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Dynamic and reliability analysis of stochastic structure system using probabilistic finite element method

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Abstract. Industrial structure systems may have nonlinearity, and are also sometimes exposed to the danger of random excitation. This paper proposes a method to analyze response and reliability design of a complex nonlinear structure system under random excitation. The nonlinear structure system which is subjected to random process is modeled by finite element method. The nonlinear equations are expanded sequentially using the perturbation theory. Then, the perturbed equations are solved in probabilistic methods. Several statistical properties of random process that are of interest in random vibration applications are reviewed in accordance with the nonlinear stochastic problem.

Key words: response analysis; modal analysis; multi-DOF system; mdeling of complex system; finite element method; structural vibration.

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1. Introduction

Recent developments in jet and rocket propulsion have given rise to new problems in mechanical and structural vibrations. The pressure fields generated by these devices fluctuated in a random manner and contain a wide spectrum of frequencies that may result in severe vibration in the aircraft or missile structure. As more data are gathered in strong motion earthquakes, it is becoming apparent that earthquakes are examples of random process that may excite severe vibration and even failure in structure. Measurements of the motion of ships in a confused sea or aircraft flying through turbulent air reveal that such motions can be described only statistically. The examples given above have two things in common: (a) they involve the response of mechanical system to random excitation; (b) in general, they involve nonlinear behavior, since almost all real physical systems exhibit non-linearity for sufficiently large motion. The theory of linear systems subjected to random

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excitation is well developed. In the case of nonlinear systems, however, the standard techniques of linear analysis cannot be applied. In many areas of random mechanics, sometimes we need to analytically obtain the exact probability densities of response processes for non-linear stochastic systems (Lin 1995, Zhu 1996, Wang 1998, 2000). However, in a general case, no exact solution can be obtained and numerical methods must be used. Especially for a complex and large structural system the numerical simulation takes much computation efforts to obtain the response when they are modeled in multi-degrees of freedom.

Therefore, this study proposes an analytical method for nonlinear vibration of mechanical system against a random excitation by applying the statistical method and numerical method using finite element analysis. The actual random excitation is approximated to the corresponding stationary random process. Then, several statistical properties including reliability analysis that are of interest in nonlinear random vibration applications are reviewed.

2. Modeling of nonlinear stochastic system using finite element method

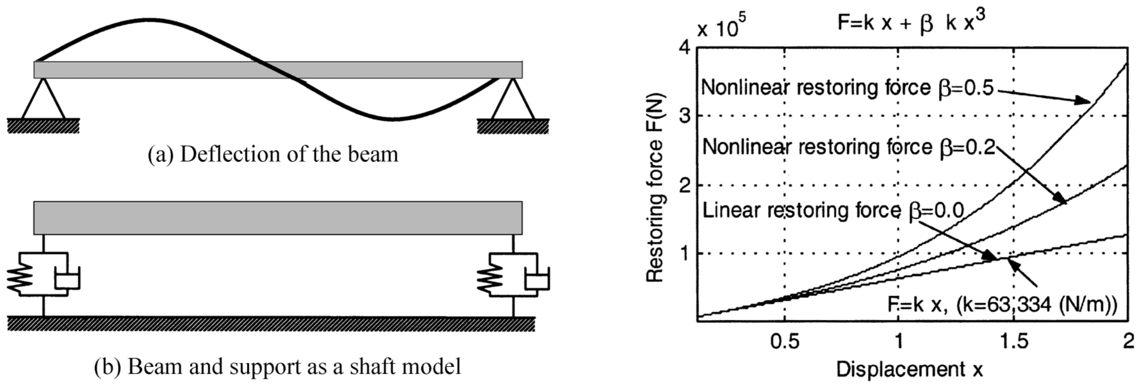
Equations of motion are derived and applied to discrete nonlinear dynamic systems subjected to random excitation. As a nonlinear system, a nonlinear mechanical system is considered, as shown in Fig. 1(a), (b). The beam has nonlinear restoring force with respect to its material property.

The nonlinear restoring forces increase with their nonlinear stiffness coefficient β, as shown in Fig. 1(c). Nevertheless, the nonlinear displacement decreases with the same restoring force. Thus it is important to show the nonlinear restoring force in the response of the system. To this end, the nonlinear restoring force parameters are set to β = 0.0, 0.2 and 0.5.

Mechanical systems, which are excited by strong-motion, show random processes that may excite severe vibration and even failure in systems. For a simple explanation to show the methodology of the new approach, a single DOF vibratory system with the nonlinear restoring force is considered.

$$\ddot{x} + 2\varsigma\omega_0\dot{x} + \omega_0^2 x + \varepsilon\omega_0^2 x^3 = \ddot{X}_0 \tag{1}$$

where ς, ω_0, ε are damping ratio, natural frequency and a small parameter, respectively. $\ddot{X}_0$ is a



(a) Deflection of the beam

(b) Beam and support as a shaft model

(c) Nonlinear restoring force characteristic

Fig. 1 Model of nonlinear system

random process; hence, it is extremely difficult to obtain an anlytical solution $x(t)$. Because of the nonlinear characteristic, the output is no longer a Gaussian random process; hence, the statistic characteristic of its vibration cannot be evaluated easily. Therefore, an adequate method to evaluate the statistical properties of the response should be developed. For this reason, the random excitation needs to be approximated to Gaussian stationary process by reasonable procedures. For instance, a simple stationary representation of ground acceleration can be expressed based on the study of frequency content of a number of strong ground-motion records (Shin 1982). Input PSD (power spectrum density) can be expressed as

$$S_N(\Omega) = \frac{\omega_g^4 + 4_\varsigma^2\omega_g^2\Omega^2}{2(\omega_g^2 - \Omega^2)^2 + 4\varsigma_g^2\omega_g^2\Omega^2} S_0 \tag{2}$$

where ω_g, ς_g and S_0 are a dominant frequency, damping ratio of filter and spectrum intensity of random process, respectively. In the case of regarding random excitation as a stationary process, the dynamic responses of nonlinear system can be obtained by using the perturbation theory.

$$\ddot{x} + 2\varsigma\omega_0\dot{x} + \omega_0^2 x + \varepsilon\omega_0^2 x^3 = W(t), \quad x = x_0 + \varepsilon x_1 \tag{3}$$

$W(t)$ is a strong motion part of random excitation $\ddot{X}_0$, which has the PSD function $S_N(\Omega)$. There is a method to solve Eq. (3) using perturbation method with white noise as an excitation, which is proposed by Crandall (1963). He used the power spectrum intensity (S_0) of white noise as a random excitation. However, if the excitation is not white noise as usual, this method cannot be applied to actual random excitation. Thus, this study proposes a new method using the PSD function and perturbation method, which can be applied to analysis of nonlinear problem. The nonlinear response of Eq. (3) is obtained by direct integration method with earthquake data, as shown in Fig. 2(a). And those PSD of response is compared to show the effectiveness of the proposed method, as shown in Fig. 2(b). It is shown that the PSD of proposed method calculated relatively in good agreement with the integration method.

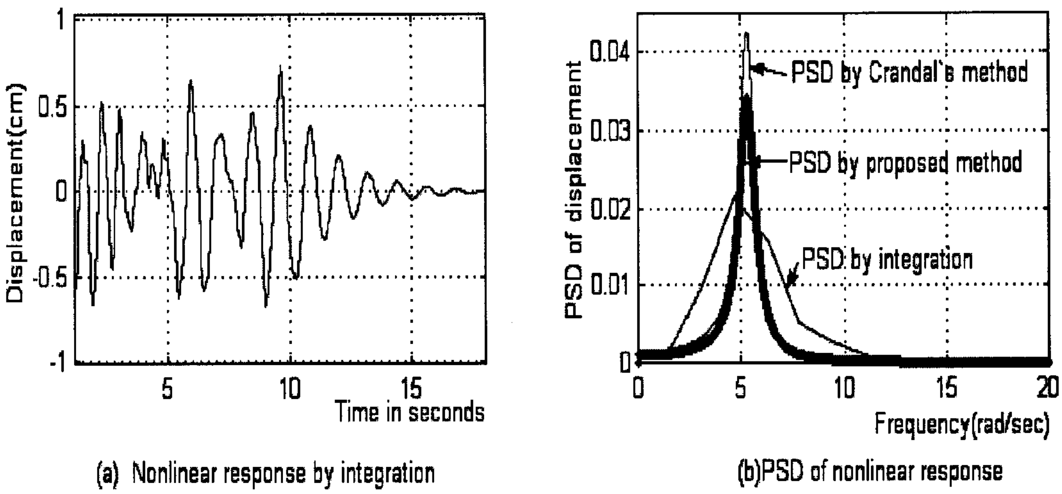


Fig. 2 Comparison of nonlinear responses with the direct integration method

2.1 Application of finite element method to nonlinear stochastic system

Response and its reliability analysis for failure problem are carried out analytically after modeling the MDOF nonlinear mechanical system. The mass and stiffness matrix of a uniform beam element such as the one shown in Fig. 3, which is of length l and mass per unit length ρ and has a bending stiffness of EI, are formulated.

The lateral displacement $w(x, t)$, which is a function of displacement in bending vibration time t, of any section a distance x along the beam element can be expressed as

$$y(x,t) = \phi_1(x)\cdot u_1 + \phi_2(x)\cdot u_2 + \phi_3(x)\cdot u_3 + \phi_4(x)\cdot u_4 \tag{4}$$

in which $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$ and $\phi_4(x)$ are appropriate shape function, and u_1, u_2, u_3 and u_4 are functions of time t defining the displacements at the ends of the beam element (joint displacements). The kinetic energy T of the beam element can be obtained and then the time derivative of that expression will result in mass matrix

$$[m]_{ij} = \rho A\int_0^l \phi_i(x)\phi_j(x)dx, \qquad (i, j = 1, 2, 3, 4) \tag{5}$$

The stiffness matrix for the beam element is determined by considering the strain energy of the beam. The strain energy can be expressed as

$$U = \frac{1}{2}EI\int_0^l \left\{\frac{\partial^2 y(x,t)}{\partial x^2}\right\}^2 dx + \frac{1}{2}\beta EI\int_0^l \left\{\frac{\partial^2 y(x,t)}{\partial x^2}\right\}^4 dx \tag{6}$$

where β is nonlinear term. Then, taking the partial derivative of U, the linear stiffness elements k_{ij} can be obtained.

$$\frac{\partial U}{\partial u_j} = EI\int_0^l [\,\phi_1''(x)\cdot u_1 + \phi_2''(x)\cdot u_2 + \phi_3''(x)\cdot u_3 + \phi_4''(x)\cdot u_4]\cdot\phi_j(x)dx \tag{7}$$

The second term of the strain energy as stated in Eq. (6), which is equivalently regarded as the nonlinear restoring force term, can be expressed as

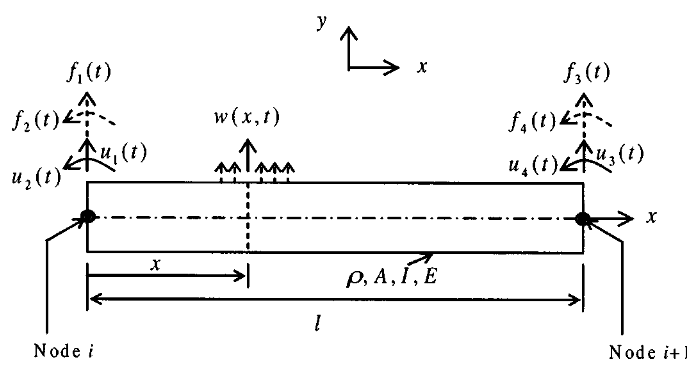


Fig. 3 Model of nonlinear system

$$\frac{\partial U_{nl}}{\partial u_j} = 2\beta EI\int_0^l [\phi_1''(x)\cdot u_1 + \phi_2''(x)\cdot u_2 + \phi_3''(x)\cdot u_3 + \phi_4''(x)\cdot u_4]^3 \cdot \phi_j''(x)dx \tag{8}$$

This equation is apparent that all the possible terms for a term-by-term integration have the form of matrix. It can be seen that the bracketed term above corresponding to the nonlinear stiffness elements k_{nij} and can be determined using $\phi_i''(x)$.

$$[k_n(x)] = 2\beta EI\int_0^l [\phi_1''(x)\cdot u_1 + \phi_2''(x)\cdot u_2 + \phi_3''(x)\cdot u_3 + \phi_4''(x)\cdot u_4]^3 \cdot \phi_j''(x)dx \tag{9}$$

The nonlinear restoring force term, which is expressed in nonlinear stiffness matrix term, is complex coupled term. This term cannot be assembled easily with the other element. Thereby, in this study, as a first step of analysis, nonlinear restoring force term is approximated by taking the term of u^3 component. Hence, the nonlinear stiffness matrix is expressed in compact form as

$$[k_n(u)]\{u^3\} = 2\beta EI\begin{bmatrix} \frac{1296}{l^7}\frac{1}{5} & \frac{48}{l^4}\frac{21}{20} & -\frac{1296}{l^7}\frac{1}{5} & \frac{48}{l^4}\frac{21}{20} \\ \frac{432}{l^6}\frac{3}{10} & \frac{16}{l^3}\frac{11}{5} & -\frac{432}{l^6}\frac{3}{10} & \frac{16}{l^3}\frac{19}{20} \\ -\frac{1296}{l^7}\frac{1}{5} & -\frac{48}{l^4}\frac{21}{20} & \frac{1296}{l^7}\frac{1}{5} & -\frac{48}{l^4}\frac{21}{20} \\ \frac{432}{l^6}\frac{3}{10} & \frac{16}{l^3}\frac{19}{20} & -\frac{432}{l^6}\frac{3}{10} & \frac{16}{l^3}\frac{11}{5} \end{bmatrix}\{u^3\} = \beta[k_N^e]\{u^3\} \tag{10}$$

Then, the number of finite element and their DOF are described according to nodal point, as illustrated in Fig. 4. The equation of motion of entire beam is obtained according to just a routine procedure by superposing each element of mass matrix, linear stiffness matrix, nonlinear stiffness matrix and forces. When the system is excited by random force, the equation of motion is Moon (2001)

$$[M]\{\ddot{x}\} + [K]\{x\} + \beta[K_n]\{x^3\} = [F] + \{F_E\}, \tag{11}$$

where $[M], [K], [K_n]$ are mass matrix, stiffness matrix and nonlinear stiffness matrix of the entire beam, respectively. $\{F\}$, $\{F_E\}$ are a force vector, a random force, respectively.

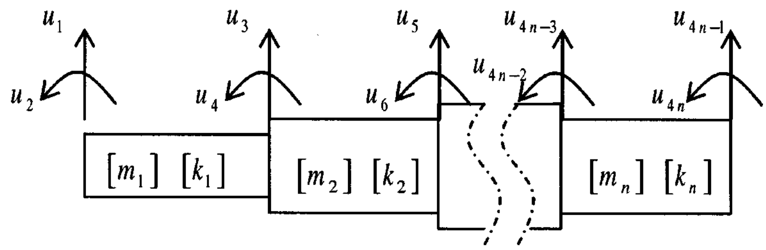


Fig. 4 Comparison of nonlinear responses with the direct integration method

$\{F_E\} = -[M]\{I\}\{\ddot{X}_0\}$, where $\{I\}$ is a vector which shows the direction of the force. The response of nonlinear random vibration is solved statistically. In order to apply modal analysis, modal coordinate system is introduced by using the modal matrix $[\Phi]$ of the linear system. Then, the displacement $\{x\}$ can be transformed into the modal coordinate system $\{\xi\}$ (Moon 1999, 2001).

$$\{\ddot{\xi}\} + [\diagdown\omega^2\diagdown]\{\xi\} + \varepsilon[\diagdown k_N\diagdown]\{\xi^3\} = \{f(t)\} + \{f_E\}, \quad \{\xi\} = \{\xi^{(0)}\} + \varepsilon\{\xi^{(1)}\} \tag{12}$$

where $\{f(t)\}$ and $\{f_E\}$ are a force, a random force in modal coordinates. The small variant ε can be regarded as the perturbation parameter, because the variant $\varepsilon[\diagdown k_N\diagdown]$ is small relative to $[\ \omega^2\ ]$. The superscripts (0), (1) denote the perturbation order. Then, the perturbed equations are evaluated. The equation of motion of order $\varepsilon^{(0)}$ is

$$\ddot{\xi}_i^{(0)} + 2\zeta\omega_i\dot{\xi}_i^{(0)} + \omega_i^2\xi_i^{(0)} = W_i^{(0)}, \quad (i = 1, 2, 3, \ldots, n) \tag{13}$$

$W_i^{(0)}$ is the external force term in modal coordinates. The damping of the system is assumed to be the proportional damping of the eigenvalue. According to the linear random vibration theory, the solution of the linear differential equation may be readily obtained. Then, the equation of motion of order $\varepsilon^{(1)}$ can be described as

$$\ddot{\xi}_i^{(1)} + 2\zeta_i\omega_i\dot{\xi}_i^{(1)} + \omega_i^2\xi_i^{(1)} = f_i^{(1)}(\xi_i^{(0)}), \quad (i = 1, 2, \ldots, n) \tag{14}$$

where $f_i^{(1)}(\xi_i^{(0)}) = -\eta_i^2\xi_i^{(0)3}$, η_i^2 is the nonlinear coefficient. Then the response is

$$\xi_i^{(1)}(t) = -\eta_i^2\int_0^\infty \xi_i^{(0)3}(t-\tau)h_i(\tau)d\tau. \tag{15}$$

$h_i(\tau)$ is the impulse response function of the linear system. The covariance of the nonlinear response can be obtained as

$$R_{\eta i}(\tau) = \int_0^\infty \left\{\frac{1}{2}S_{Ni}^{(0)}(\Omega)|H_i(\Omega)|^2 - \frac{3}{2}\varepsilon\sigma_{\eta i}^{(0)2}S_{Ni}^{(0)}(\Omega)|H_i(\Omega)|^2 H_i^*(\Omega)\cos\Omega\tau\right\}d\Omega \tag{16}$$

where $H_i^*(\Omega)$ is conjugate function of $H_i(\Omega)$. $S_{Ni}^{(0)}(\Omega)$ is the spectral density of the excitation. Then, the spectral density $S_{\eta i}(\Omega)$ of the nonlinear response is obtained by taking the Fourier transform of the covariance function as

$$S_{\eta i}(\Omega) = S_{Ni}^{(0)}(\Omega)|H_i(\Omega)|^2[1 - 6\varepsilon\beta_i^2\sigma_{\eta i}^{(0)2}\mathrm{Re}[H_i(\Omega)]] \tag{17}$$

where $\mathrm{Re}[H_i(\Omega)]$ is the real part of $H_i(\Omega)$. The corresponding variance can be obtained from the covariance $R_{\eta i}(\tau)$ of the system by letting $\tau = 0$.

$$\sigma_{\eta i}^2 = \sigma_{\eta i}^{(0)2}[1 - 6\varepsilon\beta_i^2\int_0^\infty\{R_{\eta i}(\tau)h_i(\tau)\}d\tau]. \tag{18}$$

The stationary variance $\sigma_{\eta i}^{(0)2}$ is the mean-square value of the linear response. Then, by changing the modal coordinates into physical coordinates, the response can be obtained.

2.2 Failure probability and reliability analysis of the system

Generally, the evaluation of random response of machinery is concerned with the maintenance of the operating ability against the random excitation. As a case, the possibility of failure is obtained by assuming that a failure of the system occurs when the response crosses over the safe bound. Thereby, failure possibility through the crossing theory is considered. The mean r-upcrossing rate is obtained when r is the constrained amplitude of system.

$$\upsilon_{\eta}^{+}(r) = \frac{\omega_{t0}\sigma_{\eta_0}^{(0)}\sqrt{\frac{\varepsilon}{\pi}}}{K_{1/4}\left(\frac{1}{8\varepsilon\sigma_{\eta_0}^{(0)2}}\right)}\exp\left(-\frac{1}{8\varepsilon\sigma_{\eta_0}^{(0)2}}\right)\exp\left(-\frac{1}{2\sigma_{\eta_0}^{(0)2}}\left(r^2+\frac{\varepsilon}{2}r^4\right)\right) \tag{19}$$

where w_{t0} is the first natural frequency of the overall system. $\sigma_{\eta_0}^{(0)2}$ represents the stationary variance of first modal displacement when $\varepsilon = 0$. $\sigma_{\eta_0}^{(0)}$ is the root mean square of the first modal response. $K_{1/4}$ is the modified Bessel function of order 1/4. Thereby, failure possibility of the system can be obtained by supposing that the failure of the system occurs when the system exceeds the limit amplitude. To consider this, the limit of response (***r***) at system is regarded, which is constrained to prevent the damage of the system. This condition can be used to define a safe set. In this case, performance of the system depends primarily on crossing characteristics of safe set of response. Thus, the concept for mean crossing rate of safe set is applied to demonstrate the reliability of the system. The reliability coincides with the mean upcrossing rate that the response belongs to the safe set. Also, the probability of failure can be obtained by the complement of obtained reliability, which are defined as

$$P_F[r, t_s] = 1 - \exp[-\upsilon_{\eta}^{+}(r)t_s], \quad P_R(r, t_s) = 1 - P_F(r, t_s) \tag{20}$$

3. Results of the analysis

As an example, nonlinear rotor system, which is shown in Fig. 5, is considered. The rotor is considered as a uniform beam for the simplicity of calculation. As a support, ball bearing is considered for the turbine of aircraft engine. The rotor is modeled by the twenty beam elements. The modal damping ratios of the system is given by $\zeta = 0.05$. Length of shaft is 800 (mm), Diameter of shaft is 50 (mm). Young’s modulus of shaft is 2.1×10^{11} (N/m^2). Density of shaft is 7.81×10^3 (kg/m^3). Bearing coefficients is 1.0×10^6 k_b (N/m). The rotor has 84 DOF because it has 21 nodes and there are 4 variables per node. For the response analysis, the part of earthquake $W(t)$ is used as 3-18 seconds of Taft (1952) earthquake ($\varsigma_g = 0.41$, $\omega_g = 18.75$ rad/sec, $\alpha = 1.75$ m/sec^2, $S_0 = 0.0132$ m^2/sec^4/Hz). And the properties of nonlinear responses are calculated according to the procedure of nonlinear random vibration analysis, which is computed to the first order perturbation.

In Fig. 6, the PSD of the nonlinear responses of the system at the different position of beam are presented with various nonlinear parameters. The response is calculated by taking 5 modes. Each PSD of the system shows a characteristic of seismic response. Investigation of the PSD reveals that the PSD is smaller when the nonlinear parameter becomes large, which shows the nonlinear characteristic of response caused by hard spring type restoring force.

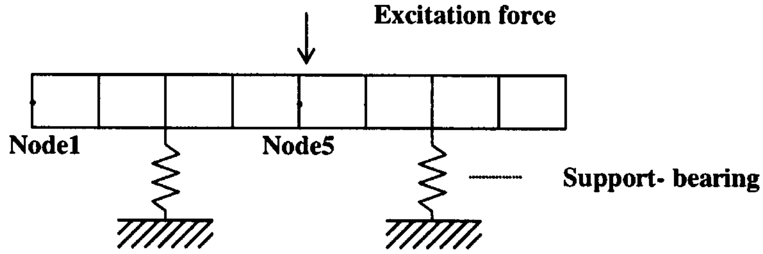


Fig. 5 Model of calculation

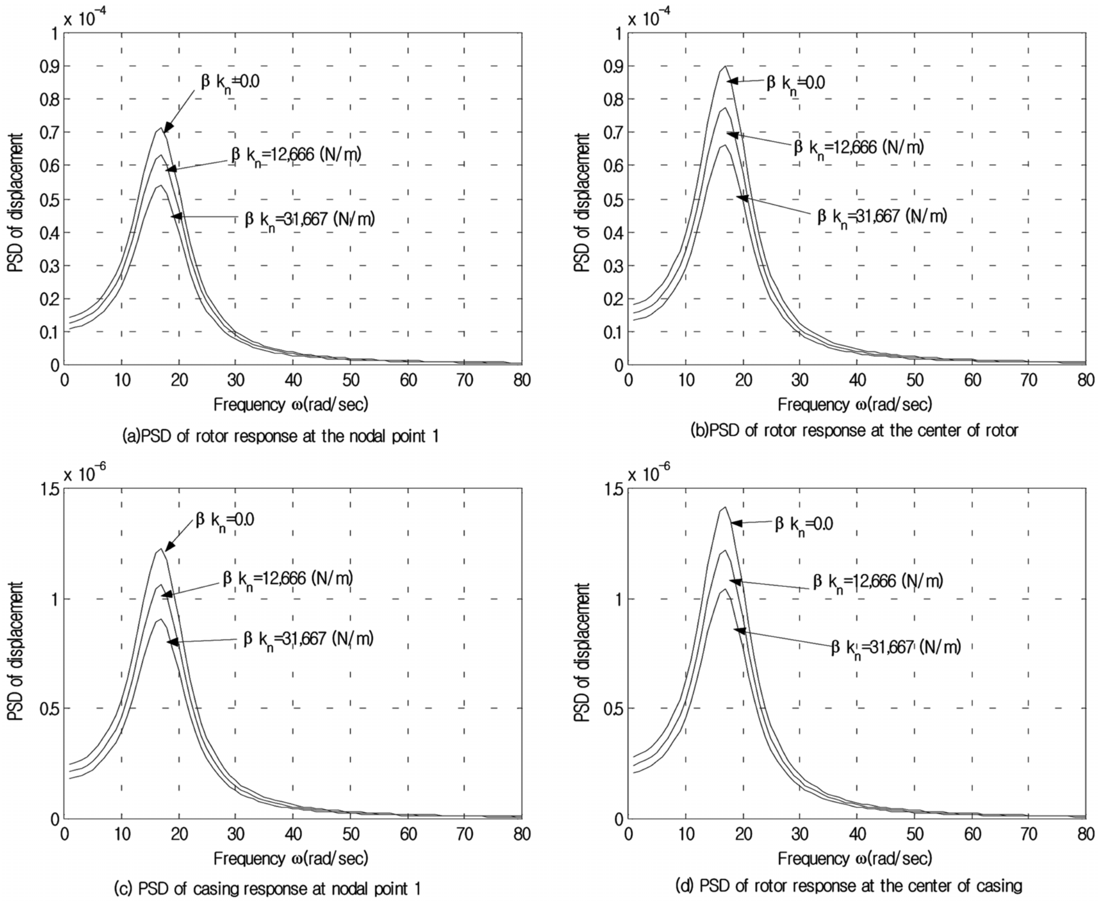


Fig. 6 Comparison of PSD of the response

Table 1 shows the variance of the nonlinear response at the middle of beam by changing its numbers of adopting modes 3, 5 and 10. The values of the nonlinear response are investigated according to the analytical methods for various values of nonlinear parameter (β = 0.0, 0.2. 0.5). To prove the computing efficiency, those values are compared with the results of direct integration method and Equivalent Linearized Method, which are obtained by same calculation condition with the proposed method. The response of integration is obtained by the direct integration of the

Table 1 Comparison of computing efficiency of the methods

Analysis method	Variance/Dev β = 0.0	Variance/Dev β = 0.2	Variance/Dev β = 0.5	Calculation Time (seconds)
3 mode	0.00531/5.7	0.00472/4.6	0.00272/5.4	440
5 mode	0.00522/3.9	0.00467/3.5	0.00269/4.2	512
10 mode	0.00511/1.9	0.00461/2.2	0.00265/2.7	687
ELM	0.00520/3.5	0.00471/4.4	0.00267/3.4	653
DIM	0.00502	0.00451	0.00258	3370

Each analysis method stands for as 3 modes, 5 modes and 10 modes. ELM (Equivalent Linearized Method), DIM (direct integration method). Dev stands for the deviation of the calculation accuracy per percentage. $\mathrm{Re}[H_i(\Omega)]$ (%).

equation of motion according to the Runge-Kutta method. Investigation of the variance reveals that the value shows a decreases with β in the spread of displacement about equilibrium point when $\beta = 0$. This is consistent with our intuition, which suggests that stiffer systems exhibit smaller displacements, and with the observation that the system stiffness increases with nonlinear parameter. This result also reveals that the variance of displacement of a hardening spring type nonlinear system is always less than that for the corresponding linear system.

Calculation accuracy of the proposed method is evaluated to show the effectiveness of the proposed method. The deviations of the result to the direct integration method are 4.6%, 3.5% and 2.2% by the proposed method where the number of adopting modes 3, 5 and 10, respectively in the case of (β = 0.2). The deviations of the result to the direct integration method are 3.5%, 4.4%, 3.4% in the case of the Equivalent Linearized Method when (β = 0.0, 0.2, 0.5). The responses within 6% deviation error are obtained by the proposed analytical method. It is believed that the accuracy of the response within 6% deviation error for the vibration analysis of the system is effective analytical method. As a case, the calculation time for the variances of response in Table 1 is examined to verify the effectiveness of the proposed method. The response of the direct integration method is obtained under the strong motion duration (= 3-18 seconds) of excitation. The proposed method takes 440, 512, 687 seconds to calculate the frequency response within 5% deviation error where the number of adopting modes are 3, 5, 10, respectively, when (β = 0.2), while the direct integration method takes 3370 seconds to compute the same response by using the personal computer Logix IBM Co.. The Equivalent Linearized method takes 653 seconds to calculate the frequency response within 4.4% deviation error. As a result, it can be observed in this study that a drastic reduction in computational time can be obtained while retaining the accuracy of the solution.

3.1 Reliability and failure probability of the system

Generally, the evaluation of random response of machinery is concerned with the maintenance of the operating ability against the excitation. This can be verified by the possibility of failure by the contact between the components. Here, the mean upcrossing rates of stationary nonlinear responses for several values of nonlinear parameter are considered according to the crossing theory. The reliability coincides with the mean upcrossing rate that the response belongs to the safe set. Also, the probability of failure can be obtained by the complement of obtained reliability. In Fig. 7(a), (b), the failure possibility and its reliability by mean ***r***- upcrossing rate of response according to

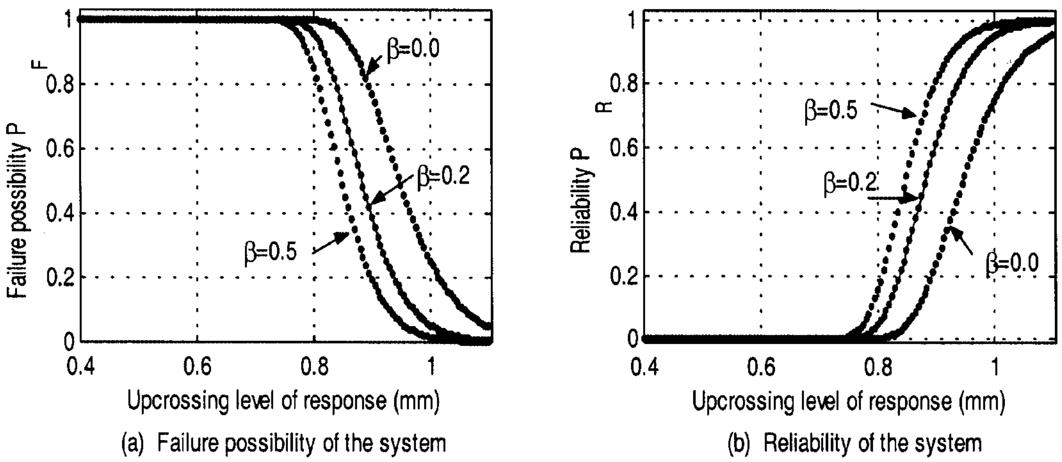


Fig. 7 Failure probability and its reliability according to nonlinear parameter

nonlinear parameter are presented. The failure possibility decreases with threshold ***r***. And they decrease more rapidly when nonlinear parameters become large.

As a result, it is shown that nonlinear random responses could be efficiently calculated according to the selected number of vibration modes. Several statistical properties of the responses that are of interest in nonlinear vibration applications are reviewed. The results reported herein will provide a better understanding of the nonlinear vibration against random excitation. Moreover, it is believed that those properties of the results can be utilized in the dynamic design of the nonlinear system.

4. Conclusions

In this study, the vibration analysis method of a nonlinear mechanical system was theoretically formulated when the random excitation is regarded as a stationary process. The formulation is concerned with reducing the number of DOF by modal substitution. It is shown that nonlinear random responses could be efficiently calculated according to the selected number of vibration modes. Several statistical properties of the responses that are of interest in nonlinear vibration applications are reviewed. The variance value of the nonlinear response which is important in evaluating the reliability of the system is obtained economically. Failure probability of system is also reviewed applying the crossing theory.

The results reported herein will provide a better understanding of the nonlinear vibration against random excitation. Moreover, it is believed that those properties of the results can be utilized in the dynamic design of the nonlinear system.

Acknowledgements

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