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Analytical solutions to magneto-electro-elastic beams

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Abstract. By means of the two-dimensional basic equations of transversely isotropic magneto-electro-elastic media and the strict differential operator theorem, the general solution in the case of distinct eigenvalues is derived, in which all mechanical, electric and magnetic quantities are expressed in four harmonic displacement functions. Based on this general solution in the case of distinct eigenvalues, a series of problems is solved by the trial-and-error method, including magneto-electro-elastic rectangular beam under uniform tension, electric displacement and magnetic induction, pure shearing and pure bending, cantilever beam with point force, point charge or point current at free end, and cantilever beam subjected to uniformly distributed loads. Analytical solutions to various problems are obtained.

Key words: general solution; magneto-electro-elastic plane; harmonic function; analytical solution.

1. Introduction

Due to its excellent piezoelectric/piezomagnetic properties, composites made of piezoelectric/piezomagnetic materials have found widespread applications. Therefore, it is necessary to make theoretical analysis and accurate quantitative descriptions of electric, magnetic and stress fields inside piezoelectric/piezomagnetic composites in the working condition caused by the joint action of mechanical loads, electric fields and magnetic fields, from the point of view of electro-magnetro-mechanical coupling. So piezoelectric and magneto-electro-elastic materials have attracted a considerable amount of research in recent years and many important achievements have been made for these materials.

In regard to piezoelectric materials, Sosa and Castro (1994) presented the solutions for the cases of concentrated loads and point charge applied at the line boundary of a piezoelectric half-plane. Kogan *et al*. (1996) gave an analytical solution of infinite body with spheroidal inclusion under the joint action of uniform loads, electric displacement, in-plane shearing and off-plane shearing. Ding *et al*. (1997a) obtained the solutions for a piezoelectric wedge subjected to concentrated forces and

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point charge. Ding *et al.* (1997b) derived Green's functions for a two-phase infinite piezoelectric plane, in which all physical quantities are expressed in three harmonic functions.

Magneto-electro-elastic materials possess simultaneously piezoelectric, piezomagnetic and magnetoelectric effects. Liu *et al*. (2001) obtained Green's functions for an infinite two-dimensional anistropic magneto-electro-elastic medium containing an elliptical cavity based on the extended Stroh Formalism. Pan (2001, 2002b) derived the exact solutions for three-dimensional anisotropy linearly magneto-electro-elastic, simply-supported, and multi-layered rectangular plates under static loads and analytical solutions for free vibrations, respectively. Pan (2002a) derived three-dimensional Green's functions in anisotropic magneto-electro-elastic full space, half space, and bi-materials based on the extended Stroh formalism by applying the two-dimensional Fourier transforms. Wang and Shen (2002) obtained the general solution expressed by five harmonic functions and applied the derived general solution to find the fundamental solution for a generalized dislocation and also to derive Green's functions for a semi-infinite magneto-electro-elastic solid. Wang and Shen (2003) presented analytic solutions for the plane problem of a inclusion of arbitrary shape in an entire plane, or within one of the two bonded dissimilar half-plane. Hou *et al.* (2003) analyzed the elliptical Hertizan contact of transversely isotropic magneto-electro-elastic bodies with the general solutions in terms of harmonic functions. Chen *et al.* (2003) obtained analytical solutions of simply supported magneto-electro-elastic circular plate under uniform loads with a general solution in forms of harmonic functions.

In this paper, the works of Ding *et al.* (1997a,b) will be generalized into transversely isotropic magneto-electro-elastic media. By means of the two-dimensional basic equations of transversely isotropic magneto-electro-elastic media and the strict differential operator theorem, the general solution in the case of distinct eigenvalues is derived, in which all mechanical, electric and magnetic quantities are expressed in four harmonic displacement functions. Then, with the trial-and-error method, exact solutions to some simple problems are acquired, which include magneto-electro-elastic rectangular beam under uniform tension, electric displacement and magnetic induction, pure shearing and pure bending. We also give out the analytical solutions in harmonic polynomials to cantilever beam with point force, point charge or point current at free end, and subjected to uniformly distributed loads on upper and bottom surfaces.

2. General solution to the plane problem of magneto-electro-elastic solid

For the transversely isotropic magneto-electro-elastic bodies, the basic equations have been given as in Pan (2001) (where *xoy* plane denotes the isotropic plane). If we introduce such assumptions for plane problems as the displacements u_i, the electric potential Φ and magnetic potential Ψ are independent of y for the plane-strain problems, the basic equations for two-dimensional magneto-electro-elastic solid in the *xoz* coordinates can be simplified as follows:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} + f_x = 0, \qquad \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma_z}{\partial z} + f_z = 0$$
$$\frac{\partial D_x}{\partial x} + \frac{\partial D_z}{\partial z} = f_e, \qquad \frac{\partial B_x}{\partial x} + \frac{\partial B_z}{\partial z} = f_m \tag{1}$$

$$\sigma_x = c_{11}\frac{\partial u}{\partial x} + c_{13}\frac{\partial w}{\partial z} + e_{31}\frac{\partial \Phi}{\partial z} + d_{31}\frac{\partial \Psi}{\partial z}$$

$$\tau_{xz} = c_{44}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + e_{15}\frac{\partial \Phi}{\partial x} + d_{15}\frac{\partial \Psi}{\partial x}, \qquad \sigma_z = c_{13}\frac{\partial u}{\partial x} + c_{33}\frac{\partial w}{\partial z} + e_{33}\frac{\partial \Phi}{\partial z} + d_{33}\frac{\partial \Psi}{\partial z}$$

$$D_x = e_{15}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) - \varepsilon_{11}\frac{\partial \Phi}{\partial x} - g_{11}\frac{\partial \Psi}{\partial x}, \qquad D_z = e_{31}\frac{\partial u}{\partial x} + e_{33}\frac{\partial w}{\partial z} - \varepsilon_{33}\frac{\partial \Phi}{\partial z} - g_{33}\frac{\partial \Psi}{\partial z}$$

$$B_x = d_{15}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) - g_{11}\frac{\partial \Phi}{\partial x} - \mu_{11}\frac{\partial \Psi}{\partial x}, \qquad B_z = d_{31}\frac{\partial u}{\partial x} + d_{33}\frac{\partial w}{\partial z} - g_{33}\frac{\partial \Phi}{\partial z} - \mu_{33}\frac{\partial \Psi}{\partial z} \tag{2}$$

where $\sigma_i(\tau_{ij})$, u_i, D_i and B_i are the components of stress, displacement, electric displacement and magnetic induction, respectively; Φ and Ψ are the electric potential and magnetic potential, respectively; f_i, f_e and f_m are body force, free charge density and current density, respectively (According to electromagnetic theorem, $f_m = 0$); c_{ij}, e_{ij}, d_{ij}, ε_{ij}, g_{ij} and μ_{ij} are the elastic, piezoelectric, piezomagnetic, dielectric, electromagnetic and magnetic constants, respectively. For the plane-stress problems, we take the stress component $\sigma_y = \tau_{xy} = \tau_{yz} = 0$, electric displacement $D_y = 0$, magnetic induction $B_y = 0$ and the plate width $b = 1$, then the basic equations can also be simplified and expressed as Eqs. (1), (2), in which c_{ij}, e_{ij}, d_{ij}, ε_{ij}, g_{ij} and μ_{ij} shall be replaced with the coefficients $\bar{c}_{ij}$, $\bar{e}_{ij}$, $\bar{d}_{ij}$, $\bar{\varepsilon}_{ij}$, $\bar{g}_{ij}$ and $\bar{\mu}_{ij}$, respectively. The coefficients are expressed in terms of material constants and listed in Appendix A.

Ding *et al*. (1997a,b) derived the solutions to piezoelectric plane problem, in which all physical quantities are expressed in three harmonic functions. With the method and the strict differential operator theorem presented in Ding *et al*. (1997a,b), the general solutions of two-dimensional magneto-electro-elastic media in the case of distinct eigenvalues can be easily derived and expressed in four harmonic functions as follows:

$$u = \sum_{j=1}^{4}\frac{\partial \psi_j}{\partial x}, \quad w_m = \sum_{j=1}^{4} s_j k_{mj}\frac{\partial \psi_j}{\partial z_j}, \quad \sigma_x = \sum_{j=1}^{4}\omega_{4j}\frac{\partial^2 \psi_j}{\partial z_j^2}$$

$$\sigma_m = \sum_{j=1}^{4}\omega_{mj}\frac{\partial^2 \psi_j}{\partial z_j^2}, \quad \tau_m = \sum_{j=1}^{4} s_j\omega_{mj}\frac{\partial^2 \psi_j}{\partial x \partial z_j} \qquad (m = 1, 2, 3) \tag{3}$$

where the generalized displacements and stresses as follows:

$$\begin{aligned} &w_1 = w, \quad w_2 = \Phi, \quad w_3 = \Psi \\ &\sigma_1 = \sigma_z, \quad \sigma_2 = D_z, \quad \sigma_3 = B_z \\ &\tau_1 = \tau_{xz}, \quad \tau_2 = D_x, \quad \tau_3 = B_x \end{aligned} \tag{4}$$

the functions ψ_j satisfy the following equations:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z_j^2}\right)\psi_j = 0 \qquad (j = 1, 2, 3, 4) \tag{5}$$

where $z_j = s_j z\,(j = 1, 2, 3, 4)$, $s_j(j = 1{\sim}4)$ are the four roots of the following equations (We take $\mathrm{Re}(s_j) > 0$):

$$a_1 s^8 - a_2 s^6 + a_3 s^4 - a_4 s^2 + a_5 = 0 \tag{6}$$

where the coefficients $k_{mj}, \omega_{mj}(m = 1{\sim}3,\ j = 1{\sim}4)$ in Eqs. (3) and $a_n(n = 1{\sim}5)$ in Eq. (6) are the same as those listed in Hou *et al.* (2003). ω_{4j} can be calculated by the following equation:

$$\omega_{4j} = -\omega_{1j} s_j^2 \qquad (j = 1, 2, 3, 4) \tag{7}$$

3. Discussion about degeneration of the above general solution

In calculating k_{mj}, ω_{mj}, a_n in Eqs. (3) and Eq. (6), taking, $d_{15} = d_{31} = d_{33} = 0$, $g_{11} = g_{33} = 0$, $\mu_{11} = 0$ and $\mu_{33} = 1$, and making j change from 1 to 3, m from 1 to 2, respectively, we will have $a_5 = 0$, $s_4 = 0$. Then, it can be seen that the degenerated general solution without magnetic quantities is the same as that of plane problem of piezoelectric media given by Ding *et al*. (1997a,b).

Furthermore, taking $e_{15} = e_{31} = e_{33} = 0$, $d_{15} = d_{31} = d_{33} = 0$, $g_{11} = g_{33} = 0$, $\varepsilon_{11} = 0$, $\mu_{11} = 0$, $\varepsilon_{33} = 1$ and $\mu_{33} = 1$, then from Eq. (6) we will have $a_4 = a_5 = 0$ and $s_3 = s_4 = 0$. Then, taking j in Eqs. (3) from 1 to 2 and $m = 1$, respectively, we can find that the general solution is generated into that of elastic problem.

4. Exact solution to some simple problems

In this section, some simple problems are studied only by use of harmonic polynomials presented in Appendix B. Because displacement functions ψ_j satisfy weighted harmonic Eq. (5), all harmonic polynomials in Appendix B can be chosen as displacement functions simply by replacing z with z_j just as illustrated in the following sections.

4.1 Rigid body displacements, identical electric and magnetic potential

Using $\varphi_1^0(x, z)$, $\varphi_1^1(x, z)$ and $\varphi_2^1(x, z)$ in Eq. (B2) in Appendix B, we constitute the displacement function

$$\psi_j = A_{1j} x + B_{1j} z_j + B_{2j} x z_j \qquad (j = 1{\sim}4) \tag{8}$$

where A_{1j}, B_{1j}, B_{2j} are unknown constants to be determined.
Substituting Eq. (8) into Eqs. (3) and selecting the suitable constants, we have

$$u = u_0 + \omega_0 z,\ \ w = w_0 - \omega_0 x,\ \ \Phi = \Phi_0,\ \ \Psi = \Psi_0 \tag{9}$$

where $u_0, w_0, \omega_0, \Phi_0$ and Ψ_0 are unknown constants denoting rigid body displacements, rigid body rotation, identical electric and magnetic potential. Substituting Eq. (9) into Eqs. (2), we have

$$\sigma_x = \sigma_z = \tau_{xz} = 0,\ \ D_x = D_z = 0,\ \ B_x = B_z = 0 \tag{10}$$

4.2 Uniform tension, electric displacement and magnetic induction and pure shearing

Using $\varphi_2^0(x, z)$ and $\varphi_2^1(x, z)$ in Eq. (B2) in Appendix B, we constitute the displacement function

$$\psi_j = A_{2j}(x^2 - z_j^2) + B_{2j}xz_j \qquad (j = 1\sim4) \tag{11}$$

where A_{2j} and B_{2j} are unknown constants to be determined.

Substituting Eq. (11) into Eqs. (3) results in

$$u = \sum_{j=1}^{4}(2A_{2j}x + B_{2j}z_j), \qquad w = \sum_{j=1}^{4} s_j k_{1j}(B_{2j}x - 2A_{2j}z_j) \tag{12a}$$

$$\Phi = \sum_{j=1}^{4} s_j k_{2j}(B_{2j}x - 2A_{2j}z_j), \qquad \Psi = \sum_{j=1}^{4} s_j k_{3j}(B_{2j}x - 2A_{2j}z_j) \tag{12b}$$

$$\sigma_x = -2\sum_{j=1}^{4}\omega_{4j}A_{2j}, \qquad \sigma_z = -2\sum_{j=1}^{4}\omega_{1j}A_{2j}, \qquad \tau_{xz} = \sum_{j=1}^{4} s_j\omega_{1j}B_{2j} \tag{12c}$$

$$D_z = -2\sum_{j=1}^{4}\omega_{2j}A_{2j}, \qquad D_x = \sum_{j=1}^{4} s_j\omega_{2j}B_{2j} \tag{12d}$$

$$B_z = -2\sum_{j=1}^{4}\omega_{3j}A_{2j}, \qquad B_x = \sum_{j=1}^{4} s_j\omega_{3j}B_{2j} \tag{12e}$$

The above equations contain seven sensible solutions, that is a magneto-electro-elastic rectangular beam subjected to uniform tension, electric displacement and magnetic induction in x and z directions and under pure shearing. The boundary conditions are

$$x = \pm\frac{L}{2}: \quad \begin{cases} \sigma_x = \sigma_1 \\ \tau_{xz} = \tau_0 \\ D_x = D_1 \\ B_x = B_1 \end{cases}, \qquad z = \pm\frac{h}{2}: \quad \begin{cases} \sigma_z = \sigma_2 \\ \tau_{xz} = \tau_0 \\ D_z = D_2 \\ B_z = B_2 \end{cases} \tag{13}$$

In additional, we take the condition of no rotation as follows:

$$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0 \tag{14}$$

Substituting Eqs. (12) into Eqs. (13)-(14) leads to:

$$\begin{Bmatrix} A_{21} \\ A_{22} \\ A_{23} \\ A_{24} \end{Bmatrix} = -\frac{1}{2}\begin{bmatrix} \omega_{11} & \omega_{12} & \omega_{13} & \omega_{14} \\ \omega_{21} & \omega_{22} & \omega_{23} & \omega_{24} \\ \omega_{31} & \omega_{32} & \omega_{33} & \omega_{34} \\ \omega_{41} & \omega_{42} & \omega_{43} & \omega_{44} \end{bmatrix}^{-1} \begin{Bmatrix} \sigma_2 \\ D_2 \\ B_2 \\ \sigma_1 \end{Bmatrix} \tag{15}$$

$$\begin{Bmatrix} B_{21} \\ B_{22} \\ B_{23} \\ B_{24} \end{Bmatrix} = \begin{bmatrix} s_1\omega_{11} & s_2\omega_{12} & s_3\omega_{13} & s_4\omega_{14} \\ s_1\omega_{21} & s_2\omega_{22} & s_3\omega_{23} & s_4\omega_{24} \\ s_1\omega_{31} & s_2\omega_{32} & s_3\omega_{33} & s_4\omega_{34} \\ s_1(1-k_{11}) & s_2(1-k_{12}) & s_3(1-k_{13}) & s_4(1-k_{14}) \end{bmatrix}^{-1} \begin{Bmatrix} \tau_0 \\ D_1 \\ B_1 \\ 0 \end{Bmatrix} \tag{16}$$

The exact solution for a magneto-electro-elastic rectangular beam subjected to uniform tension, electric displacement and magnetic induction and under pure shearing can be obtained by substituting Eqs. (15)-(16) into Eqs. (12a)-(12e), in which stress components, electric displacements and magnetic induction are

$$\sigma_x = \sigma_1, \quad \sigma_z = \sigma_2, \quad \tau_{xz} = \tau_0 \tag{17a}$$

$$D_x = D_1, \ D_z = D_2, \ B_x = B_1, \ B_z = B_2 \tag{17b}$$

The seven sensible solutions contained of Eqs. (17a)-(17b) are

4.2.1 Uniform tension in z direction:

$$\sigma_2 \neq 0, \ \sigma_1 = \tau_0 = 0, \ D_1 = D_2 = 0, \ B_1 = B_2 = 0 \tag{18}$$

4.2.2 Uniform tension in x direction:

$$\sigma_1 \neq 0, \ \sigma_2 = \tau_0 = 0, \ D_1 = D_2 = 0, \ B_1 = B_2 = 0 \tag{19}$$

4.2.3 Uniform electric displacement in x direction:

$$D_1 \neq 0, \ \sigma_1 = \sigma_2 = \tau_0 = 0, \ D_2 = 0, \ B_1 = B_2 = 0 \tag{20}$$

4.2.4 Uniform electric displacement in z direction:

$$D_2 \neq 0, \ \sigma_1 = \sigma_2 = \tau_0 = 0, \ D_1 = 0, \ B_1 = B_2 = 0 \tag{21}$$

4.2.5 Uniform magnetic induction in x direction:

$$B_1 \neq 0, \ \sigma_1 = \sigma_2 = \tau_0 = 0, \ D_1 = D_2 = 0, \ B_2 = 0 \tag{22}$$

4.2.6 Uniform magnetic induction in z direction:

$$B_2 \neq 0, \ \sigma_1 = \sigma_2 = \tau_0 = 0, \ D_1 = D_2 = 0, \ B_1 = 0 \tag{23}$$

4.2.7 Pure shearing

$$\tau_0 \neq 0, \ \sigma_1 = \sigma_2 = 0, \ D_1 = D_2 = 0, \ B_1 = B_2 = 0 \tag{24}$$

From the solution Eqs. (9)-(10) to beam with rigid body displacements, identical electric and magnetic potential, it is found that the variable x in Eqs. (12a,b) can be replaced with $x - L$ or $x - L/2$ while stress components, electric displacements and magnetic induction remain their values.

4.3 Pure bending

Using $\varphi_3^1(x, z)$ in Eq. (B2) in Appendix B, we constitute the displacement function

$$\psi_j = B_{3j}\varphi_3^1(x, z_j) = B_{3j}\left(x^2 z_j - \frac{1}{3}z_j^3\right) \quad (j = 1\sim4) \tag{25}$$

where B_{3j} are unknown constants to be determined.

Substituting Eq. (25) into Eqs. (3) results in

$$u = 2\sum_{j=1}^{4} B_{3j}x, \quad w = \sum_{j=1}^{4} s_j k_{1j} B_{3j}(x^2 - z_j^2), \quad \Phi = \sum_{j=1}^{4} s_j k_{2j} B_{3j}(x^2 - z_j^2) \tag{26a}$$

$$\Psi = \sum_{j=1}^{4} s_j k_{3j} B_{3j}(x^2 - z_j^2), \quad \sigma_x = -2\sum_{j=1}^{4} \omega_{4j} B_{3j} z_j \tag{26b}$$

$$\sigma_z = -2\sum_{j=1}^{4} \omega_{1j} B_{3j} z_j, \quad D_z = -2\sum_{j=1}^{4} \omega_{2j} B_{3j} z_j, \quad B_z = -2\sum_{j=1}^{4} \omega_{3j} B_{3j} z_j \tag{26c}$$

$$\tau_{xz} = 2\sum_{j=1}^{4} s_j \omega_{1j} B_{3j} x, \quad D_x = 2\sum_{j=1}^{4} s_j \omega_{2j} B_{3j} x, \quad B_z = 2\sum_{j=1}^{4} s_j \omega_{3j} B_{3j} x \tag{26d}$$

The above equations contain the physically sensible solution for a magneto-electro-elastic rectangular beam under pure bending as described below.

$$x = \pm L/2: \begin{cases} \sigma_x = \sigma_0 z \\ \tau_{xz} = 0 \\ D_x = 0 \\ B_x = 0 \end{cases}, \quad z = \pm h/2: \begin{cases} \sigma_z = 0 \\ \tau_{xz} = 0 \\ D_z = 0 \\ B_z = 0 \end{cases} \tag{27}$$

Substituting Eqs. (26) into Eqs. (27) gives:

$$\sum_{j=1}^{4} s_j \omega_{1j} B_{3j} = 0 \tag{28}$$

$$\sum_{j=1}^{4} s_j \omega_{2j} B_{3j} = 0 \tag{29}$$

$$\sum_{j=1}^{4} s_j \omega_{3j} B_{3j} = 0 \tag{30}$$

$$\sum_{j=1}^{4} s_j \omega_{4j} B_{3j} = -\frac{\sigma_0}{2} \tag{31}$$

The unknown constants $B_{3j} (j = 1 \sim 4)$ can be calculated from Eqs. (28)-(31) as follows:

$$\begin{Bmatrix} B_{31} \\ B_{32} \\ B_{33} \\ B_{34} \end{Bmatrix} = -\frac{6M}{h^3} \begin{bmatrix} s_1\omega_{11} & s_2\omega_{12} & s_3\omega_{13} & s_4\omega_{14} \\ s_1\omega_{21} & s_2\omega_{22} & s_3\omega_{23} & s_4\omega_{24} \\ s_1\omega_{31} & s_2\omega_{32} & s_3\omega_{33} & s_4\omega_{34} \\ s_1\omega_{41} & s_2\omega_{42} & s_3\omega_{43} & s_4\omega_{44} \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix} \tag{32}$$

where $M = \dfrac{\sigma_0 h^3}{12}$.

The analytical solutions for a magneto-electro-elastic rectangular beam under pure bending can be obtained by substituting Eq. (32) into Eqs. (26a)-(26d), in which displacements, stress components, electric displacements and magnetic induction are

$$u = 2\sum_{j=1}^{4} B_{3j} x, \quad w_m = \sum_{j=1}^{4} s_j k_{mj} B_{3j} (x^2 - z_j^2) \qquad (m = 1, 2, 3) \tag{33a}$$

$$\sigma_x = \frac{12M}{h^3} z, \quad \sigma_z = \tau_{xz} = 0, \quad D_x = D_z = 0, \quad B_x = B_z = 0 \tag{33b}$$

It can be seen from Eqs. (33a,b) that stresses, electric displacement and magnetic induction are independent of material constants.

It is apparently that solution of Eqs. (33a,b) is also the analytical solution for a cantilever rectangular beam under bending moment M at free end while the other end $x = 0$ is fixed.

5. Cantilever beam with point force in *z* direction, point charge and point current at free end

Using $\varphi_2^1(x, z)$ and $\varphi_4^1(x, z)$ in Eq. (B2) in Appendix B, we constitute the displacement function

$$\psi_j = B_{2j} x z_j + B_{4j} (x^3 z_j - x z_j^3) \qquad (j = 1 \sim 4) \tag{34}$$

where B_{2j}, B_{4j} are unknown constants to be determined.

Substituting Eq. (34) into Eqs. (3) and superposing the rigid body displacements solutions in Eqs. (9) results in

$$u = \sum_{j=1}^{4} B_{2j} z_j + \sum_{j=1}^{4} B_{4j} (3x^2 z_j - z_j^3) \tag{35a}$$

$$w = w_0 + \sum_{j=1}^{4} s_j k_{1j} B_{2j} x + \sum_{j=1}^{4} s_j k_{1j} B_{4j} (x^3 - 3x z_j^2) \tag{35b}$$

$$\Phi = \Phi_0 + \sum_{j=1}^{4} s_j k_{2j} B_{2j} x + \sum_{j=1}^{4} s_j k_{2j} B_{4j}(x^3 - 3xz_j^2) \tag{35c}$$

$$\Psi = \Psi_0 + \sum_{j=1}^{4} s_j k_{3j} B_{2j} x + \sum_{j=1}^{4} s_j k_{3j} B_{4j}(x^3 - 3xz_j^2) \tag{35d}$$

$$\sigma_x = -6\sum_{j=1}^{4} \omega_{4j} B_{4j} x z_j, \quad \sigma_z = -6\sum_{j=1}^{4} \omega_{1j} B_{4j} x z_j \tag{35e}$$

$$D_z = -6\sum_{j=1}^{4} \omega_{2j} B_{4j} x z_j, \quad B_z = -6\sum_{j=1}^{4} \omega_{3j} B_{4j} x z_j \tag{35f}$$

$$\tau_{xz} = \sum_{j=1}^{4} s_j \omega_{1j} B_{2j} + \sum_{j=1}^{4} s_j \omega_{1j} B_{4j}(3x^2 - 3z_j^2) \tag{35g}$$

$$D_x = \sum_{j=1}^{4} s_j \omega_{2j} B_{2j} + \sum_{j=1}^{4} s_j \omega_{2j} B_{4j}(3x^2 - 3z_j^2) \tag{35h}$$

$$B_x = \sum_{j=1}^{4} s_j \omega_{3j} B_{2j} + \sum_{j=1}^{4} s_j \omega_{3j} B_{4j}(3x^2 - 3z_j^2) \tag{35i}$$

The boundary conditions of a cantilever beam shown in Fig. 1 with a point force P in z direction, point charge Q and point current J at free end, are

$$z = \pm h/2: \begin{cases} \sigma_z = 0 \\ \tau_{xz} = 0 \\ D_z = 0 \\ B_z = 0 \end{cases}, \quad x = 0: \begin{cases} \sigma_x = 0 \\ \int_{-h/2}^{+h/2} \tau_{xz} dz = -P \\ \int_{-h/2}^{+h/2} D_x dz = +Q \\ \int_{-h/2}^{+h/2} B_x dz = +J \end{cases}, \quad (x = L, z = 0): \begin{cases} u = 0 \\ \dfrac{\partial w}{\partial x} = 0 \\ w = 0 \end{cases} \tag{36}$$

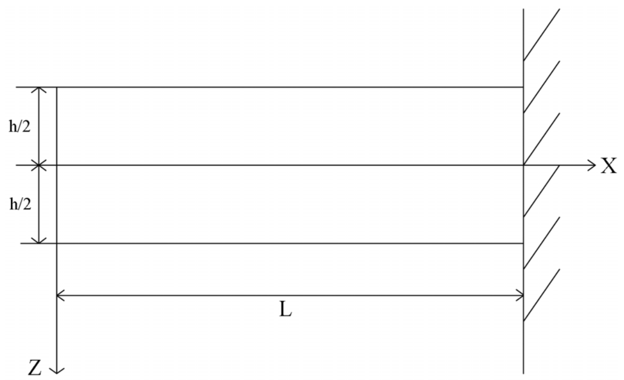

Fig. 1 A cantilever beam

Substituting Eqs. (35) into Eqs. (36) results in

$$\sum_{j=1}^{4} s_j \omega_{1j} B_{4j} = 0 \tag{37}$$

$$\sum_{j=1}^{4} s_j \omega_{2j} B_{4j} = 0 \tag{38}$$

$$\sum_{j=1}^{4} s_j \omega_{3j} B_{4j} = 0 \tag{39}$$

$$\sum_{j=1}^{4} s_j \omega_{1j} B_{2j} - \frac{3h^2}{4} \sum_{j=1}^{4} s_j^3 \omega_{1j} B_{4j} = 0 \tag{40}$$

$$h \sum_{j=1}^{4} s_j \omega_{1j} B_{2j} - \frac{h^3}{4} \sum_{j=1}^{4} s_j^3 \omega_{1j} B_{4j} = -P \tag{41}$$

$$h \sum_{j=1}^{4} s_j \omega_{2j} B_{2j} - \frac{h^3}{4} \sum_{j=1}^{4} s_j^3 \omega_{2j} B_{4j} = +Q \tag{42}$$

$$h \sum_{j=1}^{4} s_j \omega_{3j} B_{2j} - \frac{h^3}{4} \sum_{j=1}^{4} s_j^3 \omega_{3j} B_{4j} = +J \tag{43}$$

$$\sum_{j=1}^{4} s_j k_{1j} B_{2j} + 3L^2 \sum_{j=1}^{4} s_j k_{1j} B_{4j} = 0 \tag{44}$$

$$w_0 + L \sum_{j=1}^{4} s_j k_{1j} B_{2j} + L^3 \sum_{j=1}^{4} s_j k_{1j} B_{4j} = 0 \tag{45}$$

Then, B_{2j}, B_{4j}(j = 1~4) can be determined from Eqs. (37)-(44) (eight equations altogether). Then, substituting B_{2j} and B_{4j} into Eq. (45), we can obtain the unknown constants w_0. At the same time, we can calculate Φ_0 and Ψ_0 through the conditions of $\Phi = 0$ and $\Psi = 0$ at two appointed points, i.e., $\Phi(x',z') = 0, \Psi(x'',z'') = 0$, respectively. Finally, the solution is solved for a cantilever beam, which is under point force P in z direction ($Q = 0$ and $J = 0$) and the stresses are simplified as

$$\sigma_x = \frac{12P}{h^3} xz, \quad \tau_{xz} = \frac{6P}{h^3}\left(\frac{h^2}{4} - z^2\right), \quad \sigma_z = 0 \tag{46}$$

The stresses are independent of material constants.

If we replace $\frac{\partial w}{\partial x} = 0$ in Eqs. (36) with $\frac{\partial u}{\partial z} = 0$, then Eq. (44) will be changed and the other solution to the above question can be obtained. These two kinds of solutions are similar to those of elasticity problem given by Timoshenko and Goodier (1970).

6. Cantilever beam under uniform loads

The boundary conditions of a cantilever beam shown in Fig. 1 under uniform loads are

$$\text{When} \quad z = \pm h/2: \quad \sigma_z = \pm q/2, \quad \tau_{xz} = 0, \quad D_z = 0, \quad B_z = 0 \tag{47}$$

$$\text{When} \quad x = 0: \int_{-h/2}^{+h/2} \sigma_x z dz = 0, \quad \int_{-h/2}^{+h/2} \sigma_x dz = 0, \quad \tau_{xz} = 0, \quad D_x = 0, \quad B_x = 0 \tag{48}$$

$$\text{At point } (x = L, z = 0): \quad u = 0, \quad w = 0, \quad \frac{\partial w}{\partial x} = 0 \tag{49}$$

Using $\varphi_3^1(x, z)$ and $\varphi_5^1(x, z)$ in Eq. (B2) in Appendix B, we constitute the displacement function

$$\psi_j = B_{3j}\left(x^2 z_j - \frac{1}{3} z_j^3\right) + B_{5j}\left(x^4 z_j - 2x^2 z_j^3 + \frac{1}{5} z_j^5\right) \quad (j = 1 \sim 4) \tag{50}$$

where B_{3j} and B_{5j} are unknown constants to be determined.

Substituting Eq. (50) into Eqs. (3) and superposing the rigid body displacements solutions in Eq. (9) leads to

$$u = u_0 + \omega_0 z + 2\sum_{j=1}^{4} B_{3j} x z_j + 4\sum_{j=1}^{4} B_{5j}(x^3 z_j - x z_j^3) \tag{51a}$$

$$w = w_0 - \omega_0 x + \sum_{j=1}^{4} s_j k_{1j} B_{3j}(x^2 - z_j^2) + \sum_{j=1}^{4} s_j k_{1j} B_{5j}(x^4 - 6x^2 z_j^2 + z_j^4) \tag{51b}$$

$$\Phi = \Phi_0 + \sum_{j=1}^{4} s_j k_{2j} B_{3j}(x^2 - z_j^2) + \sum_{j=1}^{4} s_j k_{2j} B_{5j}(x^4 - 6x^2 z_j^2 + z_j^4) \tag{51c}$$

$$\Psi = \Psi_0 + \sum_{j=1}^{4} s_j k_{3j} B_{3j}(x^2 - z_j^2) + \sum_{j=1}^{4} s_j k_{3j} B_{5j}(x^4 - 6x^2 z_j^2 + z_j^4) \tag{51d}$$

$$\sigma_x = -2\sum_{j=1}^{4} \omega_{4j} B_{3j} z_j + \sum_{j=1}^{4} \omega_{4j} B_{5j}(-12x^2 z_j + 4z_j^3) \tag{52a}$$

$$\sigma_z = -2\sum_{j=1}^{4} \omega_{1j} B_{3j} z_j + \sum_{j=1}^{4} \omega_{1j} B_{5j}(-12x^2 z_j + 4z_j^3) \tag{52b}$$

$$D_z = -2\sum_{j=1}^{4} \omega_{2j} B_{3j} z_j + \sum_{j=1}^{4} \omega_{2j} B_{5j}(-12x^2 z_j + 4z_j^3) \tag{52c}$$

$$B_z = -2\sum_{j=1}^{4} \omega_{3j} B_{3j} z_j + \sum_{j=1}^{4} \omega_{3j} B_{5j}(-12x^2 z_j + 4z_j^3) \tag{52d}$$

$$\tau_{xz} = 2\sum_{j=1}^{4} s_j \omega_{1j} B_{3j} x + \sum_{j=1}^{4} s_j \omega_{1j} B_{5j}(4x^3 - 12x z_j^2) \tag{52e}$$

$$D_x = 2\sum_{j=1}^{4} s_j\omega_{2j}B_{3j}x + \sum_{j=1}^{4} s_j\omega_{2j}B_{5j}(4x^3 - 12xz_j^2) \quad (52f)$$

$$B_x = 2\sum_{j=1}^{4} s_j\omega_{3j}B_{3j}x + \sum_{j=1}^{4} s_j\omega_{3j}B_{5j}(4x^3 - 12xz_j^2) \quad (52g)$$

where $u_0, w_0, \omega_0, \Phi_0, \Psi_0, B_{3j}$ and $B_{5j}(j = 1\sim4)$ are unknown constants to be determined.

From Eqs. (47), we arrive at

$$\sum_{j=1}^{4} s_j\omega_{1j}B_{3j} = -\frac{3q}{4h} \quad (53)$$

$$\sum_{j=1}^{4} s_j^3\omega_{1j}B_{5j} = -\frac{q}{2h^3} \quad (54)$$

$$\sum_{j=1}^{4} s_j\omega_{1j}B_{5j} = 0 \quad (55)$$

$$-\sum_{j=1}^{4} s_j\omega_{2j}B_{3j} + \frac{h^2}{2}\sum_{j=1}^{4} s_j^3\omega_{2j}B_{5j} = 0 \quad (56)$$

$$\sum_{j=1}^{4} s_j\omega_{2j}B_{5j} = 0 \quad (57)$$

$$-\sum_{j=1}^{4} s_j\omega_{3j}B_{3j} + \frac{h^2}{2}\sum_{j=1}^{4} s_j^3\omega_{3j}B_{5j} = 0 \quad (58)$$

$$\sum_{j=1}^{4} s_j\omega_{3j}B_{5j} = 0 \quad (59)$$

From $x = 0$: $\int_{-h/2}^{+h/2}\sigma_x z dz = 0$, we have

$$-\frac{2}{3}\sum_{j=1}^{4} s_j\omega_{4j}B_{3j} + \frac{h^2}{5}\sum_{j=1}^{4} s_j^3\omega_{4j}B_{5j} = 0 \quad (60)$$

Then, the unknown constants B_{3j} and B_{5j} ($j = 1\sim4$) can be determined from the eight equations Eqs. (53)-(60). At the same time, we find that the four equations left in boundary condition Eqs. (48) are satisfied automatically. Then, the three unknown constants u_0, ω_0 and w_0 can be determined from Eqs. (49), Φ_0 and Ψ_0 can be obtained through the conditions of $\Phi = 0$ and $\Psi = 0$ at two appointed points, i.e., $\Phi(x', z') = 0, \Psi(x'', z'') = 0$, respectively. Finally, the solution is obtained completely for a cantilever rectangular beam that is uniformly loaded on the surfaces ($z = \pm h/2$) with $\pm(q/2)$, respectively.

Superposing this solution on the solution of a rectangular beam under uniform tension in z direction as discussed in Section 4.2 and letting $\sigma_2 = q/2$ and replacing x in Eqs. (12a,b) with $(x - L)$ result in the solution for a cantilever beam fixed at the end $x = L$, which is loaded with uniform q on

Table 1 Material properties (Pan 2002a)

C_{11}	C_{12}	C_{13}	C_{33}	C_{44}	C_{66}
1.66×10^{11}	7.7×10^{10}	7.8×10^{10}	1.62×10^{11}	4.3×10^{10}	4.45×10^{10}
e_{31}	e_{33}	e_{15}	d_{31}	d_{33}	d_{15}
–4.4	18.6	11.6	580.3	699.7	550
ε_{11}	ε_{33}	g_{11}	g_{11}	μ_{11}	μ_{33}
1.12×10^{-8}	1.26×10^{-8}	5.0×10^{-12}	3.0×10^{-12}	5×10^{-6}	10×10^{-6}

Unit: C–N/m^2, e–C/m^2, d–N/Am, ε–C/Vm, μ–Ns^2/C^2, g–Ns/VC.

Table 2 Cantilever beam under uniformly distributed load

Point	w	Φ	Ψ	σ_z	D_z	B_z	σ_x
A(M)	–0.4310E-6	–0.2362E+1	–0.1403E+0	–0.5000E+2	0.0000	0.0000	+0.1685E+5
A(P)	–0.4412E-6	–0.3975E+1	_____	–0.5000E+2	0.0000	_____	+0.1685E+5
A(E)	–0.4612E-6	_____	_____	–0.5000E+2	_____	_____	+0.1685E+5
B(M)	–0.4309E-6	–0.2043E+1	–0.1479E+0	–0.3437E+2	–0.2023E-8	–0.1266E-6	+0.8448E+4
B(P)	–0.4411E-6	–0.3674E+1	_____	–0.3437E+2	–0.1895E-8	_____	+0.8447E+4
B(E)	–0.4610E-6	_____	_____	–0.3438E+2	_____	_____	+0.8446E+4
C(M)	–0.4308E-6	–0.1936E+1	–0.1505E+0	0.0000	0.0000	0.0000	0.0000
C(P)	–0.4410E-6	–0.3574E+1	_____	0.0000	0.0000	_____	0.0000
C(E)	–0.4610E-6	_____	_____	0.0000	_____	_____	0.0000

the upper surface $z = +h/2$ and free on the bottom one.

Based on Eqs. (51)-(52), all the displacements, stresses, electric and magnetic quantities at any inner or boundary point of the cantilever beam can be obtained. In the calculation, we treat the problem as a plane-stress one and set $L = 0.30$ m, $h = 0.02$ m, $q = -100$ Pa, $\Phi(L, 0) = 0$ and $\Psi(L, 0) = 0$. Assume that a piezoelectric cantilever beam and an elastic cantilever beam have the same constants as those of magneto-electro-elastic beam shown in Table 1, and the same geometric dimensions and boundary conditions. The results of different beam at three reference points, i.e., A(0.150, 0.010), B(0.150,0.005) and C(0.150,0.000) are listed in Table 2 for comparison, where “M”, “P” and “E” denote magneto-electro-elastic, piezoelectric and elastic material beam, respectively. It is obvious that the displacements caused by distributed load on surface are different, whereas the stresses exhibit no noticeable difference.

7. Conclusions

Due to material anisotropy and coupling between mechanical deformation, electric field and magnetic field, analytical solutions for transversely isotropic magneto-electro-elastic materials are much more difficult to obtain and the process of solution is more complicated, compared with those in elasticity theory of isotropic materials. In general, stress components and displacements are dependent on material constants. However, in some solutions, the stresses as shown in Eqs. (10), (17a), (17b), (33b) and (46) are independent of material constants and in agree with those of the theory of elasticity for isotropic materials. The analytical solutions obtained in this paper are also useful for study of other problems relating to more complicated loads and boundary conditions by the superposition principle. Moreover, these solutions can serve as benchmarks for numerical methods such as the finite element method, the boundary element method, etc.

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Appendix A:

$$\bar{c}_{11} = \frac{c_{11}^2 - c_{12}^2}{c_{11}}, \quad \bar{c}_{13} = \frac{(c_{11} - c_{12})c_{13}}{c_{11}}, \quad \bar{c}_{33} = \frac{c_{11}c_{33} - c_{13}^2}{c_{11}}, \quad \bar{c}_{44} = c_{44}$$

$$\bar{e}_{31} = \frac{(c_{11} - c_{12})e_{31}}{c_{11}}, \quad \bar{e}_{33} = \frac{c_{11}e_{33} - c_{13}e_{31}}{c_{11}}, \quad \bar{e}_{15} = e_{15}, \quad \bar{d}_{31} = \frac{(c_{11} - c_{12})d_{31}}{c_{11}}$$

$$\bar{d}_{33} = \frac{c_{11}d_{33} - c_{13}d_{31}}{c_{11}}, \quad \bar{d}_{15} = d_{15}, \quad \bar{\varepsilon}_{11} = \varepsilon_{11}, \quad \bar{\varepsilon}_{33} = \frac{c_{11}\varepsilon_{33} + e_{31}^2}{c_{11}}, \quad \bar{g}_{11} = g_{11}$$

$$\bar{g}_{33} = \frac{c_{11}g_{33} + e_{31}d_{31}}{c_{11}}, \quad \bar{\mu}_{11} = \mu_{11}, \quad \bar{\mu}_{33} = \frac{c_{11}\mu_{33} + d_{31}^2}{c_{11}}$$

Appendix B:

Harmonic polynomials for the plane problems can be written in the following form:

$$\varphi_n^m(x, z) = x^{n-m}z^m + \sum_{i=1}^{\left[\frac{n-m}{2}\right]} (-1)^i \frac{(n-m)(n-m-1)\ldots(n-m-2i+1)}{(2i+m)!} x^{n-2i-m} z^{2i+m} \tag{B1}$$

$$(m = 0, 1; n = 1, 2, \ldots)$$

where $\left[\frac{n-m}{2}\right]$ denotes the largest integer $\leq \frac{n-m}{2}$. From Eq. (B1), the first seventeen harmonic polynomials can be written as follows:

$$\varphi_0^0(x, z) = 1$$

$$\varphi_1^0(x, z) = x, \quad \varphi_1^1(x, z) = z$$

$$\varphi_2^0(x, z) = x^2 - z^2, \quad \varphi_2^1(x, z) = xz$$

$$\varphi_3^0(x, z) = x^3 - 3xz^2, \quad \varphi_3^1(x, z) = x^2 z - \frac{1}{3}z^3$$

$$\varphi_4^0(x, z) = x^4 - 6x^2z^2 + z^4, \quad \varphi_4^1(x, z) = x^3 z - xz^3$$

$$\varphi_5^0(x, z) = x^5 - 10x^3z^2 + 5xz^4, \quad \varphi_5^1(x, z) = x^4 z - 2x^2z^3 + \frac{1}{5}z^5$$

$$\varphi_6^0(x, z) = x^6 - 15x^4z^2 + 15x^2z^4 - z^6, \quad \varphi_6^1(x, z) = x^5 z - \frac{10}{3}x^3z^3 + xz^5$$

$$\varphi_7^0(x, z) = x^7 - 21x^5z^2 + 35x^3z^4 - 7xz^6, \quad \varphi_7^1(x, z) = x^6 z - 5x^4z^3 + 3x^2z^5 - \frac{1}{7}z^7$$

$$\varphi_8^0(x, z) = x^8 - 28x^6z^2 + 70x^4z^4 - 28x^2z^6 + z^8$$

$$\varphi_8^1(x, z) = x^7 z - 7x^5z^3 + 7x^3z^5 - xz^7 \tag{B2}$$