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A numerical method for the limit analysis of masonry structures

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Abstract. The paper presents a numerical method for the limit analysis of structures made of a rigid no-tension material. Firstly, we formulate the constrained minimum problem resulting from the application of the kinematic theorem, which characterizes the collapse multiplier as the minimum of all kinematically admissible multipliers. Subsequently, by using the finite element method, we derive the corresponding discrete minimum problem in which the objective function is linear and the inequality constraints are linear as well as quadratic. The method is then applied to some examples for which the collapse multiplier and a collapse mechanism are explicitly known. Lastly, the solution to the minimum problem calculated via numerical codes for quadratic programming problems, is compared to the exact solution.

Key words: limit analysis; masonry structures; finite elements.

1. Introduction

In Kooharian (1952) and Heyman (1966) the methods of the limit analysis usually employed for structures made of an ideally plastic material were extended to the study of masonry arches. This extension is based on the hypotheses that the masonry does not withstand tension and has infinite compressive strength.

Subsequently, many authors have addressed the problem of modelling the mechanical response of masonries (Del Piero 1989). One of the main results of this research has been the constitutive equation for *masonry-like* or *no-tension* materials. This equation is based on the hypotheses that the infinitesimal strain is the sum of an *elastic part* and of a positive semidefinite *inelastic part* (also called *fracture strain*), and that the stress is negative semidefinite, depends linearly on the elastic part and is orthogonal to the inelastic strain. As it is extremely difficult to determine explicit solutions to equilibrium problems involving solids made of such materials, methods that would enable determining the value of the collapse load (if any) for an assigned class of loads would be of great value in applications.

Recently, Del Piero (1998) has shown that the classical static and kinematic theorems of limit analysis, already proved for ideally plastic materials (Drucker *et al.* 1952), also hold for a broader

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class of materials, including masonry-like materials. Definition of the collapse and subsequent proof of the limit analysis theorems provided in Del Piero (1998) are based on two hypotheses, which are specific not only to ideally plastic materials, but to masonry-like materials as well. In fact, both these material types share two distinguish features: they both have a constraint on the stresses, in particular the set of admissible stresses is convex, as well as an orthogonality property linking these admissible stresses to the fracture strains.

In Lucchesi *et al.* (1997) and (1999) the collapse load and corresponding collapse mechanism have been determined for two different cases: the first deal with circular arches subjected to their own weight and a point load applied somewhere along the extrados, while the second analyses a toroidal tunnel subjected to its own weight and a vertical load acting on its top circle.

The approach followed in such works cannot however be applied when the problems involved are more complex in terms of either geometry or loads distribution. In these more difficult cases it may thus be useful to resort to a finite-element formulation in order to solve the problem of limit analysis of masonry structures numerically, as performed in Capsoni and Corradi (1997) for ideally plastic structures.

In this paper we set forth the minimum problem resulting from the application of the kinematic theorem to a structure made of a rigid no-tension material. The collapse multiplier, which is the minimum of all kinematically admissible multipliers, is determined by solving a constrained minimum problem in which we must minimize a linear functional defined on the set of virtual displacements, under the constraint that the infinitesimal strain tensors associated to these displacements, are positive semidefinite. We then consider a discretization of the structure into finite elements and derive the corresponding discrete minimum problem. In the case of rigid ideally plastic materials (Capsoni and Corradi 1997), the kinematic theorem is formulated as a convex minimum problem with equality constraints, whose solution is calculated via the Lagrange multiplier method. On the contrary, in the case of rigid no-tension materials we obtain a minimum problem in which the objective function is linear and the constrains, both linear and quadratic, are inequalities. As both the objective function and the admissible region are convex, any local minimum is also a global minimum (Cea 1978).

The method has been applied to some examples for which the collapse multiplier and a collapse mechanism are explicitly known. The solution to the minimum problem has been calculated via numerical codes for non-linear programming problems and then compared to the exact solution, as well as to the solution obtained via an incremental static analysis performed with the finite element code NOSA (Lucchesi *et al.* 2000).

2. The constitutive equation

Let $\mathcal{V}_k$ be a k-dimensional linear space, and Lin_k the space of all linear applications from $\mathcal{V}_k$ into $\mathcal{V}_k$ (second-order tensors) with the inner product $\mathbf{A} \cdot \mathbf{B} = tr(\mathbf{A}\mathbf{B}^T)$, where $\mathbf{A}$, $\mathbf{B} \in Lin_k$ and $\mathbf{B}^T$ is the transpose of $\mathbf{B}$. Let Sym_k be the subspace of Lin_k of all symmetric tensors; we denote by Sym_k^+ and Sym_k^- the convex cones of Sym_k of all positive and negative semidefinite tensors, respectively. Let $\mathbf{T} \in Sym_3$ be the Cauchy stress tensor and $\mathbf{E} \in Sym_3$ the infinitesimal strain tensor, $\mathbf{E} = 1/2\ (\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ where $\mathbf{u}$ is the displacement vector.

A rigid no-tension material is a material in which the Cauchy stress $\mathbf{T}$ is constrained to be negative semidefinite,

$$\mathbf{T} \in Sym_3^- \tag{1}$$

Moreover, the infinitesimal strain $\mathbf{E}$ satisfies the following condition of normality

$$(\mathbf{T} - \mathbf{T}^*) \cdot \mathbf{E} \geq 0, \quad \forall \mathbf{T}^* \in Sym_3^- \tag{2}$$

in other words, $\mathbf{E}$ is zero, if $\mathbf{T}$ belongs to the interior of Sym_3^-, while it must belong to the normal cone to Sym_3^- at $\mathbf{T}$, if $\mathbf{T}$ lies on the boundary of Sym_3^-.

It is simple matter to prove that (1)-(2) are equivalent to the conditions

$$\begin{aligned} &\mathbf{T} \in Sym_3^- \\ &\mathbf{E} \in Sym_3^+ \\ &\mathbf{T} \cdot \mathbf{E} = 0 \end{aligned} \tag{3}$$

If $\mathbf{E}$ and $\mathbf{T}$ satisfy (3), then they are coaxial. Tensor $\mathbf{E}$ can be interpreted as fracture strain. In fact, if $\mathbf{E}$ is non-null in any region of a structure made of a no-tension material, then we can expect fractures to be present in that region.

Now, let us consider a body Ω (a closed region of the Euclidean space with piecewise C^1 boundary $\partial\Omega$) made of a material having constitutive Eq. (3). Moreover, let us assume that $\partial\Omega = \partial\Omega_u \cup \partial\Omega_f$, with $\partial\Omega_u \cap \partial\Omega_f = \emptyset$ and that volume-force field $\mathbf{b}$ and surface-force field $\mathbf{f}$ are assigned on Ω and $\partial\Omega_f$, respectively. Finally, we require that

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \partial\Omega_u \tag{4}$$

Here we briefly recall the definitions and results described in Del Piero (1998), where the problem of the choice of functional spaces for stress and displacement fields has been discussed in detail. Let $\mathcal{B}$ denote a suitable normed vector space consisting of tensor fields $\mathbf{A}$: $\Omega \rightarrow Sym_3$ and let $\mathcal{K}$ be the subset of $\mathcal{B}$ constituted by negative semidefinite tensor fields

$$\mathcal{K} = \{\mathbf{A} \in \mathcal{B} | A(x) \in Sym_3^- \ \forall x \in \Omega\} \tag{5}$$

it should be noted that $\mathcal{K}$ is a convex cone of $\mathcal{B}$, whose elements are called *admissible stress fields.*

Let $\mathcal{M}$ be the dual space of $\mathcal{B}$, for each $\mathbf{T}$ belonging to the boundary $\partial\mathcal{K}$ of $\mathcal{K}$, let us consider the *normal cone* to $\partial\mathcal{K}$ at $\mathbf{T}$

$$\mathcal{N}(\mathbf{T}) = \{\mathbf{B} \in \mathcal{M} | \int_\Omega (\mathbf{T} - \tilde{\mathbf{T}}) \cdot \mathbf{B} dV \geq 0, \forall \tilde{\mathbf{T}} \in \mathcal{K}\} \tag{6}$$

If the fields are sufficiently smooth, then $\mathbf{E}$ satisfies the constitutive Eq. (3) in each point of Ω if and only if $\mathbf{E}$ belongs to $\mathcal{N}(\mathbf{T})$.

For $\mathcal{S}$ the set of *virtual* displacement fields, namely, the infinitesimal displacement fields $\mathbf{u}$ that satisfy (4), let $D(\mathcal{S})$ be the set of the infinitesimal strains associated to displacements belonging to $\mathcal{S}$, via the operator

$$D(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T) \tag{7}$$

An element $\mathbf{T}_c$ belonging to $\partial\mathcal{K}$ is a *collapse stress* field if

$$D(\mathcal{S}) \cap \mathcal{N}(\mathbf{T}_c) \neq \{0\} \tag{8}$$

The non-null elements $\mathbf{E}_c$ belonging to $D(\mathcal{S}) \cap \mathcal{N}(\mathbf{T}_c)$ are *collapse strains*, and the displacement field $\mathbf{u}_c$ belonging to $\mathcal{S}$, such that $\mathbf{E}_c = 1/2(\nabla\mathbf{u}_c + \nabla\mathbf{u}_c^T)$, is a *collapse mechanism* associated to $\mathbf{T}_c$.

Now, let us consider the volume-force field $\mathbf{b}$ and the surface-force field $\mathbf{f}$ acting on $\partial\Omega_f$. The stress field $\mathbf{T} \in \mathcal{B}$ is *equilibrated* with the load $(\mathbf{b}, \mathbf{f})$ if

$$\int_\Omega \mathbf{T}\cdot\mathbf{E}\,dV = \int_\Omega \mathbf{b}\cdot\mathbf{u}\,dV + \int_{\partial\Omega_f}\mathbf{f}\cdot\mathbf{u}\,dS,\ \ \forall\mathbf{u}\in\mathcal{S}, \mathbf{E} = D(\mathbf{u}) \tag{9}$$

Let us consider the following loading process $\mathbf{f} = \mathbf{f}(\lambda) = \mathbf{f}_p + \lambda\mathbf{f}_v$, where $\mathbf{f}_p$ is the permanent part of the load, $\mathbf{f}_v$ the variable part and $\lambda \geq 0$ is the *loading multiplier.* The multiplier λ is *safe* if there exists a stress field $\mathbf{T}$ equilibrated with $(\mathbf{b}, \mathbf{f})$ that belongs to the interior of $\mathcal{K}$. λ is *statically admissible* if there exists a stress field $\mathbf{T}$ equilibrated with $(\mathbf{b}, \mathbf{f})$ belonging to $\mathcal{K}$. Finally, λ is a *collapse multiplier* if there is a collapse stress field $\mathbf{T}$ equilibrated with $(\mathbf{b}, \mathbf{f})$.

In this paper we assume that for the given volume-force field $\mathbf{b}$ and the loading process $\mathbf{f}(\lambda)$, a collapse multiplier $\lambda_c > 0$ exists. The static theorem dictates that if $\lambda = 0$ is a safe multiplier, then the collapse multiplier is the maximum of all statically admissible multipliers.

Let us consider a collapse stress field $\mathbf{T}^*$ and let $\mathbf{u}^*$ be an associated collapse mechanism, such that

$$\int_{\partial\Omega_f}\mathbf{f}_v\cdot\mathbf{u}^*\,dS > 0 \tag{10}$$

The scalar

$$\lambda^* = -\frac{\int_\Omega \mathbf{b}\cdot\mathbf{u}^*\,dV + \int_{\partial\Omega_f}\mathbf{f}_p\cdot\mathbf{u}^*\,dS}{\int_{\partial\Omega_f}\mathbf{f}_v\cdot\mathbf{u}^*\,dS} \tag{11}$$

is a *kinematically admissible* multiplier. According to the kinematic theorem, then, the collapse multiplier is the minimum of all kinematically admissible multipliers.

3. Finite element modelling and kinematic theorem

Let λ_c and $\mathbf{u}_c$ denote the collapse multiplier and a collapse mechanism, respectively (note that the collapse multiplier is unique, while the collapse mechanism can be not unique). Our aim is to determine λ_c and $\mathbf{u}_c$ numerically by using the kinematic theorem. To this end, let us introduce the linear functional

$$\Phi(\mathbf{u}) = -\int_\Omega \mathbf{b}\cdot\mathbf{u}\,dV - \int_{\partial\Omega_f}\mathbf{f}_p\cdot\mathbf{u}\,dS \tag{12}$$

defined on $\mathcal{S}$.

In view of Eq. (10), in order to determine λ_c and $\mathbf{u}_c$, we must solve the following constrained minimum problem

$$\underset{\mathbf{u} \in \mathcal{S}}{Min} \Phi(u) \tag{13}$$

$$\int_{\partial\Omega_f} \mathbf{f}_v \cdot \mathbf{u} dS = 1 \tag{14}$$

$$D(\mathbf{u})(x) \in Sym_3^+ \quad \forall x \in \Omega \tag{15}$$

$$\mathbf{u}(x) = \mathbf{0} \quad \forall x \in \partial\Omega_u \tag{16}$$

The set of all displacement fields $\mathbf{u}$ belonging to $\mathcal{S}$ such that $\int_{\partial\Omega_f} \mathbf{f}_v \cdot \mathbf{u} dS = 1$ and $D(\mathbf{u})(x) \in Sym_3^+$ $\forall x \in \Omega$, is a convex set.

In order to numerically solve the problem (13)-(16), it has to be discretized. For a sake of simplicity, we shall limit ourselves to the case in which Ω is a plane region, subdivided into m four-nodes isoparametric finite elements with four Gauss points, having bilinear shape functions

$$N_1(\xi, \eta) = \frac{1}{4}(1-\xi)(1-\eta) \tag{17}$$

$$N_2(\xi, \eta) = \frac{1}{4}(1+\xi)(1-\eta) \tag{18}$$

$$N_3(\xi, \eta) = \frac{1}{4}(1+\xi)(1+\eta) \tag{19}$$

$$N_4(\xi, \eta) = \frac{1}{4}(1-\xi)(1+\eta) \tag{20}$$

with $(\xi, \eta) \in [-1, 1] \times [-1, 1]$, Hinton and Owen (1977). For $\mathbf{a}^{(i)} \in \mathcal{V}_8$, the nodal displacements vector of the i-th element, the displacement (u, v) of a point of the i-th element, with co-ordinates (ξ, η), is

$$u(\xi, \eta) = \sum_{j=1}^{4} N_j(\xi, \eta) u_j, \qquad v(\xi, \eta) = \sum_{j=1}^{4} N_j(\xi, \eta) v_j \tag{21}$$

where u_j and v_j denote the horizontal and vertical displacements of the j-th node.

For n the number of nodes of the finite element mesh, we denote by $\mathbf{a} \in \mathcal{V}_{2n}$, the vector of nodal displacements, and $\mathbf{c}_b$, $\mathbf{c}_p$ and $\mathbf{c}_v \in \mathcal{V}_{2n}$ as the vectors of the equivalent nodal loads corresponding to $\mathbf{b}$, $\mathbf{f}_p$ and $\mathbf{f}_v$, respectively. The vector of the engineering components of the strain tensor at the Gauss point (ξ_k, η_k) of the i-th element is

$$\mathbf{h}^{(i)}(\xi_k, \eta_k) = \begin{pmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{pmatrix} = \mathbf{B}^{(i)}(\xi_k, \eta_k) \mathbf{a}^{(i)} \tag{22}$$

where u and v are the components of the displacement vector, and $\mathbf{B}^{(i)}(\xi_k, \eta_k)$ is the matrix of the Cartesian derivatives of the shape functions, calculated in the Gauss point (ξ_k, η_k) of the i-th element

$$\mathbf{B}^{(i)} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix} \tag{23}$$

The positive semidefiniteness constraint on the strain tensor field (15) can be expressed by the conditions

$$\frac{\partial u}{\partial x} \geq 0, \quad \frac{\partial v}{\partial y} \geq 0, \quad \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{1}{4}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 \geq 0 \tag{24}$$

which must be satisfied in each Gauss point of each mesh element. For each fixed element i, conditions (24) can be expressed in terms of the nodal displacements vector $\mathbf{a}^{(i)}$ and the matrix $\mathbf{B}^{(i)}$. Let us put

$$\mathbf{q}_1^{(i)T} = \left(\frac{\partial N_1}{\partial x} \;\; 0 \;\; \frac{\partial N_2}{\partial x} \;\; 0 \;\; \frac{\partial N_3}{\partial x} \;\; 0 \;\; \frac{\partial N_4}{\partial x} \;\; 0\right) \tag{25}$$

$$\mathbf{q}_2^{(i)T} = \left(0 \;\; \frac{\partial N_1}{\partial y} \;\; 0 \;\; \frac{\partial N_2}{\partial y} \;\; 0 \;\; \frac{\partial N_3}{\partial y} \;\; 0 \;\; \frac{\partial N_4}{\partial y}\right) \tag{26}$$

$$\mathbf{q}_3^{(i)T} = \left(\frac{\partial N_1}{\partial y} \;\; \frac{\partial N_1}{\partial x} \;\; \frac{\partial N_2}{\partial y} \;\; \frac{\partial N_2}{\partial x} \;\; \frac{\partial N_3}{\partial y} \;\; \frac{\partial N_3}{\partial x} \;\; \frac{\partial N_4}{\partial y} \;\; \frac{\partial N_4}{\partial x}\right) \tag{27}$$

$$\mathbf{A}^{(i)} = \frac{1}{2}(\mathbf{q}_1^{(i)} \otimes \mathbf{q}_2^{(i)} + \mathbf{q}_2^{(i)} \otimes \mathbf{q}_1^{(i)}) - \frac{1}{4}\mathbf{q}_3^{(i)} \otimes \mathbf{q}_3^{(i)} \tag{28}$$

where the derivatives of the shape functions are calculated in the current Gauss point of the i-th element; thus, conditions (24) can be rewritten as

$$\mathbf{q}_1^{(i)} \cdot \mathbf{a}^{(i)} \geq 0, \quad \mathbf{q}_2^{(i)} \cdot \mathbf{a}^{(i)} \geq 0, \quad \mathbf{a}^{(i)} \cdot \mathbf{A}^{(i)}\mathbf{a}^{(i)} \geq 0 \tag{29}$$

Finally, discretizing the minimum problem (13)-(16) leads to the following problem

$$\underset{\mathbf{a} \in \mathcal{V}_{2n}}{Min} - (\mathbf{c}_b + \mathbf{c}_p) \cdot a \tag{30}$$

$$\mathbf{c}_v \cdot \mathbf{a} = 1 \tag{31}$$

$$\mathbf{q}_1^{(i)}(\xi_k, \eta_k) \cdot \mathbf{a}^{(i)} \geq 0, \qquad k = 1, \ldots, 4, \quad i = 1, \ldots, m \tag{32}$$

$$\mathbf{q}_2^{(i)}(\xi_k, \eta_k) \cdot \mathbf{a}^{(i)} \geq 0, \qquad k = 1, \ldots, 4, \quad i = 1, \ldots, m \tag{33}$$

$$\mathbf{a}^{(i)} \cdot \mathbf{A}^{(i)}(\xi_k, \eta_k)\mathbf{a}^{(i)} \geq 0, \qquad k = 1, \ldots, 4, \quad i = 1, \ldots, m \tag{34}$$

with the further constraint that the displacements of nodes belonging to the boundary portion $\partial\Omega_u$ are zero.

Let us now introduce the connectivity matrices $\mathbf{L}^{(i)}$, linking the nodal displacements vector $\mathbf{a}$ of the structure with the nodal displacements $\mathbf{a}^{(i)}$ of the i-th element,

$$\mathbf{a}^{(i)} = \mathbf{L}^{(i)}\mathbf{a} \tag{35}$$

The inequalities (32), (33) and (34) become

$$\mathbf{L}^{(i)T}\mathbf{q}_1^{(i)}(\xi_k, \eta_k) \cdot \mathbf{a} \geq 0, \qquad k = 1, \ldots, 4, \quad i = 1, \ldots, m \tag{36}$$

$$\mathbf{L}^{(i)T}\mathbf{q}_2^{(i)}(\xi_k, \eta_k) \cdot \mathbf{a} \geq 0, \qquad k = 1, \ldots, 4, \quad i = 1, \ldots, m \tag{37}$$

$$\mathbf{a} \cdot \mathbf{L}^{(i)T}\mathbf{A}^{(i)}(\xi_k, \eta_k)\mathbf{L}^{(i)}\mathbf{a} \geq 0, \qquad k = 1, \ldots, 4, \quad i = 1, \ldots, m \tag{38}$$

Now, we want to prove some properties of tensor $\mathbf{A}^{(i)} \in Sym_8$ defined in Eq. (28). To this end, it should be noted that vectors $\mathbf{q}_1^{(i)}$ and $\mathbf{q}_2^{(i)}$ are orthogonal, that $\mathbf{q}_1^{(i)} \cdot \mathbf{q}_3^{(i)} = \mathbf{q}_2^{(i)} \cdot \mathbf{q}_3^{(i)}$, and $\|\mathbf{q}_1^{(i)}\|^2 + \|\mathbf{q}_2^{(i)}\|^2 = \|\mathbf{q}_3^{(i)}\|^2$. From (28) it follows that for each $\mathbf{n} \in Span\{\mathbf{q}_1^{(i)}, \mathbf{q}_2^{(i)}, \mathbf{q}_3^{(i)}\}^{\perp}$, we have $\mathbf{A}^{(i)}\mathbf{n} = \mathbf{0}$; thus $\mathbf{A}^{(i)}$ has a zero eigenvector with multiplicity of at least five. To determine the remaining eigenvalues of $\mathbf{A}^{(i)}$, we must account for the fact that the corresponding eigenvectors belong to the subspace $Span\{\mathbf{q}_1^{(i)}, \mathbf{q}_2^{(i)}, \mathbf{q}_3^{(i)}\}$. By using the relations

$$\mathbf{A}^{(i)}\mathbf{q}_1^{(i)} = \frac{1}{2}\|\mathbf{q}_1^{(i)}\|^2\mathbf{q}_2^{(i)} - \frac{1}{4}(\mathbf{q}_1^{(i)} \cdot \mathbf{q}_3^{(i)})\mathbf{q}_3^{(i)} \tag{39}$$

$$\mathbf{A}^{(i)}\mathbf{q}_2^{(i)} = \frac{1}{2}\|\mathbf{q}_2^{(i)}\|^2\mathbf{q}_1^{(i)} - \frac{1}{4}(\mathbf{q}_1^{(i)} \cdot \mathbf{q}_3^{(i)})\mathbf{q}_3^{(i)} \tag{40}$$

$$\mathbf{A}^{(i)}\mathbf{q}_3^{(i)} = \frac{1}{2}(\mathbf{q}_1^{(i)} \cdot \mathbf{q}_3^{(i)})(\mathbf{q}_1^{(i)} + \mathbf{q}_2^{(i)}) - \frac{1}{4}\|\mathbf{q}_3^{(i)}\|^2\mathbf{q}_3^{(i)} \tag{41}$$

simple calculations reveal that the remaining eigenvalues of $\mathbf{A}^{(i)}$ are the three real roots $\omega_1^{(i)}, \omega_2^{(i)}$ and $\omega_3^{(i)}$ of the third-degree polynomial

$$\begin{aligned} p(\omega) = {} & \omega^3 + \frac{1}{4}\|\mathbf{q}_3^{(i)}\|^2\omega^2 + \frac{1}{4}((\mathbf{q}_1^{(i)} \cdot \mathbf{q}_3^{(i)})^2 - \|\mathbf{q}_1^{(i)}\|^2\|\mathbf{q}_2^{(i)}\|^2)\omega \\ & \frac{1}{16}\|\mathbf{q}_3^{(i)}\|^2((\mathbf{q}_1^{(i)} \cdot \mathbf{q}_3^{(i)})^2 - \|\mathbf{q}_1^{(i)}\|^2\|\mathbf{q}_2^{(i)}\|^2) \end{aligned} \tag{42}$$

or, in particular,

$$\omega_1^{(i)} = -\frac{1}{4}\|\mathbf{q}_3^{(i)}\|^2 \tag{43}$$

$$\omega_2^{(i)} = -\frac{1}{2}\sqrt{\|\mathbf{q}_1^{(i)}\|^2\|\mathbf{q}_2^{(i)}\|^2 - (\mathbf{q}_1^{(i)} \cdot \mathbf{q}_3^{(i)})^2} \tag{44}$$

$$\omega_3^{(i)} = \frac{1}{2}\sqrt{\|\mathbf{q}_1^{(i)}\|^2\|\mathbf{q}_2^{(i)}\|^2 - (\mathbf{q}_1^{(i)} \cdot \mathbf{q}_3^{(i)})^2} \tag{45}$$

for which it holds that $\omega_1^{(i)} \le 0$, $\omega_2^{(i)} \le 0$, $\omega_3^{(i)} \ge 0$. Let us now calculate the eigenvectors $\mathbf{p}_1^{(i)}, \mathbf{p}_2^{(i)}, \mathbf{p}_3^{(i)}$ of $\mathbf{A}^{(i)}$ corresponding to $\omega_1^{(i)}$, $\omega_2^{(i)}$ and $\omega_3^{(i)}$. In order to simplify calculation of the eigenvectors, it is convenient to distinguish the following four cases.

Case 1: $\mathbf{q}_3^{(i)} = 0$.

Thus, $\mathbf{q}_1^{(i)} = \mathbf{q}_2^{(i)} = \mathbf{0}$ and $\mathbf{A}^{(i)} = \mathbf{0}$.

Case 2: $\mathbf{q}_3^{(i)} \neq \mathbf{0}, \mathbf{q}_2^{(i)} = \mathbf{0}, \mathbf{q}_1^{(i)} = \mathbf{0}$, or $\mathbf{q}_3^{(i)} \neq \mathbf{0}, \mathbf{q}_2^{(i)} = \mathbf{0}, \mathbf{q}_1^{(i)} \neq \mathbf{0}$.

In this case, $\mathbf{A}^{(i)} = -\frac{1}{4}\mathbf{q}_3^{(i)} \otimes \mathbf{q}_3^{(i)}$, $\omega_2^{(i)} = \omega_3^{(i)} = 0$, and the eigenvectors corresponding to $\omega_1^{(i)}$, $\omega_2^{(i)}$, $\omega_3^{(i)}$ are

$$\mathbf{p}_1^{(i)} = \mathbf{q}_3^{(i)} \tag{46}$$

$$\mathbf{p}_2^{(i)}, \mathbf{p}_3^{(i)} \text{ mutually orthogonal and orthogonal to } \mathbf{p}_1^{(i)} \tag{47}$$

Case 3: $\mathbf{q}_3^{(i)} \neq \mathbf{0}, \mathbf{q}_2^{(i)} \neq \mathbf{0}, \mathbf{q}_1^{(i)} \neq \mathbf{0}, \mathbf{q}_1^{(i)} \cdot \mathbf{q}_3^{(i)} = 0$.

From (44)-(45) it follows that $\omega_2^{(i)} = -\frac{1}{2}\|\mathbf{q}_1^{(i)}\|\|\mathbf{q}_2^{(i)}\|$, $\omega_3^{(i)} = \frac{1}{2}\|\mathbf{q}_1^{(i)}\|\|\mathbf{q}_2^{(i)}\|$. In view of $\|\mathbf{q}_3^{(i)}\|^2 = \|\mathbf{q}_1^{(i)}\|^2 + \|\mathbf{q}_2^{(i)}\|^2 \ge 2\|\mathbf{q}_1^{(i)}\|\|\mathbf{q}_2^{(i)}\|$, the eigenvalues of $\mathbf{A}^{(i)}$ are ordered as follows

$$\omega_1^{(i)} \le \omega_2^{(i)} < 0 < \omega_3^{(i)} \tag{48}$$

Hence, it is a simple matter to show that the eigenvectors of $\mathbf{A}^{(i)}$ are respectively

$$\mathbf{p}_1^{(i)} = \mathbf{q}_3^{(i)} \tag{49}$$

$$\mathbf{p}_2^{(i)} = -\frac{\|\mathbf{q}_2^{(i)}\|}{\|\mathbf{q}_1^{(i)}\|}\mathbf{q}_1^{(i)} + \mathbf{q}_2^{(i)} \tag{50}$$

$$\mathbf{p}_3^{(i)} = \frac{\|\mathbf{q}_2^{(i)}\|}{\|\mathbf{q}_1^{(i)}\|}\mathbf{q}_1^{(i)} + \mathbf{q}_2^{(i)} \tag{51}$$

Case 4: $\mathbf{q}_3^{(i)} \neq \mathbf{0}, \mathbf{q}_2^{(i)} \neq \mathbf{0}, \mathbf{q}_1^{(i)} \neq \mathbf{0}, \mathbf{q}_1^{(i)} \cdot \mathbf{q}_3^{(i)} \neq 0$.

The eigenvectors corresponding to the eigenvalues $\omega_1^{(i)}, \omega_2^{(i)}, \omega_3^{(i)}$ are respectively

$$\mathbf{p}_1^{(i)} = \mathbf{q}_1^{(i)} - \mathbf{q}_2^{(i)} + \frac{\|\mathbf{q}_3^{(i)}\|^2 - 2\|\mathbf{q}_1^{(i)}\|^2}{2(\mathbf{q}_1^{(i)} \cdot \mathbf{q}_3^{(i)})}\mathbf{q}_3^{(i)} \tag{52}$$

$$\mathbf{p}_2^{(i)} = \frac{\sqrt{\|\mathbf{q}_1^{(i)}\|^2\|\mathbf{q}_2^{(i)}\|^2 - (\mathbf{q}_1^{(i)} \cdot \mathbf{q}_3^{(i)})^2} - \|\mathbf{q}_1^{(i)}\|^2}{\mathbf{q}_1^{(i)} \cdot \mathbf{q}_3^{(i)}}\mathbf{q}_1^{(i)}$$
$$+ \frac{\sqrt{\|\mathbf{q}_1^{(i)}\|^2\|\mathbf{q}_2^{(i)}\|^2 - (\mathbf{q}_1^{(i)} \cdot \mathbf{q}_3^{(i)})^2} - \|\mathbf{q}_1^{(i)}\|^2}{\mathbf{q}_1^{(i)} \cdot \mathbf{q}_3^{(i)}}\mathbf{q}_2^{(i)} + \mathbf{q}_3^{(i)} \tag{53}$$

$$\mathbf{p}_3^{(i)} = -\frac{\sqrt{\|\mathbf{q}_1^{(i)}\|^2\|\mathbf{q}_2^{(i)}\|^2 - (\mathbf{q}_1^{(i)} \cdot \mathbf{q}_3^{(i)})^2} + \|\mathbf{q}_2^{(i)}\|^2}{\mathbf{q}_1^{(i)} \cdot \mathbf{q}_3^{(i)}}\mathbf{q}_1^{(i)}$$
$$- \frac{\sqrt{\|\mathbf{q}_1^{(i)}\|^2\|\mathbf{q}_2^{(i)}\|^2 - (\mathbf{q}_1^{(i)} \cdot \mathbf{q}_3^{(i)})^2} + \|\mathbf{q}_1^{(i)}\|^2}{\mathbf{q}_1^{(i)} \cdot \mathbf{q}_3^{(i)}}\mathbf{q}_2^{(i)} + \mathbf{q}_3^{(i)} \tag{54}$$

We are now in a position to prove that the admissible region defined by the conditions (32), (33) and (34) is a closed convex set of $\mathcal{V}_{2n}$, as it is the intersection of a finite number of closed convex sets. In order to prove that for each $k = 1, \ldots, 4$ and $i = 1, \ldots, m$, the inequalities (32), (33) and (34) define a convex set of $\mathcal{V}_{2n}$, it is sufficient to prove that the subset $\mathcal{C} = \{\mathbf{v} \in \mathcal{V}_8 | \mathbf{q}_1 \cdot \mathbf{v} \geq 0, \mathbf{v} \cdot \mathbf{A}\mathbf{v} \geq 0\}$ with $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{A}$ given in (25), (26) and (28), is convex. For the sake of simplicity, we omit the index i.

Let us designate $\tilde{\mathbf{p}}_j$, $j = 4, 5, 6, 7, 8$, as an orthonormal basis of $Span\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}^{\perp}$, and put

$$\tilde{\mathbf{p}}_i = \frac{\mathbf{p}_i}{\|\mathbf{p}_i\|}, \quad i = 1, 2, 3 \tag{55}$$

where the vectors $\mathbf{p}_i$ are given in Eqs. (46)-(47), (49), (50), (51) or (52), (53), (54) respectively for cases 1 to 4. As the spectral theorem dictates that

$$\mathbf{A} = \sum_{j=1}^{3} \omega_j \tilde{\mathbf{p}}_j \otimes \tilde{\mathbf{p}}_j \tag{56}$$

for a vector $\mathbf{v} = \sum_{j=1}^{8} v_j \tilde{\mathbf{p}}_j$, the condition $\mathbf{v} \cdot \mathbf{A}\mathbf{v} \geq 0$ is equivalent to the inequality

$$\sum_{j=1}^{3} \omega_j v_j^2 \geq 0 \tag{57}$$

and the conditions $\mathbf{q}_1 \cdot \mathbf{v} \geq 0, \mathbf{q}_2 \cdot \mathbf{v} \geq 0$, become

$$\sum_{j=1}^{3} v_j \mathbf{q}_1 \cdot \tilde{\mathbf{p}}_j \geq 0, \qquad \sum_{j=1}^{3} v_j \mathbf{q}_2 \cdot \tilde{\mathbf{p}}_j \geq 0 \tag{58}$$

Thus, the set $\mathcal{C}$ can be represented as follows

$$\mathcal{C} = \left\{ \mathrm{x} \in \mathbb{R}^8 \middle| \sum_{j=1}^{3} \omega_j x_j^2 \geq 0, \sum_{j=1}^{3} x_j \mathbf{q}_1 \cdot \tilde{\mathbf{p}}_j \geq 0, \sum_{j=1}^{3} x_j \mathbf{q}_2 \cdot \tilde{\mathbf{p}}_j \geq 0 \right\} \tag{59}$$

Let us now analyze the preceding four cases separately.

In *Case 1* we have $\mathcal{C} = \mathbb{R}^8$.

In *Case 2*, as it holds that $\mathcal{C} = \{\mathrm{x} \in \mathbb{R}^8 | x_1 = 0, (\mathbf{p}_2 \cdot \mathbf{q}_2) x_2 + (\mathbf{p}_3 \cdot \mathbf{q}_2) x_3 \geq 0\}$, $\mathcal{C}$ is therefore convex.

In *Case 3*, $\mathcal{C} = \left\{ \mathrm{x} \in \mathbb{R}^8 \middle| \frac{\|\mathbf{q}_3\|^2}{2\|\mathbf{q}_1\|\|\mathbf{q}_2\|} x_1^2 + x_2^2 - x_3^2 \leq 0, -x_2 + x_3 \geq 0, x_2 + x_3 \geq 0 \right\}$, which is also convex.

Finally, in *Case 4*, from Eqs. (43), (44), (45) and (52), (53), (54) it follows that

$$\begin{aligned}
\mathcal{C} = \Bigg\{ \mathrm{x} \in \mathbb{R}^8 \Big| & \|\mathbf{q}_3\|^2 x_1^2 + 2\sqrt{\|\mathbf{q}_1\|^2\|\mathbf{q}_2\|^2 - (\mathbf{q}_1 \cdot \mathbf{q}_3)^2}\, x_2^2 \\
& -2\sqrt{\|\mathbf{q}_1\|^2\|\mathbf{q}_2\|^2 - (\mathbf{q}_1 \cdot \mathbf{q}_3)^2}\, x_3^2 \leq 0 \\
& \frac{\|\mathbf{q}_3\|^2}{2\|\mathbf{p}_1\|} x_1 + \frac{(\mathbf{q}_1 \cdot \mathbf{q}_3)^2 + \|\mathbf{q}_1\|^2\sqrt{\|\mathbf{q}_1\|^2\|\mathbf{q}_2\|^2 - (\mathbf{q}_1 \cdot \mathbf{q}_3)^2} - \|\mathbf{q}_1\|^2\|\mathbf{q}_2\|^2}{(\mathbf{q}_1 \cdot \mathbf{q}_3)\|\mathbf{p}_2\|} x_2 \\
& + \frac{(\mathbf{q}_1 \cdot \mathbf{q}_3)^2 - \|\mathbf{q}_1\|^2\sqrt{\|\mathbf{q}_1\|^2\|\mathbf{q}_2\|^2 - (\mathbf{q}_1 \cdot \mathbf{q}_3)^2} - \|\mathbf{q}_1\|^2\|\mathbf{q}_2\|^2}{(\mathbf{q}_1 \cdot \mathbf{q}_3)\|\mathbf{p}_3\|} x_3 \geq 0, \\
& -\frac{\|\mathbf{q}_3\|^2}{2\|\mathbf{p}_1\|} x_1 + \frac{(\mathbf{q}_1 \cdot \mathbf{q}_3)^2 + \|\mathbf{q}_2\|^2\sqrt{\|\mathbf{q}_1\|^2\|\mathbf{q}_2\|^2 - (\mathbf{q}_1 \cdot \mathbf{q}_3)^2} - \|\mathbf{q}_1\|^2\|\mathbf{q}_2\|^2}{(\mathbf{q}_1 \cdot \mathbf{q}_3)\|\mathbf{p}_2\|} x_2 \\
& + \frac{(\mathbf{q}_1 \cdot \mathbf{q}_3)^2 - \|\mathbf{q}_2\|^2\sqrt{\|\mathbf{q}_1\|^2\|\mathbf{q}_2\|^2 - (\mathbf{q}_1 \cdot \mathbf{q}_3)^2} - \|\mathbf{q}_1\|^2\|\mathbf{q}_2\|^2}{(\mathbf{q}_1 \cdot \mathbf{q}_3)\|\mathbf{p}_3\|} x_3 \geq 0 \Bigg\}
\end{aligned} \tag{60}$$

where

$$\|\mathbf{p}_1\| = \frac{\|\mathbf{q}_3\|}{2|\mathbf{q}_1 \cdot \mathbf{q}_3|}\sqrt{4(\mathbf{q}_1 \cdot \mathbf{q}_3)^2 + (\|\mathbf{q}_2\|^2 - \|\mathbf{q}_1\|^2)^2} \tag{61}$$

$$\begin{aligned}
\|\mathbf{p}_2\| = \Bigg(& \left(\frac{\sqrt{\|\mathbf{q}_1\|^2\|\mathbf{q}_2\|^2 - (\mathbf{q}_1 \cdot \mathbf{q}_3)^2} - \|\mathbf{q}_2\|^2}{(\mathbf{q}_1 \cdot \mathbf{q}_3)} \right)^2 \|\mathbf{q}_1\|^2 \\
& + \left(\frac{\sqrt{\|\mathbf{q}_1\|^2\|\mathbf{q}_2\|^2 - (\mathbf{q}_1 \cdot \mathbf{q}_3)^2} - \|\mathbf{q}_1\|^2}{(\mathbf{q}_1 \cdot \mathbf{q}_3)} \right)^2 \|\mathbf{q}_2\|^2 \\
& + 2\left(\frac{2\sqrt{\|\mathbf{q}_1\|^2\|\mathbf{q}_2\|^2 - (\mathbf{q}_1 \cdot \mathbf{q}_3)^2} - \|\mathbf{q}_3\|^2}{(\mathbf{q}_1 \cdot \mathbf{q}_3)} \right)(\mathbf{q}_1 \cdot \mathbf{q}_3) + \|\mathbf{q}_3\|^2 \Bigg)^{1/2}
\end{aligned} \tag{62}$$

$$\|\mathbf{p}_3\| = \left(\left(\frac{\sqrt{\|\mathbf{q}_1\|^2 \|\mathbf{q}_2\|^2 - (\mathbf{q}_1 \cdot \mathbf{q}_3)^2} + \|\mathbf{q}_2\|^2}{(\mathbf{q}_1 \cdot \mathbf{q}_3)} \right)^2 \|\mathbf{q}_1\|^2 \right.$$
$$+ \left(\frac{\sqrt{\|\mathbf{q}_1\|^2 \|\mathbf{q}_2\|^2 - (\mathbf{q}_1 \cdot \mathbf{q}_3)^2} + \|\mathbf{q}_1\|^2}{(\mathbf{q}_1 \cdot \mathbf{q}_3)} \right)^2 \|\mathbf{q}_2\|^2$$
$$\left. -2 \left(\frac{2(\sqrt{\|\mathbf{q}_1\|^2 \|\mathbf{q}_2\|^2 - (\mathbf{q}_1 \cdot \mathbf{q}_3)^2} + \|\mathbf{q}_3\|^2)}{(\mathbf{q}_1 \cdot \mathbf{q}_3)} \right) (\mathbf{q}_1 \cdot \mathbf{q}_3) + \|\mathbf{q}_3\|^2 \right)^{1/2} \tag{63}$$

It can be verified that, if $\mathbf{q}_1 \cdot \mathbf{q}_3 > 0$, then Eq. (60) becomes

$$\mathcal{C} = \left\{ \mathrm{x} \in \mathbb{R}^8 \middle| \|\mathbf{q}_3\|^2 x_1^2 + 2\sqrt{\|\mathbf{q}_1\|^2 \|\mathbf{q}_2\|^2 - (\mathbf{q}_1 \cdot \mathbf{q}_3)^2}\, x_2^2 \right.$$
$$\left. -2\sqrt{\|\mathbf{q}_1\|^2 \|\mathbf{q}_2\|^2 - (\mathbf{q}_1 \cdot \mathbf{q}_3)^2}\, x_3^2 \le 0,\ x_3 \le 0 \right\} \tag{64}$$

while, on the contrary, if $\mathbf{q}_1 \cdot \mathbf{q}_3 < 0$, then

$$\mathcal{C} = \left\{ \mathrm{x} \in \mathbb{R}^8 \middle| \|\mathbf{q}_3\|^2 x_1^2 + 2\sqrt{\|\mathbf{q}_1\|^2 \|\mathbf{q}_2\|^2 - (\mathbf{q}_1 \cdot \mathbf{q}_3)^2}\, x_2^2 \right.$$
$$\left. -2\sqrt{\|\mathbf{q}_1\|^2 \|\mathbf{q}_2\|^2 - (\mathbf{q}_1 \cdot \mathbf{q}_3)^2}\, x_3^2 \le 0,\ x_3 \ge 0 \right\} \tag{65}$$

which are convex sets of $\mathbb{R}^8$.

As the objective function and the admissible region are convex, any local minimum of (30)-(34) is also a global minimum Cea (1978).

One further property of the minimum problem (30)-(34) stems from the fact that the inequalities (32), (33) and (34) define a cone. In fact, if we put $\mathbf{c}_v = \mathbf{c}_v(\alpha) = \alpha\bar{\mathbf{c}}$, with $\alpha > 0$, it is a simple matter to prove that, if $\bar{\mathbf{a}}$ is a solution to (30)-(34) corresponding to $\mathbf{c}_v(1)$, then $\alpha^{-1}\bar{\mathbf{a}}$ is a solution to (30)-(34) corresponding to $\mathbf{c}_v(\alpha)$.

4. Numerical examples

In this section we apply the numerical method proposed in the foregoing to some plane problems for which the collapse multiplier and collapse mechanism are both known. The minimum problem (30)-(34) has been solved by using the free software FILTER available at the server for optimization NEOS (Czyzyk *et al.* 1998, Gropp and Moré 1997, Dolan 2001). The first example regards a thick-walled cylinder subjected to two uniform radial pressures, and the resulting numerical solution is compared to the exact one. For the other two cases, a circular arch and a rectangular panel, the numerical solution to the respective minimum problem (30)-(34) is compared with both the corresponding exact solution and the numerical solution obtained via an incremental non-linear analysis performed with the finite element code NOSA (Lucchesi *et al.* 2000).

In the NOSA implementation, the masonry is considered to be a non-linear elastic material with zero tensile strength and infinite compressive strength. Once the permanent load has been assigned, the value of the collapse load and the corresponding mechanism are then obtained by progressively increasing the variable load assigned to the structure, until it is no longer possible to determine an admissible equilibrated solution.

4.1 Thick-walled cylinder subjected to two uniform radial pressures

An unbounded cylindrical body, whose cross section is a circular ring, with inner radius a and outer radius b, made of a rigid no-tension material, is subjected to two uniform pressures p_2 and λp_1, where $\lambda \geq 0$, acting on the outer and inner surfaces, respectively. A cylindrical reference system $\{O, \rho, \theta, z\}$ is chosen with origin at the centre of the ring and the z-axis orthogonal to its plane (Fig. 1). It has been demonstrated (Bennati and Padovani 1997) that for $\lambda \leq \frac{bp_2}{ap_1}$, there exists an admissible stress field equilibrated with loads p_2 and λp_1. For values of λ greater than $\frac{bp_2}{ap_1}$, on the contrary, there are no stress fields that are both in equilibrium with the external pressures and semidefinite negative. The value of the collapse multiplier is (Bennati and Padovani 1997).

$$\lambda_c = \frac{bp_2}{ap_1} \tag{66}$$

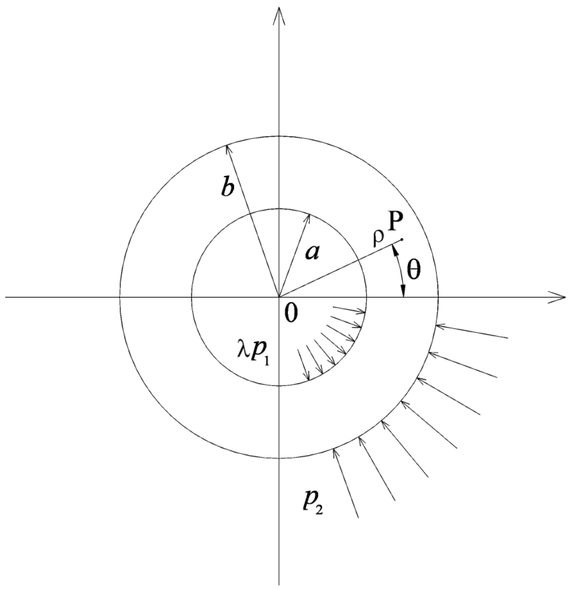

Fig. 1 Thick-walled cylinder subjected to radial pressures

In fact, the stress field **T** having components

$$\sigma_\rho(\rho) = -\lambda_c p_1 \frac{a}{\rho}, \quad \sigma_\theta(\rho) = 0, \quad \tau_{\rho\theta}(\rho) = 0 \tag{67}$$

with respect to the chosen reference system, is admissible and equilibrated with loads p_2 and $\lambda_c p_1$. Moreover, **T** is a collapse stress field, as $D(\mathcal{S}) \cap \mathcal{N}(\mathbf{T})$ contains the strain tensor **E** with components

$$\varepsilon_\rho(\rho) = 0, \quad \varepsilon_\theta(\rho) = \frac{\alpha}{\rho}, \quad \varepsilon_{\rho\theta}(\rho) = 0 \tag{68}$$

where $\alpha > 0$, corresponding to collapse mechanism **u** with components

$$u_\rho(\rho) = \alpha, \quad u_\theta(\rho) = 0 \tag{69}$$

One quarter of the circular ring has been discretized with 16 four-node plane elements (the number of nodal points is $n = 25$), adopting the following values:

$$a = 1 \text{ m}, \quad b = 2 \text{ m}, \quad p_1 = 1\ N/\text{m}^2, \quad p_2 = 10000\ N/\text{m}^2 \tag{70}$$

The solution to the minimum problem (30)-(34) calculated with the FILTER code is

$$\lambda^* = 20047 \tag{71}$$

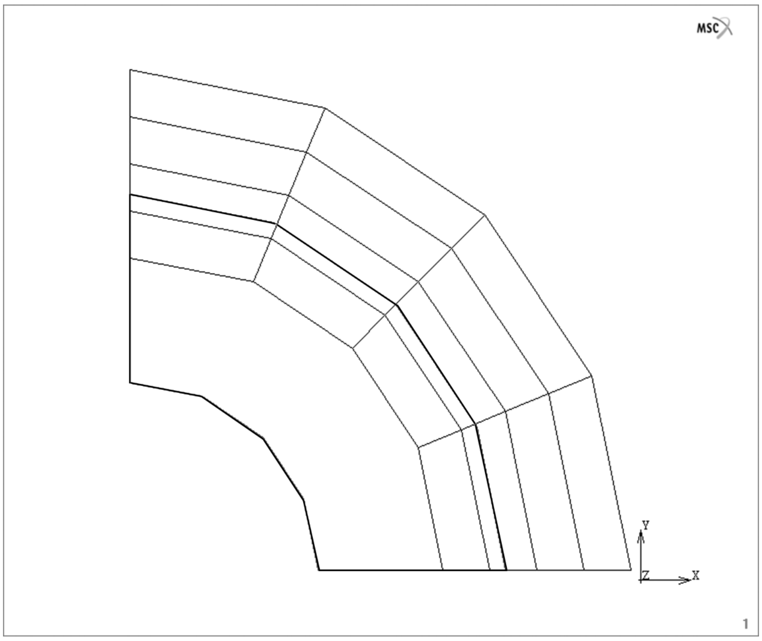

Fig. 2 One quarter of circular ring - collapse mechanism corresponding to λ^* (grey line) superimposed on the undeformed configuration (black line)

and, from Eq. (66) we get

$$\lambda_c = 20000 \tag{72}$$

Fig. 2 shows the collapse mechanisms corresponding to λ^*; the displacement yielded by FILTER is of the type (69).

4.2 Circular arch subjected to its own weight and a point load on the extrados

Let us consider a circular arch with mean radius r and thickness h. We assume that the arch, clamped at the springings, is made of a rigid no-tension materials with specific weight $\mathbf{b}$ and is subjected to a vertical load $\lambda\mathbf{f}_v$ applied at the keystone (Fig. 3). Our aim is to determine the collapse multiplier and a collapse mechanism.

We choose the following parameters,

$$r = 1 \text{ m}, \quad h = 0.14 \text{ m}, \quad |\mathbf{b}| = 20000 \text{ N/m}^3, \quad |\mathbf{f}_v| = 1\ N \tag{73}$$

by taking into account that the width of the arch is 0.01 m, the weight of the arch is 87.965 N. In Lucchesi *et al.* (1997) the collapse load for a circular arch such as that in Fig. 3 has been calculated explicitly as function of the ratio $t = h/2r$. In the case at hand, we have $t = 0.07$, and the collapse multiplier is

$$\lambda_c = 7.794 \tag{74}$$

One half of the arch has been discretized with 144 four-node plane elements (the number of nodal points is $n = 175$), and the solution to the minimum problem (30)-(34) calculated with FILTER is

$$\lambda^* = 7.68 \tag{75}$$

for which the relative error is $\varepsilon = \dfrac{7.794 - 7.68}{7.794} \simeq 1.0\%$

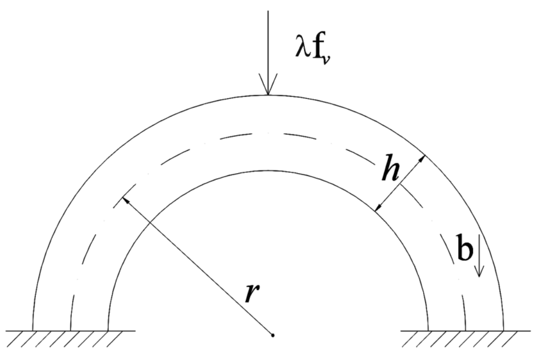

Fig. 3 Circular arch

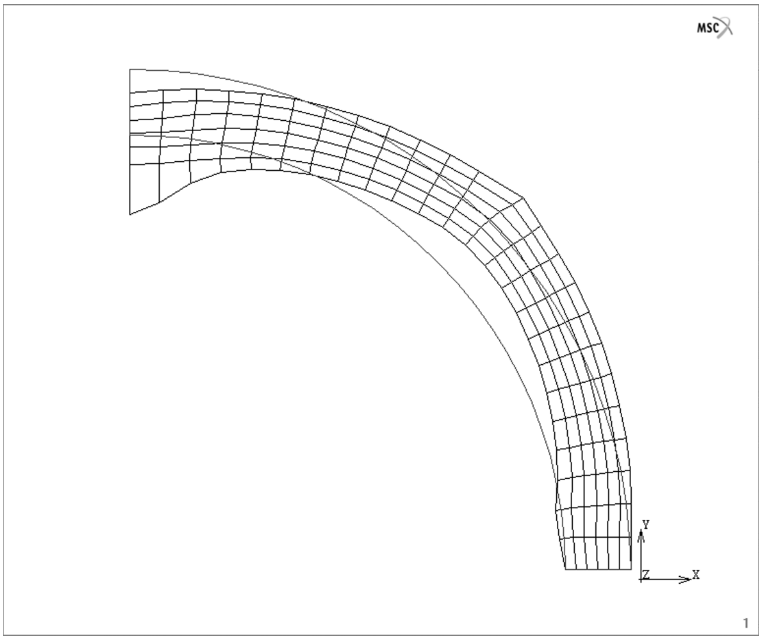

Fig. 4 Collapse mechanism corresponding to λ^* as calculated by FILTER superimposed on the undeformed configuration

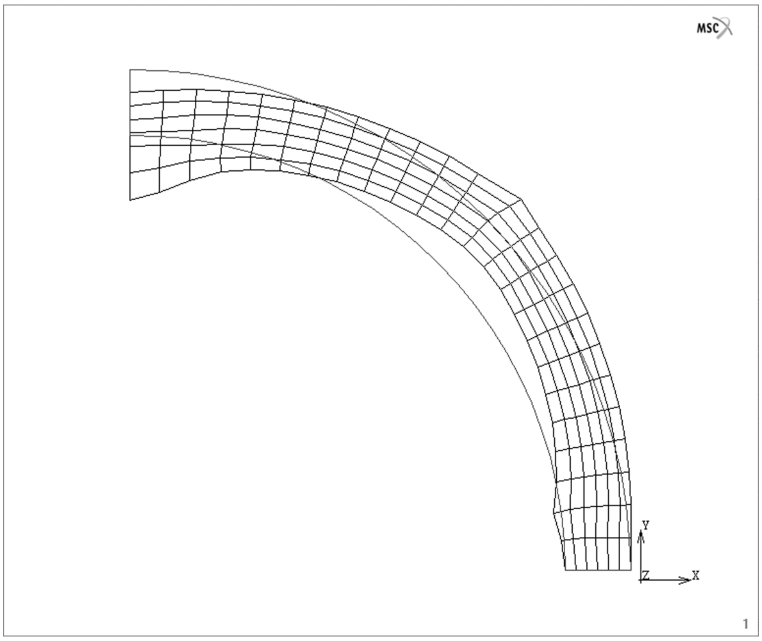

Fig. 5 Collapse mechanism corresponding to λ^N as calculated by NOSA superimposed on the undeformed configuration

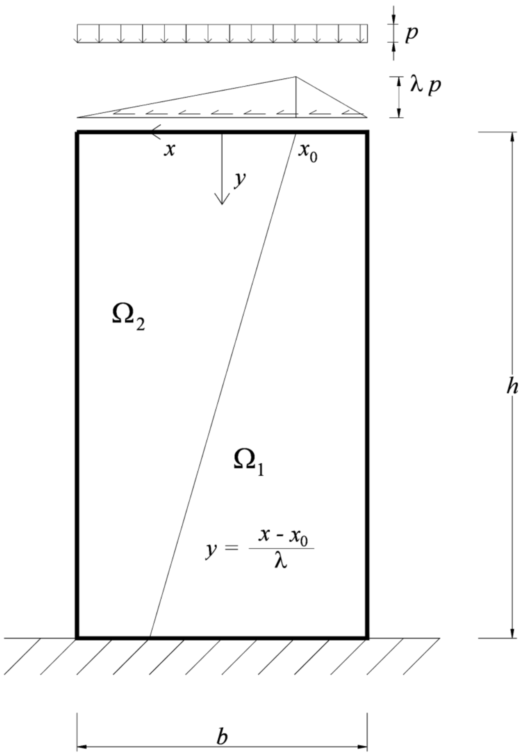

Fig. 6 Rectangular panel

Figs. 4 and 5 show the collapse mechanism corresponding to λ^*, respectively calculated by FILTER and via an incremental analysis performed with the NOSA code using the same discretization. The collapse multiplier resulting from the NOSA analysis is $\lambda^N = 7.8$.

4.3 Rectangular panel

Considering now a rectangular panel made of a rigid no-tension material, we assume that the panel is clamped at the base and subjected to the load $\mathbf{f}_p + \lambda \mathbf{f}_v$ (Fig. 6), with

$$\mathbf{f}_\mathrm{p} = (0, p), \quad \mathbf{f}_v = (f_v, 0) \tag{76}$$

where

$$f_v(x) = \begin{cases} \dfrac{p(b+2x)}{b+2x_0}, & -\dfrac{b}{2} \le x \le x_0 \\ \dfrac{p(b-2x)}{b-2x_0}, & x_0 \le x \le \dfrac{b}{2} \end{cases}$$

An admissible stress field equilibrated with the load has been determined in Lucchesi and Zani (2001) it is regular in the two regions Ω_1 and Ω_2, delineated by the isostatic line $y = \dfrac{x - x_0}{\lambda}$. Moreover, the collapse multiplier λ_c and a collapse mechanism (u, v) have also been explicitly calculated

$$\lambda_c = \frac{b - 2x_0}{2h} \tag{77}$$

$$u(x, y) = \begin{cases} 0, & (x, y) \in \Omega_1 \\ x - \lambda_c y - x_0, & (x, y) \in \Omega_2 \end{cases} \tag{78}$$

$$v(x, y) = \begin{cases} 0, & (x, y) \in \Omega_1 \\ \dfrac{\lambda_c(x - \lambda_c y - x_0)(b - 2x)}{2\lambda_c y - b + 2x_0}, & (x, y) \in \Omega_2 \end{cases} \tag{79}$$

The panel has been discretized using 256 plane elements (in this case the nodal points are $n = 289$) and the following parameters

$$h = 0.6 \text{ m}, \quad b = 0.6 \text{ m}, \quad p = 1\ N/\text{m}^2 \tag{80}$$

For $x_0 = 0$, by (77) we obtain a collapse multiplier of $\lambda_c = 0.5$. The minimum problem has been solved with FILTER, for which the resulting value of the collapse multiplier is $\lambda^* = 0.676$. For the same discretization the NOSA code instead furnished a collapse multiplier $\lambda^N = 0.686$.

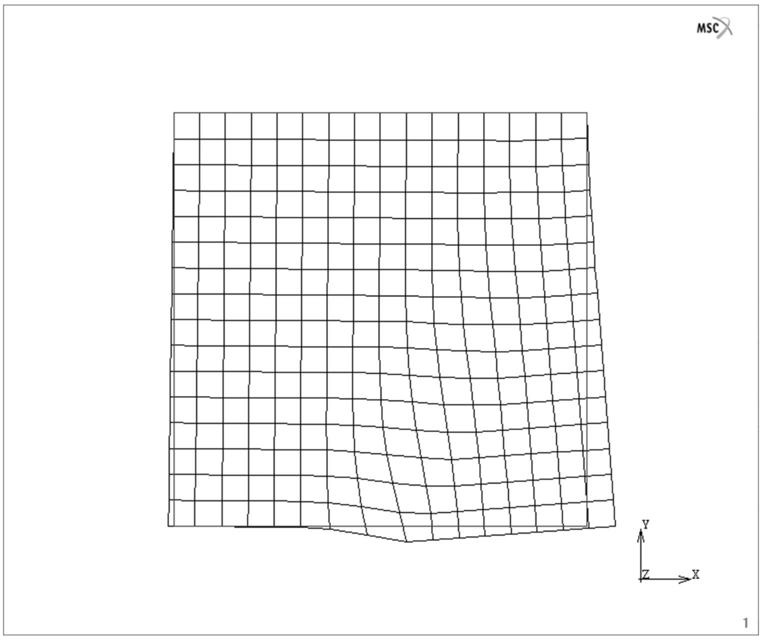

Fig. 7 Collapse mechanism corresponding to λ^* as calculated by FILTER superimposed to the undeformed configuration

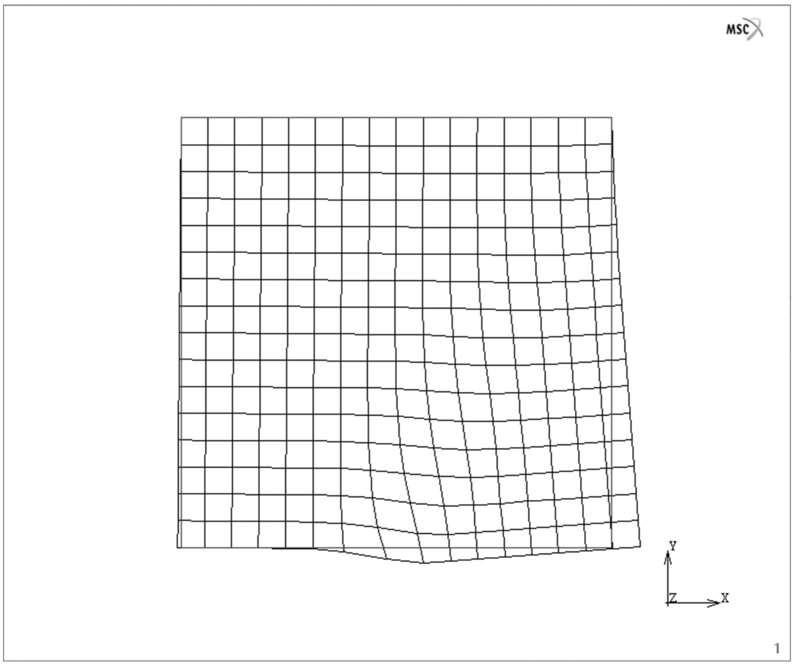

Fig. 8 Collapse mechanism corresponding to λ^N as calculated by NOSA superimposed on the undeformed configuration

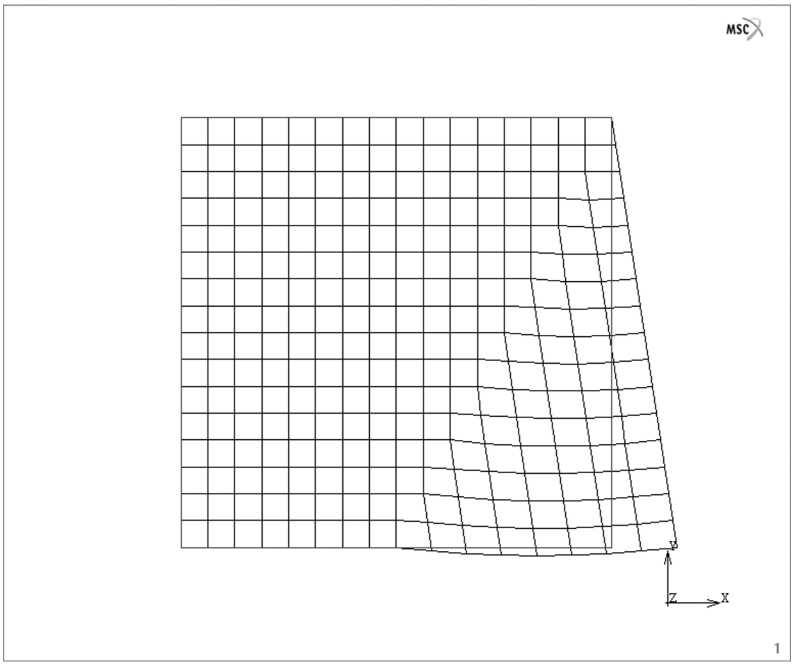

Fig. 9 Collapse mechanism (78)-(79) corresponding to λ_c superimposed on the undeformed configuration

Thus, we can conclude that for a fixed discretization of the panel, the numerical method proposed herein yields the same approximated solution to the collapse problem as an incremental analysis conducted via the NOSA code. Of course, with both methods the accuracy of the solution depends on the number of elements used in discretizing the panel. Seeing that the solution calculated by NOSA tends towards the exact solution as the number of elements increases, we would expect the solution of (30)-(34) to behave in a similar way. Figs. 7 and 8 show the collapse mechanism calculated via FILTER and NOSA respectively. For the sake of comparison, the exact collapse mechanism reported in (78)-(79) has been plotted in Fig. 9.

5. Conclusions

This paper presents a numerical method for the limit analysis of structures made of a rigid no-tension material. After formulation of the constrained minimum problem resulting from application of the kinematic theorem, the corresponding discrete minimum problem is derived via the finite element method. We thereby obtain a minimum problem in which the objective function is linear and the inequality constraints are linear as well as quadratic. In particular, even if the functions describing the quadratic constraints are not convex, the admissible region is convex; thus, any local minimum is a global minimum as well. The method has been applied to some examples for which the collapse multiplier and a collapse mechanism are explicitly known. The solution to the minimum problem has been calculated via numerical codes for quadratic programming problems and then compared to both the exact solution and the numerical solution obtained with the finite element code NOSA. The examples described in Section 4 demonstrates that the results of the proposed numerical procedure and those of an incremental static analysis conducted via the NOSA code are essentially equivalent.

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