_{(n-1)}\xi'^{(n-1)} \tag{7a}$$

where $U_y(\xi')$ is a polynomial of order $(n - 1)$. For 8-noded quarter-point element (QPE), the displacement variation along OB can be expressed as

$$U_y(\xi') = a_0 + a_1(1 + \xi') + a_2(1 + \xi')^2 \tag{7b}$$

The constants a_0, a_1,…., $a_{(n-1)}$ can be evaluated by matching the displacements at the nodes j, $(j - 1)$, …., $(j - n + 1)$ in element ① . A set of simultaneous equations of order n is formed, which can be solved for obtaining the constants a_0, a_1,…., $a_{(n-1)}$.

Considering element ②, stress (σ_{yy}) distribution along OA can be expressed as a function of ξ for non-singular elements as

$$\sigma_{yy}(\xi) = b_0 + b_1\xi + \ldots + b_{(n-1)}\xi^{(n-1)} \tag{8a}$$

where $\sigma_{yy}(\xi)$ is a polynomial of order $(n - 1)$. For 8-noded QPE element, stress distribution along OA can be expressed as

$$\sigma_{yy}(\xi) = b_0/(1 + \xi) + b_1 + b_2(1 + \xi) \tag{8b}$$

The constants b_0, b_1,…., $b_{(n-1)}$ can be computed by matching the nodal forces with the derived consistent load vector from FE analysis. The nodal forces $F_{y,j}$, $F_{y,(j+1)}$,…., $F_{y,(j+n-1)}$ shown in Fig. 2 are the forces exerted at node j, $(j + 1)$,….., $(j + n - 1)$ by the structure below OA on the structure above OA. In FEA, these forces are obtained by summing up the forces at nodes j, $(j + 1)$,…., $(j + n - 1)$ from the elements on the edge above OA. These forces should be consistent with the stress distribution given in Eq. (8), which can be expressed as

$$F_i = \int_0^{\Delta a} [N_i]^T \sigma_{yy}(\xi) dx \quad i = 1, \ldots, n \tag{9}$$

where N_i are shape functions of the respective element obtained by substituting $\eta = -1$. By using the transformation between the global and natural coordinate system, dx can be expressed in terms of $d\xi$ (Eq. 6).

By substituting the expressions for displacement and stress variation given by Eqs. (7) and (8) respectively in CCI Eqs. (3) and (4), G_I and G_{II} can be expressed as

$$G_I = \underset{\Delta a \to 0}{Lt} \frac{1}{2\Delta a} \int_0^{\Delta a} \sigma_y(\xi) U_y(\xi') dx \tag{10}$$

$$G_{II} = \underset{\Delta a \to 0}{Lt} \frac{1}{2\Delta a} \int_0^{\Delta a} \sigma_{xy}(\xi) U_x(\xi') dx \tag{11}$$

Hitherto, these integrals given are evaluated in closed form for several simple elements. But the procedure becomes complicated for higher order elements and in particular for 3-D problems with Hexa8, Hexa20 and Hexa27 solid elements. So, in the present study, Gauss integration technique has been proposed for evaluating these integrals. For different finite elements employed in the basic stress analysis, one may use different rules for integration. For non-singular elements the rule of integration for accurate evaluation of these integrals can be easily worked out depending on the value of n. For example, Guass integration rule of 2, 3 and 4 will be required for 4-noded, 8-noded/ 9-noded and 12-noded elements respectively. However, for 8-noded QPE, in view of the function chosen for $\sigma_{yy}(\xi)$ (Eq. 8(b)), the required Guass integration rule has to be arrived at by conducting numerical studies.

As it can be observed from Eqs. (5)-(11), the proposed NI-MVCCI is a generalized technique and is independent of the type of finite elements used, except for assuming the appropriate expressions for displacement and stress variation (Eqs. (7) and (8)). In the present study, this new procedure is demonstrated for 4-noded, 8-noded (regular and quarter-point), 9-noded and 12-noded elements. Closed form MVCCI equations for computing G_I and G_{II} for 4-noded and 8-noded (regular and quarter-point) are presented in Table 1. It may be noted that the MVCCI equations for 9-noded element will be same as that of regular 8-noded element. Closed form MVCCI equations for 12-noded elements can easily be developed by procedures similar to that for 8-noded element. The expressions for these elements for appropriate substitution in Eqs. (5)-(11) for computation of SERR using NI-MVCCI technique have also been presented in Table 1.

Table 1 MVCCI equations and expressions required for evaluation of NI-MVCCI

Element	Shape functions along edge OB(for edge OA replace ξ' by ξ)	Relation between x & ξ'	Displacement variation $(U_y(\xi'))$ along OB	Stress variation $(\sigma_{yy}(\xi'))$ along OA	Displacement and force conditions	Relation between ξ' & ξ	MVCCI equations
4-noded	$\frac{1}{2}(1+\xi'\xi_i')$	$-\frac{\Delta a}{2}(1+\xi')$	$a_0+a_1\xi'$	$b_0+b_1\xi$	$U_y=0$ at $\xi'=-1$ $U_y=U_{y,j-1}$ at $\xi'=1$ $F_y=F_{y,j}$ at $\xi=-1$ $F_y=F_{y,j+1}$ at $\xi=1$	$\xi'=-\xi$	$G_I=\frac{1}{2\Delta a}[F_{y,j}(U_{y,j-1}-U_{y,(j-1)'})]$ $G_{II}=\frac{1}{2\Delta a}[F_{x,j}(U_{x,j-1}-U_{x,(j-1)'})]$
8-noded/ 9-noded	$\frac{1}{2}(1+\xi'\xi_i')\xi'\xi_i'$ (end nodes) $\frac{1}{2}(1-\xi'^2)$ (mid node)	$-\frac{\Delta a}{2}(1+\xi')$	$a_0+a_1\xi'$ $+a_2\xi'^2$	$b_0+b_1\xi$ $+b_1\xi^2$	$U_y=0$ at $\xi'=-1$ $U_y=U_{y,j-1}$ at $\xi'=0$ $U_y=U_{y,j-2}$ at $\xi'=1$ $F_y=F_{y,j}$ at $\xi=-1$ $F_y=F_{y,j+1}$ at $\xi=0$ $F_y=F_{y,j+2}$ at $\xi=1$	$\xi'=-\xi$	$G_I=\frac{1}{2\Delta a}\begin{bmatrix}F_{y,j}(U_{y,j-2}-U_{y,(j-2)'})+\\F_{y,j+1}(U_{y,j-1}-U_{y,(j-1)'})\end{bmatrix}$ $G_{II}=\frac{1}{2\Delta a}\begin{bmatrix}F_{x,j}(U_{x,j-2}-U_{x,(j-2)'})+\\F_{x,j+1}(U_{x,j-1}-U_{x,(j-1)'})\end{bmatrix}$
8-noded QPE	$\frac{1}{2}(1+\xi'\xi_i')\xi'\xi_i$ (end nodes) $\frac{1}{2}(1-\xi'^2)$ (mid node)	$\frac{\Delta a}{4}(1+\xi')^2$	$a_0+a_1(1+\xi')$ $+a_2(1+\xi')^2$	$b_0/(1+\xi)$ $+b_1$ $+b_2(1+\xi)$	$U_y=0$ at $\xi'=-1$ $U_y=U_{y,j-1}$ at $\xi'=0$ $U_y=U_{y,j-2}$ at $\xi'=1$ $F_y=F_{y,j}$ at $\xi=-1$ $F_y=F_{y,j+1}$ at $\xi=0$ $F_y=F_{y,j+2}$ at $\xi=1$	$(1+\xi)^2$ $+(1+\xi')^2$ $=4$	$G_I=\frac{1}{2\Delta a}\begin{bmatrix}(C_{11}F_{y,j}+C_{12}F_{y,j+1}+C_{13}F_{y,j+2})\\(U_{y,j-1}-U_{y,(j-1)'})\\+(C_{21}F_{y,j}+C_{22}F_{y,j+1}+C_{23}F_{y,j+2})\\(U_{y,j-2}-U_{y,(j-2)'})\end{bmatrix}^{**}$ $G_{II}=\frac{1}{2\Delta a}\begin{bmatrix}(C_{11}F_{x,j}+C_{12}F_{x,j+1}+C_{13}F_{x,j+2})\\(U_{x,j-1}-U_{x,(j-1)'})\\+(C_{21}F_{x,j}+C_{22}F_{x,j+1}+C_{23}F_{x,j+2})\\(U_{x,j-2}-U_{x,(j-2)'})\end{bmatrix}^{**}$
12-noded	$\frac{1}{16}(1+\xi'\xi_i')$ $(-1+9\xi'^2)$ (end nodes) $\frac{9}{16}(1+9\xi'\xi_i')$ $(1-\xi'^2)$ (mid nodes)	$-\frac{\Delta a}{2}(1+\xi')$	$a_0+a_1\xi'$ $+a_2\xi'^2+a_3\xi'^3$	$b_0+b_1\xi$ $+b_2\xi^2$ $+b_3\xi^3$	$U_y=0$ at $\xi'=-1$ $U_y=U_{y,j-1}$ at $\xi'=-1/3$ $U_y=U_{y,j-2}$ at $\xi'=1/3$ $U_y=U_{y,j-3}$ at $\xi'=1$ $F_y=F_{y,j}$ at $\xi=-1$ $F_y=F_{y,j+1}$ at $\xi=-1/3$ $F_y=F_{y,j+2}$ at $\xi=1/3$ $F_y=F_{y,j+3}$ at $\xi=1$	$\xi'=-\xi$	$G_I=\frac{1}{2\Delta a}\begin{bmatrix}F_{y,j}(U_{y,j-3}-U_{y,(j-3)'})+\\F_{y,j+1}(U_{y,j-2}-U_{y,(j-2)'})+\\F_{y,j+2}(U_{y,j-1}-U_{y,(j-1)'})\end{bmatrix}$ $G_{II}=\frac{1}{2\Delta a}\begin{bmatrix}F_{x,j}(U_{x,j-3}-U_{x,(j-3)'})+\\F_{x,j+1}(U_{x,j-2}-U_{x,(j-2)'})+\\F_{x,j+2}(U_{x,j-1}-U_{x,(j-1)'})\end{bmatrix}$

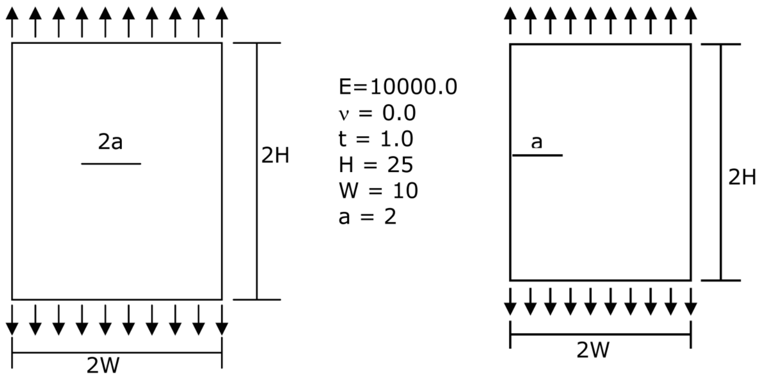
**- $C_{11}=33\pi/2-52$; $C_{12}=17-21\pi/4$; $C_{13}=21\pi/2-32$; $C_{21}=14-33\pi/8$; $C_{22}=21\pi/16-7/2$; $C_{23}=8-21\pi/8$

3. Numerical studies

Fracture analysis of 2-D crack (mode I and II) problems has been conducted to validate the proposed NI-MVCCI technique. Basic stress analysis of the plate has been carried out by employing 4-noded, 8-noded (regular and quarter-point), 9-noded and 12-noded finite elements. SERR has been evaluated by using NI-MVCCI technique. For evaluating the integrals associated with NI-MVCCI technique, Gauss integration technique has been used with rules of 2, 3 and 4 for 4-noded, 8-noded/ 9-noded and 12-noded elements respectively, while for 8-noded QPE different rules have been employed. Plane strain conditions have been assumed at the crack tip to compute SIF by using SERR value obtained using NI-MVCCI technique.

3.1 Example-1: Rectangular plate with center crack under uniaxial tension

A rectangular plate with center crack subjected to uniaxial tensile loading (mode I) as shown in Fig. 3 has been analysed to compute SERR and SIF at the crack tip. One quarter of the plate with symmetric boundary conditions has been idealized. FE idealization of the plate using 4-noded element is shown Fig. 4. Table 2 presents SERR and SIF values obtained in the present study along with the results obtained by using MVCCI technique (closed form equations) and the finite plate solution available in the literature (Rooke and Cartwright 1976). The variation of SIF with respect to $\Delta a/a$ and W/a is shown in Fig. 5.



(a) Center Crack (b) Edge Crack

Fig. 3 Rectangular plate under uniaxial tension

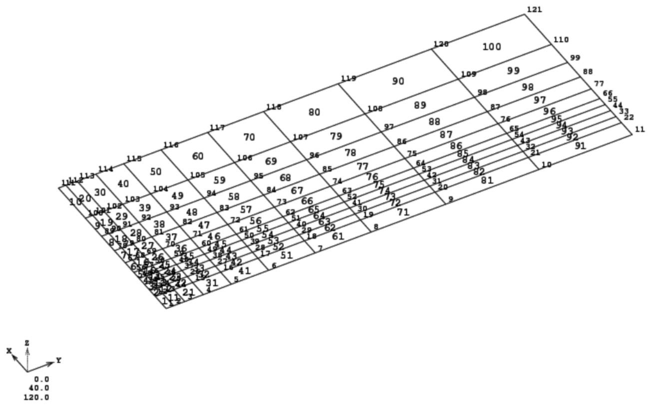
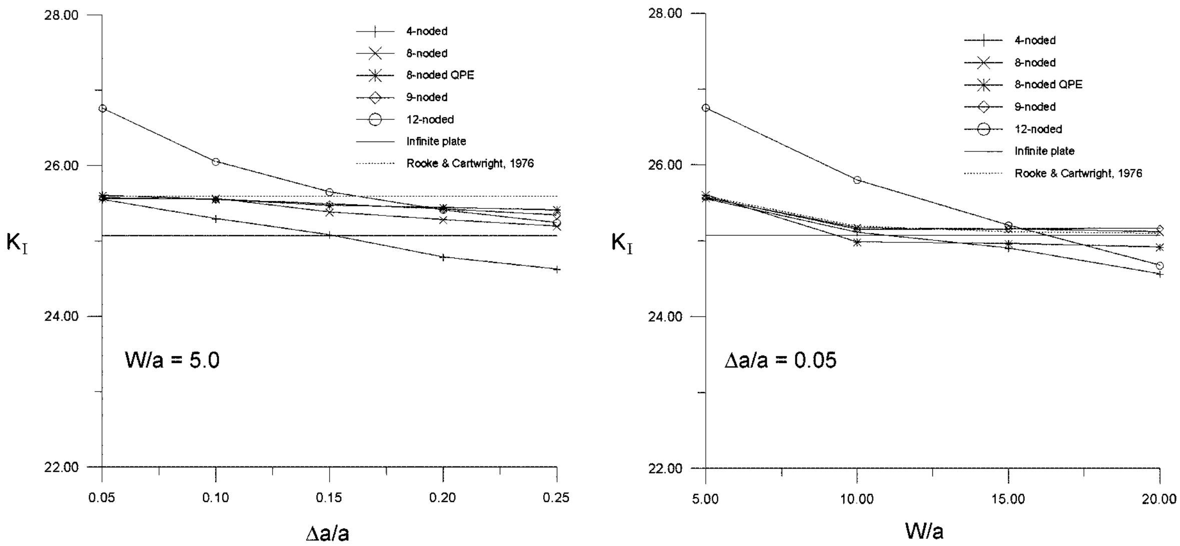


Fig. 4 FE Idealization of rectangular plate (Quarter symmetry)

Table 2 SERR and SIF for rectangular plate with center crack under uniaxial tension (Comparison for different rules of Gauss integration for NI-MVCCI) ($\Delta a/a$ = 0.05 and W/a = 5)

Gauss rule	4-noded element		8-noded element		9-noded element		8-noded QPE		12-noded element			
									$\Delta a/a$ = 0.15		$\Delta a/a$ = 0.05	
	G_I	K_I	G_I	K_I	G_I	K_I	G_I	K_I	G_I	K_I	G_I	K_I
2	0.06529	25.5524 (0.15)*	0.10303	32.09787	0.09473	30.7777	0.04675	21.6224	0.33447	57.8336	0.36255	60.2120
3			0.06531	25.5559 (0.13)*	0.06539	25.5708 (0.08)*	0.06612	25.7138	0.13197	36.3271	0.14124	37.5824
4							0.06578	25.6480	0.06579	25.6505 (0.24)*	0.07160	26.7588 (4.57)*
5							0.06567	25.6253				
6							0.06561	25.6151				
7							0.06559	25.6097				
8							0.06557	25.6069				
9							0.06556	25.6050				
10							0.06556	25.6038 (0.05)*				
MVCCI	0.06529	25.5524	0.06531	25.5559	0.06539	25.5708	0.06554	25.6005	0.06579	25.6505	0.07160	26.7588

K_I: 25.59 − with finite plate correction (Rooke and Cartwright 1976), 25.07 − infinite plate solution.
* − % deviation from analytical K_I



(a) Variation w.r.t. Δa/a (b) Variation w.r.t. W/a

Fig. 5 Variation of SIF for rectangular plate with center crack (Mode I)

Table 3 SERR and SIF for rectangular plate with edge crack under uniaxial tension (Comparison for different rules of Gauss integration for NI-MVCCI) ($\Delta a/a$ = 0.05 and W/a = 5)

Gauss rule	4-noded element		8-noded element		9-noded element		8-noded QPE		12-noded element			
									$\Delta a/a$ = 0.1		$\Delta a/a$ = 0.05	
	G_I	K_I	G_I	K_I	G_I	K_I	G_I	K_I	G_I	K_I	G_I	K_I
2	0.10917	33.0411 (3.98)*	0.20508	45.2856	0.17188	41.4578	0.07813	27.9509	0.52002	72.1124	0.76660	87.5558
3			0.11529	33.9546 (1.32)*	0.11577	34.0254 (1.12)*	0.11724	34.2339	0.72711	85.2706	0.13669	36.9716
4							0.11666	34.1548	0.11839	34.4072 (0.01)*	0.12368	35.1681 (2.20)*
5							0.11646	34.1253				
6							0.11636	34.1122				
7							0.11632	34.1051				
8							0.11629	34.1015				
9							0.11628	34.0991				
10							0.11626	34.0975 (0.92)*				
MVCCI	0.10917	33.0411	0.11529	33.9546	0.11577	34.0254	0.11623	34.0932	0.11839	34.4072	0.12368	35.1681

K_I: 34.41 – with finite plate correction (Rooke and Cartwright 1976), 28.08 – semi-infinite plate solution
* – % deviation from analytical K_I

3.2 Example-2: Rectangular plate with edge crack under uniaxial tension

A rectangular plate with an edge crack subjected to uniaxial tensile loading (mode I) as shown in Fig. 3(b) has been analysed to compute SERR and SIF at the crack tip. FE idealization as shown in Fig. 4 has been used in the studies, considering half symmetry, with appropriate changes for the boundary conditions. Table 3 presents SERR and SIF values obtained in the present study along with the results obtained by using MVCCI technique and the finite plate solution available in the literature (Rooke and Cartwright 1976). The variation of SIF with respect to $\Delta a/a$ and W/a is shown in Fig. 6.

3.3 Example-3: Rectangular plate with center crack under shear load

A rectangular plate with a center crack subjected to shear load (mode II) has been analysed to compute SERR and SIF at the crack tip. The plate geometry and attributes are the same as that of example-1. FE idealization as shown in Fig. 4 has been used in the studies, considering quarter symmetry, with appropriate changes for the loading and boundary conditions. Table 4 presents SERR and SIF values obtained in the present study along with the results obtained by using MVCCI technique and the finite plate solution available in the literature (Rooke and Cartwright 1976). The variation of SIF with respect to $\Delta a/a$ and W/a is shown in Fig. 7.

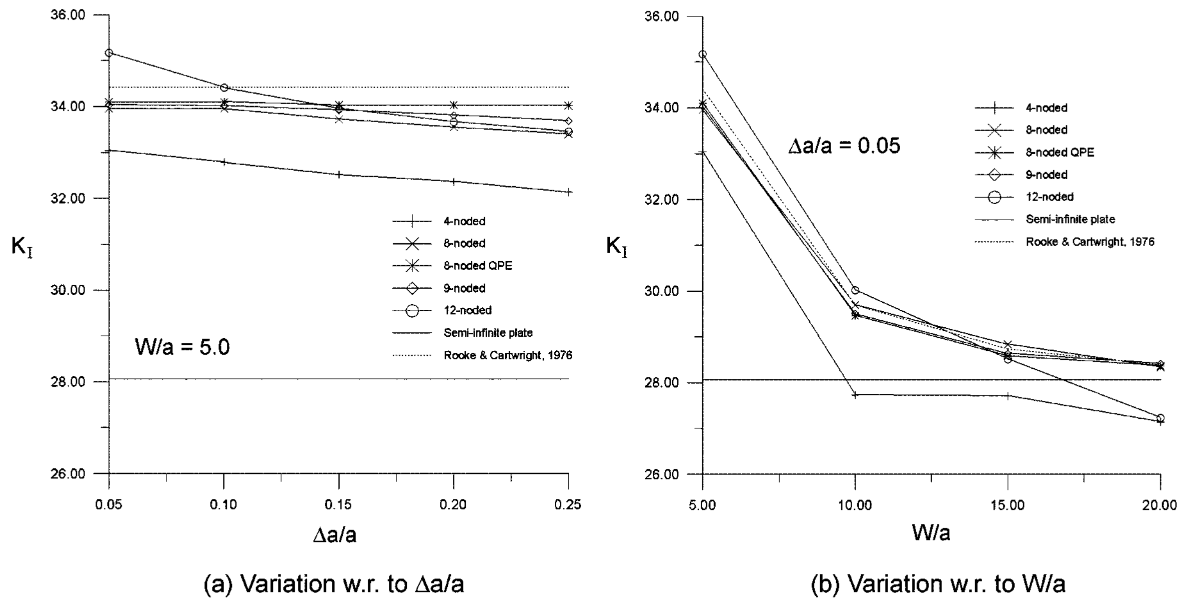


Fig. 6 Variation of SIF for rectangular plate with edge crack (Mode I)

Table 4 SERR and SIF for rectangular plate with center crack under shear load (Comparison for different rules of Gauss integration for NI-MVCCI) ($\Delta a/a$ = 0.05 and W/a = 20)

Gauss rule	4-noded element		8-noded element		9-noded element		8-noded QPE		12-noded element	
	G_{II}	K_{II}	G_{II}	K_{II}	G_{II}	K_{II}	G_{II}	K_{II}	G_{II}	K_{II}
2	0.06123	24.7441 (1.30)*	0.10205	31.9454	0.09863	31.4059	0.07965	28.2225	0.33862	58.1913
3			0.06101	24.7001 (1.48)*	0.06206	24.9109 (0.63)*	0.06108	24.7152	0.25891	50.8831
4							0.06095	24.6894	0.05874	24.2372 (3.32)*
5							0.06089	24.6768		
6							0.06087	24.6719		
7							0.06086	24.6691		
8							0.06085	24.6677		
9							0.06085	24.6668		
10							0.06084	24.6662 (1.61)*		
MVCCI	0.06123	24.7441	0.06101	24.7001	0.06206	24.9109	0.06083	24.6645	0.05874	24.2372

K_{II}: 25.10 – with finite plate correction (Rooke and Cartwright 1976), 25.07 – infinite plate solution
* – % deviation from analytical K_I

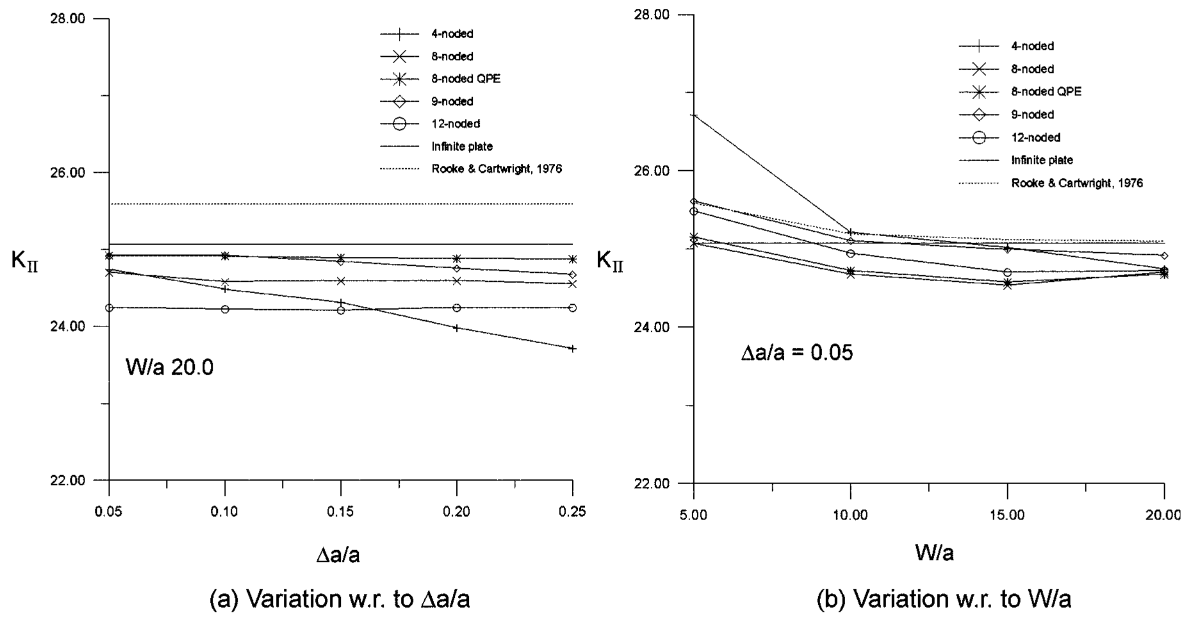


Fig. 7 Variation of SIF for rectangular plate with center crack (Mode II)

3.4 Discussion of results

It is observed from the studies that SIF computed in the present study by employing NI-MVCCI technique along with 4-noded, 8-noded, 9-noded and 12-noded elements are generally in close agreement with the reference solutions for all the problems considered. In all the cases, except for singular QPE, NI-MVCCI technique serves the purpose of performing MVCCI exactly with appropriate rules of Gauss integration. NI-MVCCI technique using 9-point integration along with 8-noded QPE at the crack tip produced results within 1% of the reference solution for all the cases. For this element, lower order of integration is acceptable if one is willing to accept higher deviation with respect to reference solution.

NI-MVCCI technique shows excellent convergence for 4-noded, 8-noded and 9-noded elements as $\Delta a/a \to 0$. It is interesting to note that 9-noded element performs well and converges faster than the other two elements to the reference solution as $\Delta a/a \to 0$. 9-noded Lagrangian element was not used much in the past, but the present study shows that it performs better than 8-noded Serendipity element. This can be attributed to better stress recovery with 9-noded element. 12-noded element shows a significant deviation (3 to 4 percent) from reference solution as $\Delta a/a \to 0$. Here too, the reason appears to be inaccurate stress recovery in 12-noded isoparametric element. In all the cases the infinite plate solutions are achieved for W/a is of the order of 20.

4. Conclusions

NI-MVCCI technique for computing SERR and SIF using for 2-D crack problems has been proposed. NI-MVCCI is a generalized technique and will remove the dependence of MVCCI equation on the type of finite elements employed in the basic stress analysis. NI-MVCCI is a post-processing technique to FEA for computing SERR and SIF. The efficacy of NI-MVCCI technique has been demonstrated for 4-noded bilinear, 8-noded Serendipity (regular & quarter-point), 9-noded Lagrangian and 12-noded cubic isoparametric finite elements. Based on the numerical studies conducted on cracked plate panels the following conclusions are drawn:

- SIF computed in the present study by employing 4-noded, 8-noded (regular & quarter-point), 9-noded and 12-noded elements generally compare well with the reference solutions.
- For 8-noded QPE Gauss integration rule of 9 is recommended for evaluation of SERR and SIF within 1% accuracy using NI-MVCCI technique.
- As $\Delta a/a \rightarrow 0$, results with 8-noded (regular and QPE) and 9-noded elements converge well to the reference solution. Post-processing from the results with 9-noded element is superior compared to 4-noded, 8-noded (regular) and 12-noded elements.
- In general SIF obtained employing 8-noded (regular and QPE) and 9-noded elements converge to the analytical solution for an infinite plate as W/a is of the order of 20. SIF obtained employing 4-noded element has about 2% deviation from the infinite plate solution for W/a = 20.
- There is scope for development of NI-MVCCI technique for 3-D crack problems and cracked stiffened and unstiffened plate/shell structural components subjected to bending and shear loads.

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