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The effective depth of soil stratum for plates resting on elastic foundation

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Abstract. The purpose of this paper is to determine the subsoil depth affected from the load on the plate resting on elastic foundation using stress distribution within the subsoil that will be occurred depending on the loading and dimension of the plate. An iterative method is developed in order to determine the effective depth of the subsoil under the plate. Numerical examples from the technical literature are solved by means of the method suggested herein and displacements, bending moments and shear forces are presented in graphical and tabular forms to evaluate the effects of the limit depth considered in the study. Results showed the efficiency and simplicity of the present approach for the plate resting on an elastic foundation.

Key words: effective depth; plates on elastic foundation; Winkler model; Vlasov model; parameters of soil.

1. Introduction

Beams or plates resting on elastic foundations are very common structural forms in civil and geotechnical engineering. Numerous works in this area of research are available in the technical literature. Researches have been trying to develope more realistic models to solve the practical problems with the reasonable accuracy.

Many researchers use the Winkler model where the vertical displacement of the beam are assumed to be proportional at every point to the contact pressure at that point to solve soil-structure interaction problems (Hetenyi 1950). In the Winkler model, it is assumed that the foundation soil consist of linear elastic springs which are closely spaced and independent of each other. One of the most important shortcomings of this model is that it assumes no interaction between the springs.

To overcome this problem, the springs in the Winkler Model are connected through a special device at the top of the springs by some other researchers. This device includes a thin elastic membrane; an elastic plate and a layer consisting of incompressible vertical elements that deform by lateral shear only. Vlasov develop a two-parameter model that accounts the effect of the neglected shear strain energy in the soil and shear forces that come from surrounding soil by introducing an arbitrary parameter, γ, to characterize the vertical distribution of the deformation in the subsoil (Selvaduari 1979).

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All these models are shown to lead to same differential equation. Basically all these models are equivalent and defer only in the definition of the second parameter. The Vlasov model requires the estimation of the γ parameter. Jones and Xenophontos (1977) established a relationship between the γ parameter and the displacement characteristics, but didn't suggest a computational method. Vallabhan and Das (1988) determined the γ parameter as a function of the characteristic of the beam and the foundation, using an iterative procedure. They named this model as modified Vlasov model and they emphasized that the parameters are depend on the properties of the soil and the structure as well as the type and magnitude of the loading and the depth of the soil.

Ayvaz *et al.* (1998) investigated the effects of the subsoil depth, the plate dimensions and their rations on the dynamic response of plates resting on an elastic foundation. Daloğlu *et al.* (1999) investigated the effects of the subsoil depth, plate dimensions and their rations on the dynamic response of rectangular plates on elastic foundations subjected to both uniformly distributed load and concentrated load at the center of the plates. Vallabhan and Daloğlu (1999), using rectangular elements developed a consistent finite element model to analyze plate resting on a layered soil medium. But they solved the problem using three different depths of the soil stratum. Ayvaz and Özgan (2002) analyzed the effects of the subsoil depth, the beam length, their ratio and the value of the vertical deformation parameter, γ, within the subsoil on the frequency parameters of beams on elastic foundations.

As seen from these studies, researches investigated the change in the soil parameters and the internal forces with the subsoil depth. The idea of determining the subsoil depth from the stress distribution depending on the type of the loading and dimension of the plate is an interesting way of solving these kinds of problems.

2. Development of the theory of Vlasov model

The total potential energy in the soil-structure system may be written as

$$\Pi = U + V \tag{1}$$

where $U = \Pi_p + \Pi_s$, in which Π_p is the strain energy stored in the plate, Π_s is the strain energy stored in the soil and V is the potential energy of the external loads.

The subsoil considered has a finite depth with a rigid boundary at the bottom (Fig. 1). The total potential energy in Eq. (1) can be expanded as

$$\Pi = \frac{1}{2}\int_{\Omega}\left(\frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial y^2}, 2\frac{\partial^2 w}{\partial x\partial y}\right)[D]\left(\frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial y^2}, 2\frac{\partial^2 w}{\partial x\partial y}\right)^T dxdy$$
$$+\frac{1}{2}\int_0^H\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{xz}\gamma_{xz})dxdydz - \int_{\Omega} qwdxdy \tag{2}$$

where σ_x, σ_y, σ_z, τ_{xz}, τ_{xy}, τ_{yz}, ε_x, ε_y, ε_z, γ_{xy}, γ_{yz} are γ_{xz} the stresses and corresponding strains in the subsoil, w, D, q, H are displacement of the plate in z direction, the flexural rigidity of the plate, distributed load on the plate and depth of the subsoil respectively. The displacement of the soil in x, y and z direction can be defined as $\bar{u}, \bar{v}$ and $\bar{w}$. To simplify the model following Vlasov and Leont'ev, it is assumed that,

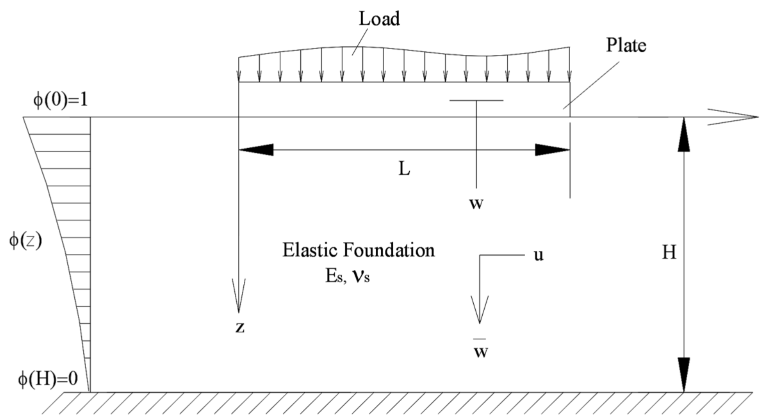

Fig. 1 A loaded plate on elastic foundation

$$\bar{u}(x, y, z) = 0 \tag{3}$$

$$\bar{v}(x, y, z) = 0 \tag{4}$$

and

$$\bar{w}(x, y, z) = w(x, y)\phi(z) \tag{5}$$

where $\phi(z)$ is the mode shape that gives the variation of the deflection in the z direction such that $\phi(0) = 1$ and $\phi(H) = 0$.

Substituting strain-displacement equations of elasticity into Eq. (2), with the assumptions above, total potential energy of the plate-soil system will be

$$\Pi = \frac{1}{2}\int_{\Omega}\left(\frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial y^2}, 2\frac{\partial^2 w}{\partial x \partial y}\right)[D]\left(\frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial y^2}, 2\frac{\partial^2 w}{\partial x \partial y}\right)^T dxdy$$

$$+\frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\{kw^2 + 2t(\nabla w)^2\}dxdy - \int_{\Omega} qwdxdy \tag{6}$$

k and $2t$ in above expression are the soil parameters and may be defined as

$$k = \int_0^H \bar{E}\left(\frac{\partial \phi}{\partial z}\right)^2 dz \tag{7}$$

$$2t = \int_0^H G\phi^2 dz \tag{8}$$

where $\bar{E} = \dfrac{E_s(1 - \nu_s)}{(1 + \nu_s)(1 - 2\nu_s)}$ and $G = \dfrac{E_s}{2(1 + \nu_s)}$ in which E_s and ν_s are the modulus of elasticity and poisson ratio of the soil stratum.

Using variational principle and minimizing the total potential energy of Eq. (6) by taking variations in w and ϕ yields (Turhan 1992)

$$\delta\Pi = \int_{\Omega}(D\nabla^4 w - 2t\nabla^2 w + kw - q)\delta w dx dy$$
$$+\int_0^H\left(-m\frac{\partial^2\phi}{\partial z^2} + n\phi\right)\delta\phi dz + \text{boundary conditions} = 0 \tag{9}$$

where $m = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\bar{E}w^2 dx dy$ and $n = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}G\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]dx dy$

Since the variations δw and $\delta\phi$ are not equal to zero, the terms in the parentheses and boundary conditions must be equal to zero. Therefore the field equation in the domain of the plate, Ω, can be written as

$$D\nabla^4 w - 2t\nabla^2 w + kw = q \tag{10}$$

where ∇^2 is the Laplace and ∇^4 is the biharmonic operators. Displacement of the soil surface outside the plate domain will be evaluated by solving the field equation of the soil domain that will be evaluated by equating D and q of Eq. (10) to zero. The second expression within the parentheses in Eq. (9) is the field equation for the deformation pattern of the soil in the vertical direction. The equation is

$$-m\frac{\partial^2\phi}{\partial z^2} + n\phi = 0 \tag{11}$$

with the boundary conditions $\phi(0) = 1$ and $\phi(H) = 0$. Solution of Eq. (11) with the given boundary conditions yields,

$$\phi(z) = \frac{\sinh\gamma\left(1 - \frac{z}{H}\right)}{\sinh\gamma} \tag{12}$$

and $(\gamma/H)^2 = n/m$, where γ represents the vertical deformation parameter within the subsoil.

The important point here is that the modulus of subgrade reaction, k, and the second parameter t which represents the shear deformation of the soil, are both dependent on the vertical deformation function ϕ and the depth of the soil H as can be seen in Eqs. (7) and (8). Furthermore the value of γ varies with the displacement of the plate and the depth of the subsoil. Therefore the variables w, q, k, $2t$, H and γ are all connected to each other for a plate on elastic foundation.

3. Application of the finite element method

In this study MZC rectangle finite element is used to develop the element stiffness matrix. Nodal displacements at each node are

$$w_i, \frac{\partial w_i}{\partial y}, -\frac{\partial w_i}{\partial x} \quad (i = 1, 2, 3, 4) \tag{13}$$

And the displacement function is

$$w = [N]\{w_e\} \tag{14}$$

where $\{w_e\}$ is the nodal displacement vector containing all 12 components of the type shown in Eq. (13).

The matrix $[N]$ contains the displacement shape functions (Weawer and Johnston 1984). By substituting Eq. (14) into Eq. (1) the stiffness matrices of the plate-soil system can be evaluated as

$$U = \frac{1}{2}\{w_e\}^t([k_p] + [k_w] + [k_v])\{w_e\} \tag{15}$$

where $[k_p]$, $[k_w]$ are $[k_v]$ stiffness matrix of the plate, the Winkler foundation stiffness matrix and the second parameter foundation stiffness matrix (Vallabhan and Daloğlu 1999).

Assembling each element stiffness matrix obtained from above equations, global stiffness matrix is evaluated as

$$[K] = \sum_{i=1}^{n_e}([k_p] + [k_w] + [k_v]) \tag{16}$$

where n_e is the total number of plate finite elements. Finally the equation to be solved is

$$[K]\{W\} = \{F\} \tag{17}$$

Here $[K]$ is the global stiffness matrix, $\{W\}$ represents the global nodal displacement, and $\{F\}$ is the applied equivalent load vector of the system. Boundary conditions need to be applied before solving equation system. The effect of the infinite soil domain outside the plate is applied as equivalent stiffness parameters on the plate boundary. Equivalent forces due to surrounding soil domain on the boundary of the plate are computed as a function of the displacement on the boundary (Turhan 1992). While representing the effect of the soil domain on the plate boundary, there are two type of stiffness to consider. One is axial stiffness related to the displacement of the plate in the z-direction, and the other type is a rotational stiffness related to the rotation of the plate at its edge. After the boundary forces for the discrete points are calculated, it is necessary to concentrate the continuous boundary forces into equivalent boundary forces at the nodes of the finite element.

3.1 The iterative procedure

In this model the solution technique is an iterative process dependent on the γ parameter. By assuming an approximate value of γ initially, the values of k and $2t$ are calculated using Eqs. (7) and (8). These values are used into Eq. (17) and the deflection of the plate are evaluated. Then using these values, the new value of γ is again calculated. The procedure is repeated until two successive values of γ are approximately equal. Further details can be found in the references Vallabhan and Das (1988, 1991).

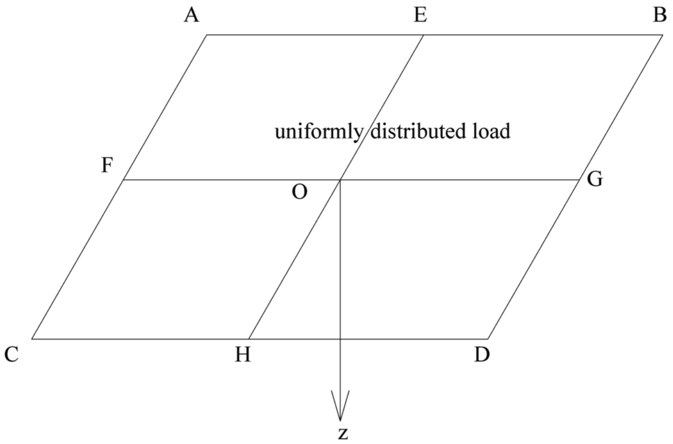

Fig. 2 Point under the center of the area subjected to uniformly distributed load

4. Determination of the effective depth of the subsoil

As mentioned before, an approach has been developed for the evaluation of stresses in the subsoil below the plate. Based on the approach, a computer program also has been developed to calculate stresses at different subsoil depth under rectangular area.

According to this approach, vertical stress at a depth z under a corner of the area subjected to uniform load can be evaluated as

$$\sigma_z = \frac{q}{4\pi}\left[\frac{2AB\sqrt{A^2+B^2+z^2}}{z^2(A^2+B^2+z^2)+A^2B^2}\frac{A^2+B^2+2z^2}{A^2+B^2+z^2}+\left(\sin^{-1}\frac{2AB\sqrt{A^2+B^2+z^2}}{z^2(A^2+B^2+z^2)-A^2B^2}\right)\right] \tag{18}$$

where A and B are dimensions of the plate in the y and x directions respectively, z is the soil depth where the stress is being calculated.

The stresses occurred under center of the plate can be evaluated using superposition rule (Fig. 2). The stress under point O due to the uniform load on the area $ABCD$ may be obtained from various rectangles as follows,

$$\sigma_{z(O)} = \sigma_{AEFO} + \sigma_{FCHO} + \sigma_{EBGO} + \sigma_{HDGO}$$

As seen from the above expression, the stress under the center of the plate is obtained by adding the stresses for these four small rectangles (Spangler 1963, Uzuner 1992).

5. Numerical examples

A convergence study is thought to be useful to decide the number of elements to be used in the finite element mesh. The number of elements is increased until the maximum displacement and the bending moment values calculated from the last two finite element meshes are almost equal. It is concluded that results have acceptable error when using equally spaced 9 elements for quarter plate.

5.1 Numerical example-1

In order to check the accuracy of this approach; the plate on the elastic foundation solved by Vallabhan *et al.* (1991) and Çelik and Saygun (1999) for various soil depth is considered. The same problem is analyzed by using the presented method and results are presented in tabular and graphical forms. The properties of the plate-soil system are as follows. The modulus of elasticity of the subsoil is 68950 kN/m^2, the poisson ratio of the subsoil is 0.25, the modulus of elasticity of the plate is 20685000 kN/m^2, the poisson ratio is 0.20, the thickness of the plate is 0.1524 m, the dimensions are 9.144 × 12.192 m and uniformly distributed load on the plate is 23.94 kN/m^2.

For this example, the effective depth of the subsoil is evaluated as 36 m. The problem is solved using modified Vlasov model initially. The displacements and the bending moments occurred in the middle of the plate are given in Table 1 and the variation of the displacements, the bending moments and the shear forces along the centerline of the plate are plotted in Fig. 3 for different values of subsoil depth such as 3.048, 9.144, 15.204, 36, 113 and 357. As it can be seen from Table 1, the changes in the displacement and the bending moment are not significant for the higher values of subsoil depth beyond H = 36 m. It is seen in Fig. 3 that the curves are very close to each other for H = 36 m and the higher values of the depth of the subsoil.

Table 1 Soil parameters, maximum displacements and the bending moments of the plate on elastic foundation subjected to uniformly distributed load using the Vlasov model

H (m)	Ref	k (kN/m^3)	$2t$ (kN/m)	γ	w (cm)	Mx (kNm)
3.048	V.S.D.	27206	26904	0.5724	0.0872	0.0529
	Çelik, Saygun	27192	26826	0.5766	0.0853	0.0445
	Present Study	27206	26858	0.5707	0.0873	0.0344
6.096	V.S.D.	13757	50282	0.9297	0.1524	0.3113
	Çelik, Saygun	13757	50410	0.9194	0.1526	0.2880
	Present Study	13743	50628	0.8993	0.1533	0.2638
9.144	V.S.D.	9430	69506	1.2644	0.1890	0.4224
	Çelik, Saygun	9377	70586	1.2064	0.1893	0.4109
	Present Study	9342	71302	1.1674	0.1909	0.3799
15.24	V.S.D.	6366	94732	1.9419	0.2070	0.4892
	Çelik, Saygun	5964	104664	1.6193	0.2212	0.4671
	Present Study	5948	105134	1.6044	0.2240	0.4253
36.00	Present Study	3464	166480	2.8930	0.2408	0.3690
113.00	Present Study	3164	180310	8.6422	0.2407	0.3521
357.00	Present Study	3164	180307	27.3035	0.2407	0.3521

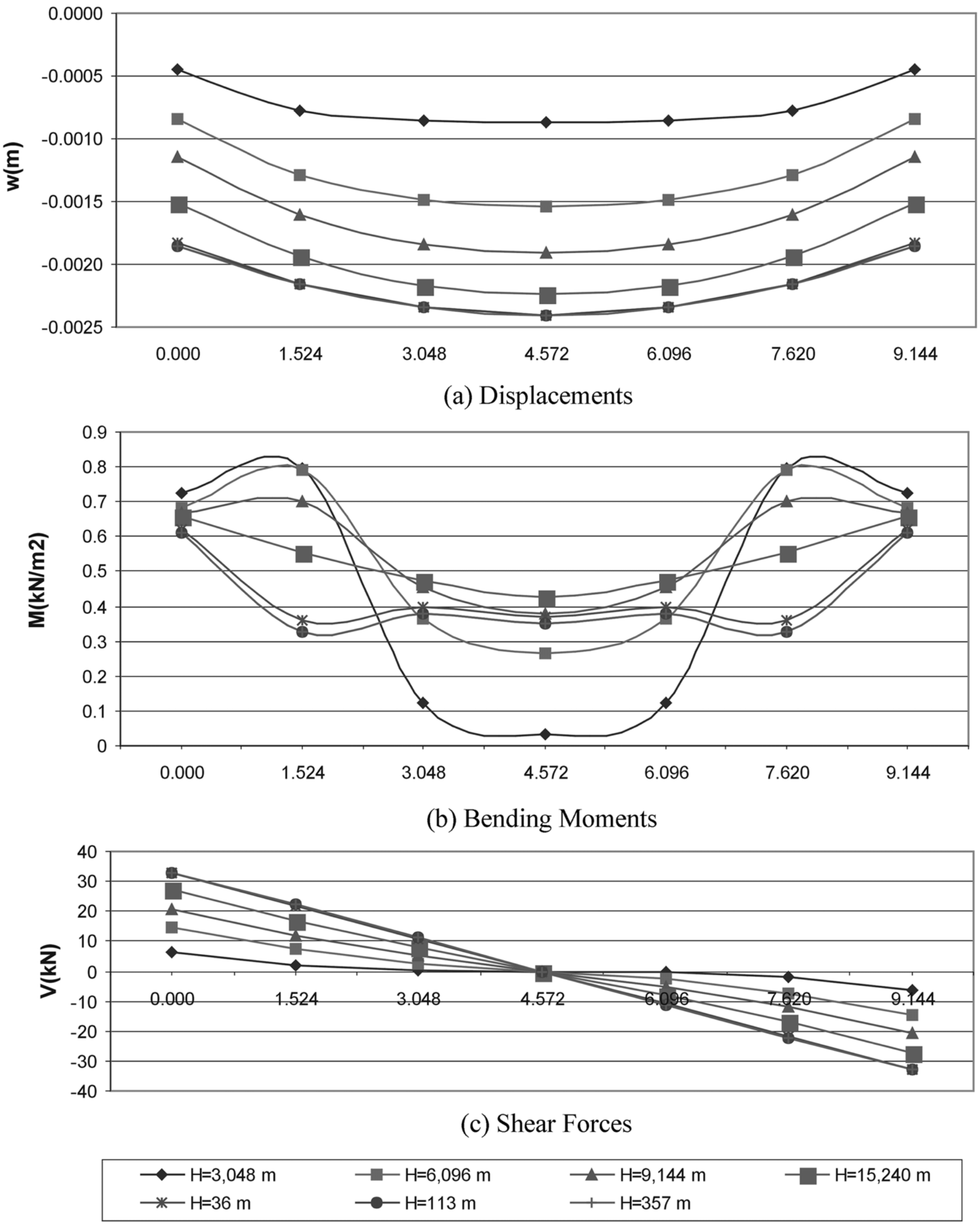

Fig. 3 Variation of displacement, bending moment, and shear force of the plate for various values of subsoil depth for uniformly distributed load using the Vlasov model

Later, the same example is analyzed using the Winkler foundation model for the same subsoil depth. The modulus of subgrade reactions to be used in the Winkler model are taken from Table 1. As it can be seen from Table 2 and Fig. 4, the value of the displacements are not changed much for the higher values of subsoil depth beyond H = 36 m.

Table 2 Soil parameters, maximum displacements and the bending moments of the plate on elastic foundation subjected to uniformly distributed load using the Winkler model

H (m)	Ref	*k* (kN/m^3)	*w* (cm)	*Mx* (kNm)
3.048	Present Study	27206	0.0879	0.0000
6.096	Present Study	13743	0.1741	0.0000
9.144	Present Study	9342	0.2562	0.0000
15.24	Present Study	5948	0.4024	0.0000
36.00	Present Study	3465	0.6849	0.0000
133.00	Present Study	3164	0.7566	0.0000
357.00	Present Study	3164	0.7566	0.0000

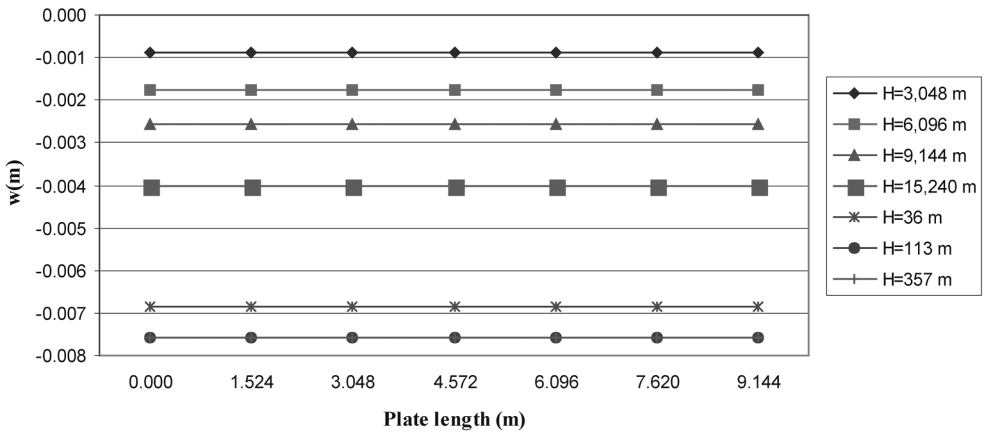

Fig. 4 Displacements along the centerline of the plate subjected to uniformly distributed load obtained with Winkler model

Table 3 Effective depth of the subsoil, soil parameters and maximum displacements of the plate for various dimensions

Length of the plate (m)	Effective depth *H* (m)	*k* (kN/m^3)	2*t* (kN/m)	γ	*w* (cm)
12.192	36	3464	166480	2.8930	0.2408
24.384	45	2557	227305	2.6025	0.3127
36.576	51	2170	269288	2.4597	0.3426
48.768	55	1962	299054	2.3661	0.3638
60.960	58	1831	321447	2.3055	0.3737
73.152	61	1728	341273	2.2756	0.3802

Further, the example is extended for various dimension of the plate in order to see the effects of the plate dimensions on the effective depth of subsoil. The length of the plate is taken as 12.192, 24.384, 36.576, 48.768, 60.960 and 73.152 while the width is kept constant as 9.144 m and the Vlasov model used for the calculation. The results are presented in Table 3. The effective depth of the plate is increasing with the increasing length of the plate as expected.

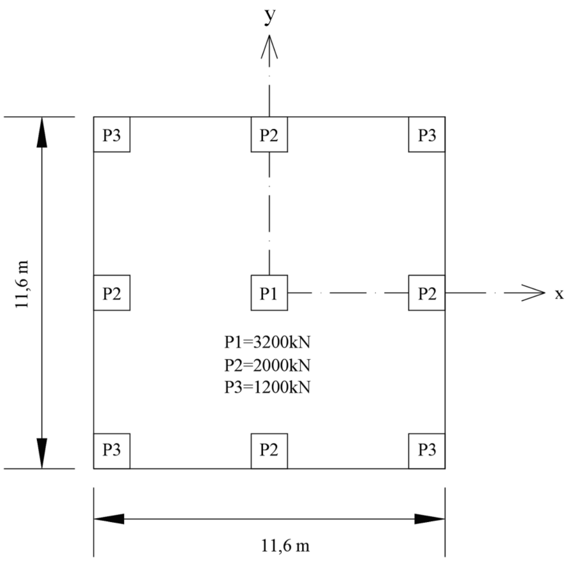

Fig. 5 Geometry and loading of the foundation plate

Table 4 Soil parameters, maximum displacements and the bending moments of the plate on elastic foundation subjected to concentrated load using the Vlasov model

H (m)	Ref	*k* (kN/m^3)	2*t* (kN/m)	γ	*w* (cm)	*Mx* (kNm)
5	Present Study	20147	43487	1.3162	0.4570	512.23
	Çelik, Saygun	19733	45803	1.1180		
10	Present Study	10298	83142	1.4756	0.7127	513.40
	Çelik, Saygun	10087	86696	1.3270		
15	Present Study	7161	116011	1.7176	0.8378	504.34
30	Present Study	4397	178230	2.5585	0.9420	475.53
88	Present Study	3702	207483	6.7860	0.9551	460.55
250	Present Study	3701	207498	19.2774	0.9551	460.54

Table 5 Soil parameters, maximum displacements and the bending moments of the plate on elastic foundation subjected to concentrated load using the Winkler model

H (m)	Ref	*k* (kN/m^3)	*w* (cm)	*Mx* (kNm)
5	Present Study	20147	0.3056	420.80
10	Present Study	10298	0.6510	323.91
15	Present Study	7161	1.0365	268.35
30	Present Study	4397	1.9338	199.51
88	Present Study	3702	2.3960	177.97
250	Present Study	3701	2.3967	177.94

5.2 Numerical example-2

As a second example, the plate under a vertical column load as shown in Fig. 5 is analyzed. The displacement, the bending moments, the shear forces and soil parameters are presented in Table 4,

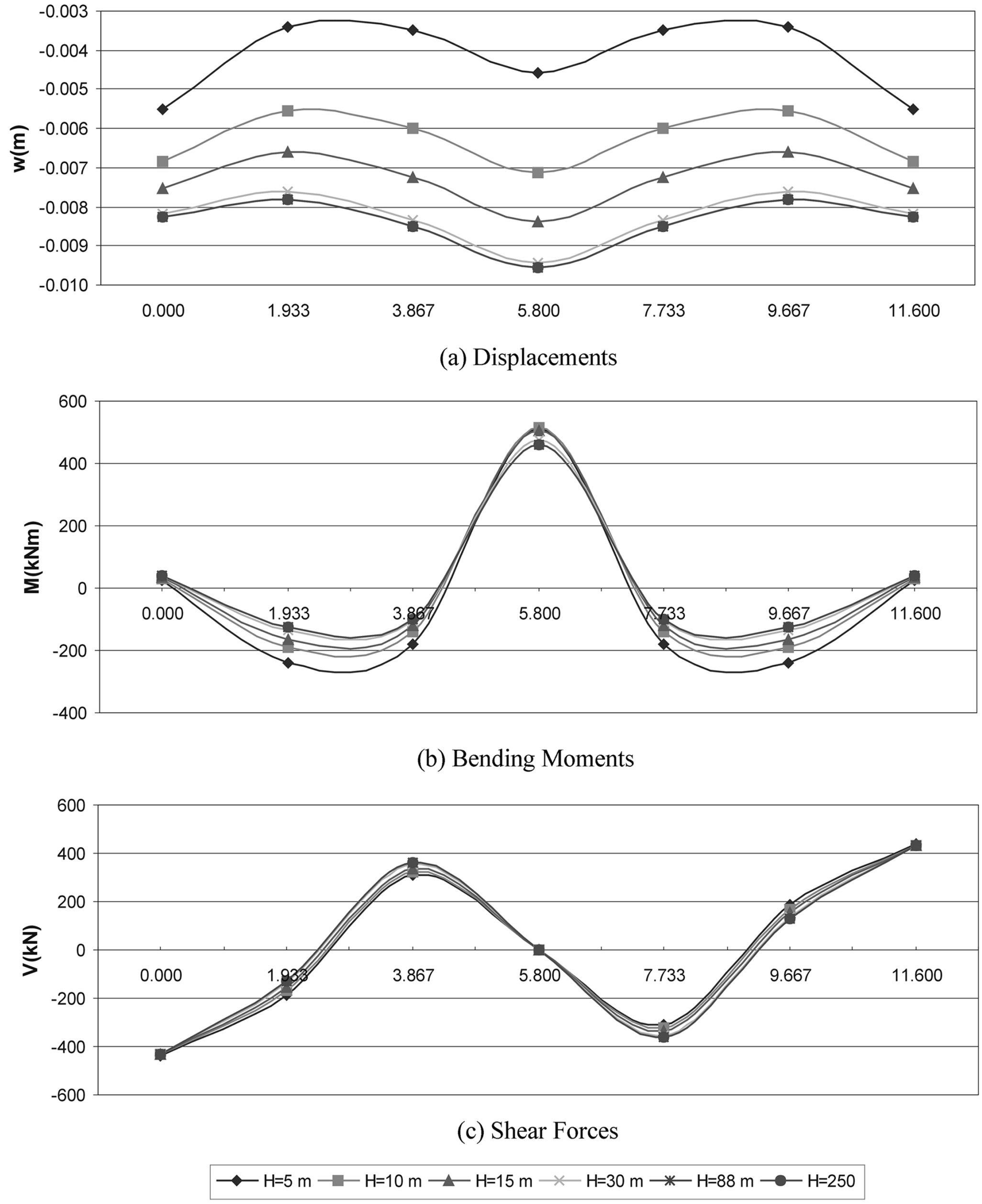

Fig. 6 Variation of displacement, bending moment, and shear force of the plate for various values of subsoil depth for concentrated load using the Vlasov model

Fig. 6 for the Vlasov Model, Table 5, Fig. 7 for the Winkler model. The modulus of elasticity of the subsoil is 80000 kN/m^2, the poisson ratio of the subsoil is 0.125, the elasticity modulus of the plate is 20000000 kN/m^2, the poisson ratio is 0.16, the thickness of the plate is 0.6 m, and the

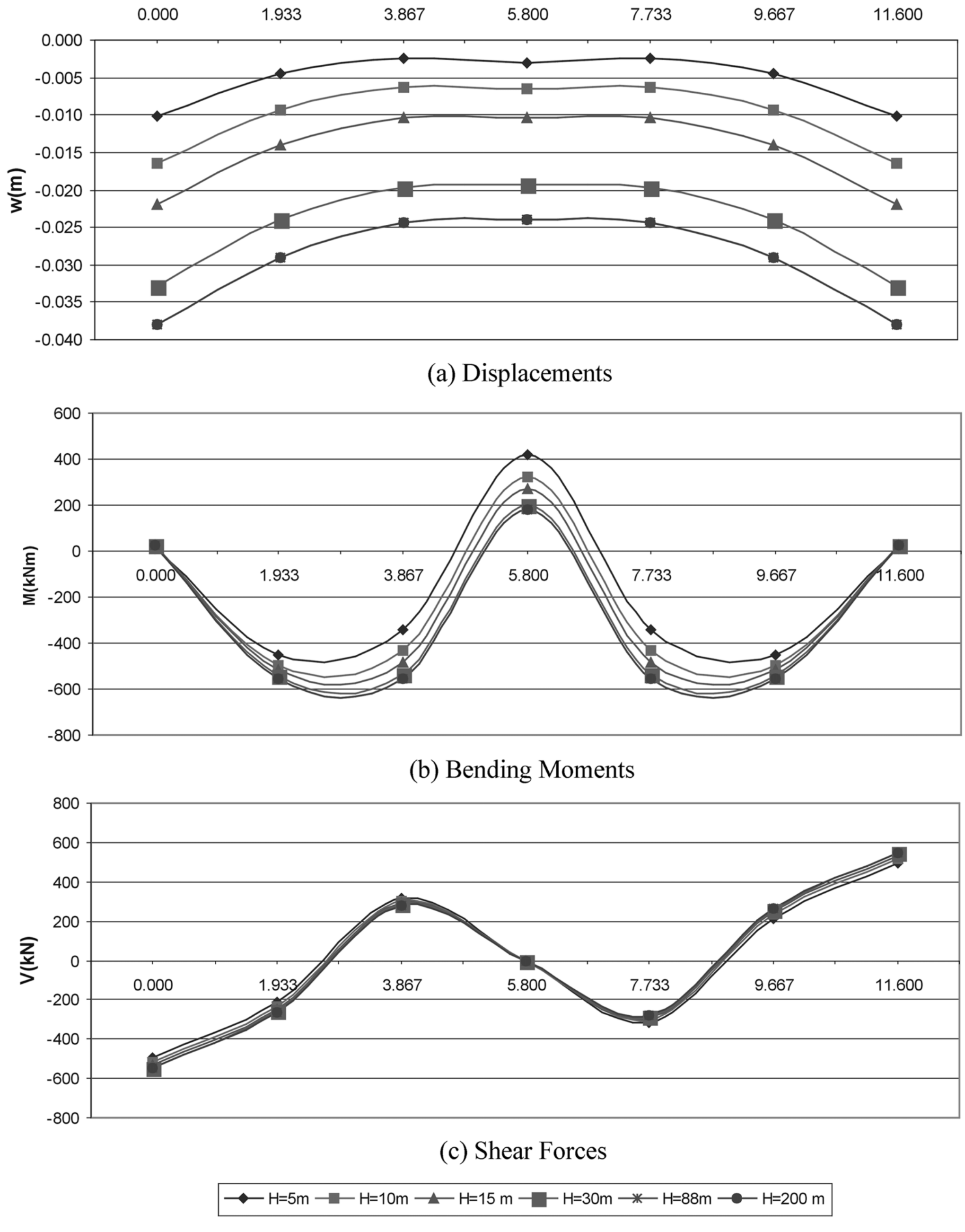

Fig. 7 Variation of displacement, bending moment, and shear force of the plate for various values of subsoil depth for concentrated load using the Winkler model

dimensions are 11.6 × 11.6 m.

The effective depth of the subsoil is computed as 88 m for the example. Studying Table 4, Table 5, Fig. 6, Fig. 7, it can easily be seen that taking the depth of the subsoil as H = 88 m and considering as a rigid boundary at the bottom will be sufficient for the analysis since the difference in the values of the displacements, the bending moments and the shear forces are negligible for the higher values of the subsoil depth.

6. Conclusions

Determination of the effective depth of the soil stratum for a plate resting on an elastic foundation was aimed in the study. An iterative method is developed in order to evaluate the internal forces of the plate on an elastic foundation as well as the effective depth of the soil stratum under the plate. The stress distribution within the soil medium depending on the loading and the dimensions of the plate is evaluated for the purpose, and the effective depth of the soil stratum is calculated. It can be concluded that using a deeper depth of soil stratum doesn't affect the results much but taking a shallow depth may result with an unrealistic solution if there is not a relatively rigid foundation at a certain depth.

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