Structural Engineering and Mechanics, Vol. 17, No. 6 (2004) 735-749
DOI: http://dx.doi.org/10.12989/sem.2004.17.6.735

A fourth order finite difference method applied to elastodynamics: Finite element and boundary element formulations

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(Received April 2, 2003, Accepted November 25, 2003)

Abstract. This work presents a direct integration scheme, based on a fourth order finite difference approach, for elastodynamics. The proposed scheme was chosen as an alternative for attenuating the errors due to the use of the central difference method, mainly when the time-step length approaches the critical time-step. In addition to eliminating the spurious numerical oscillations, the fourth order finite difference scheme keeps the advantages of the central difference method: reduced computer storage and no requirement of factorisation of the effective stiffness matrix in the step-by-step solution. A study concerning the stability of the fourth order finite difference scheme is presented. The Finite Element Method and the Boundary Element Method are employed to solve elastodynamic problems. In order to verify the accuracy of the proposed scheme, two examples are presented and discussed at the end of this work.

Key words: direct integration methods; fourth order finite difference method; D-BEM; FEM.

1. Introduction

Direct integration methods, or step-by-step methods, are powerful tools for the solution of elastodynamic problems and have been applied with the Finite Element Method (FEM) for quite some time (Bathe 1996, Cook *et al*. 1989, Weaver and Johnston 1987). Explicit integration methods, such as the central difference method (Bathe 1996, Cook *et al*. 1989, Weaver and Johnston 1987), present some characteristics that turn their use very attractive, e.g., easy computational implementation and accurate treatment of general non-linearities. Besides, they are very efficient when used with lumped mass matrix. In this case, there is no need of calculating the complete stiffness matrix and the solution can be acquired on the element level. Consequently, only modest computer storage

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requirements are necessary in order to perform the analysis. These favourable characteristics, however, are counterbalanced by the fact that explicit methods are conditionally stable, i.e., in order to avoid numerical instability, it becomes necessary to adopt a time-step smaller than a critical one (Δt_{cr}).

Implicit integration methods (Bathe 1996, Cook *et al*. 1989, Weaver and Johnston 1987), on the other hand, are unconditionally stable, which means that to obtain accuracy in the integration the time-step can be selected without the limitation predicted by the central difference method. In fact, in many cases the time-step length can be orders of magnitude greater than the critical time-step of the central difference method (Δt_{cr}). The Houbolt method (1974) was commonly used for transient analysis but, as it provides high artificial damping for low frequency response, it was supplanted by methods with better algorithmic damping properties, such as the Wilson-θ method (1973), the Newmark method (1959), and the α-method proposed by Hilber, Hughes & Taylor (1977). Implicit integration methods, on the other hand, when compared with explicit methods are more difficult to implement in a computer code, specially for non-linear problems, and require considerably more computer storage.

This work presents a direct FEM integration scheme based on fourth order Lagrange interpolation of displacements (Abramowitz and Stegun 1984, Kreyszig 1999, Cohen and Joly 1990, Souza and Moura 1997). The interpolation is taken from the time $(n-3)\Delta t$ to the time $(n+1)\Delta t$ (n is the time interval). Due to the fact of being an explicit method, it keeps all the favourable characteristics of the central difference method and, of course, its disadvantages. At a first glance, the most unfavourable characteristic of the fourth order method is a critical time-step (Δt_{crf}) smaller than the corresponding one from the central difference method: $\Delta t_{crf} = 0.8\Delta t_{cr}$ (the computation of Δt_{crf} will be presented in details). However, a favourable characteristic, that justifies the development presented in this work, has to be mentioned: the use of the proposed method practically eliminates the spurious numerical oscillations introduced by the use of the central difference method, as can be observed in the numerical examples presented in a subsequent section.

The fourth order method was also successfully applied to the Boundary Element Method (BEM), called D-BEM (Beskos 1977) (D stands for domain). This formulation presents as a fundamental solution (in usual BEM notation or, for general purposes, the Green function of the corresponding singular problem) the solution of the static problem. As a consequence, a domain integral, associated to the accelerations, appears in the final integral equation of the method. The question that appears at this point is: can the accelerations be approximated by any of the finite difference expressions (Bathe 1996, Cook *et al*. 1989, Weaver and Johnston 1987) employed in the FEM? As long as the authors have knowledge, only the Houbolt method (1974) has been successfully employed to perform the step-by-step analysis. It seems that the numerical damping introduced by the method brings advantages for its use in the D-BEM (Carrer and Telles 1992, Hatzigeorgiou and Beskos 2001) formulation. Even the DR-BEM formulation (Partridge *et al*. 1992, Kontoni and Beskos 1993) (DR stands for dual reciprocity), that has received a great deal of attention during the last years until nowadays, employs the Houbolt method for the step-by-step analysis. Other implicit method, such as the Newmark method (Bathe 1996, Cook *et al*. 1989, Weaver and Johnston 1987, Newmark 1959), did not provide reliable results when used together with the D-BEM. The fourth order method, however, was successfully implemented with the D-BEM formulation, producing the reliable results presented in this work. It is authors' opinion that the use of finite difference methods in the D-BEM (or DR-BEM) formulation is a matter that deserves attention and that a lot of research work has still to be done, and also that this work can bring some insight into this area of research. It is important to mention that the D-BEM formulation is very easy to be implemented and provides accurate results. The main difficulty, common to the FEM, falls in the analysis of problems with infinite domains.

Details concerning FEM and D-BEM formulations are discussed in the corresponding sections along this work.

With the aim of demonstrating the accuracy of the proposed formulation and its applicability to FEM and D-BEM formulations, two examples are analysed at the end of the article by comparing the numerical solutions with the analytical ones (when available) and between them.

2. Fourth order finite difference method

In the FEM context, the dynamic equilibrium equations for an assembled structure can be written as (Bathe 1996, Cook *et al.* 1989, Weaver and Johnston 1987):

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{r}(t) \tag{1}$$

In Eq. (1) $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, damping and stiffness matrices; $\mathbf{r}$ is the vector of externally applied loads, and $\ddot{\mathbf{u}}$, $\dot{\mathbf{u}}$ and $\mathbf{u}$ are the acceleration, velocity and displacement vectors.

The fourth order method is based on the fourth order Lagrange interpolation (Abramowitz and Stegun 1984, Kreyszig 1999) of the displacement $\mathbf{u}$ from time $(n-3)\Delta t$ to time $(n+1)\Delta t$ (n is the time interval). Exact differentiation of the interpolated variable yields (Souza and Moura 1997):

$$\dot{\mathbf{u}}^{t} = \frac{1}{2\Delta t}(\mathbf{u}^{t+\Delta t} - \mathbf{u}^{t-\Delta t}) - \frac{1}{12\Delta t}(3\mathbf{u}^{t+\Delta t} - 10\mathbf{u}^{t} + 12\mathbf{u}^{t-\Delta t} - 6\mathbf{u}^{t-2\Delta t} + \mathbf{u}^{t-3\Delta t}) \tag{2}$$

and

$$\ddot{\mathbf{u}}^{t} = \frac{1}{\Delta t^{2}}(\mathbf{u}^{t+\Delta t} - 2\mathbf{u}^{t} + \mathbf{u}^{t-\Delta t}) - \frac{1}{12\Delta t^{2}}(\mathbf{u}^{t+\Delta t} - 4\mathbf{u}^{t} + 6\mathbf{u}^{t-\Delta t} - 4\mathbf{u}^{t-2\Delta t} + \mathbf{u}^{t-3\Delta t}) \tag{3}$$

After the substitution of Eqs. (2) and (3) into Eq. (1), written at time t, the following equation arises:

$$\left(\frac{1}{4\Delta t}\mathbf{C} + \frac{11}{12\Delta t^{2}}\mathbf{M}\right)\mathbf{u}^{t+\Delta t} = \mathbf{r}^{t} - \left(-\frac{5}{3\Delta t^{2}}\mathbf{M} + \frac{5}{6\Delta t}\mathbf{C} + \mathbf{K}\right)\mathbf{u}^{t} - \left(\frac{1}{2\Delta t^{2}}\mathbf{M} - \frac{3}{2\Delta t}\mathbf{C}\right)\mathbf{u}^{t-\Delta t} - \left(\frac{1}{3\Delta t^{2}}\mathbf{M} + \frac{1}{2\Delta t}\mathbf{C}\right)\mathbf{u}^{t-2\Delta t} + \left(\frac{1}{12\Delta t^{2}}\mathbf{M} + \frac{1}{12\Delta t}\mathbf{C}\right)\mathbf{u}^{t-3\Delta t} \tag{4}$$

If velocity-dependent damping is negligible, Eq. (4) is reduced to:

$$\mathbf{u}^{t+\Delta t} = \frac{12\Delta t^{2}}{11}\mathbf{M}^{-1}\hat{\mathbf{r}} + \frac{1}{11}(20\mathbf{u}^{t} - 6\mathbf{u}^{t-\Delta t} - 4\mathbf{u}^{t-2\Delta t} + \mathbf{u}^{t-3\Delta t}) \tag{5}$$

where:

$$\hat{\mathbf{r}} = \mathbf{r}^{t} - \mathbf{K}\mathbf{u}^{t} \tag{6}$$

By assuming a lumped mass matrix, the system of equations represented by Eq. (5) above can be solved without any factorisation, i.e., only matrix multiplications are required to obtain the answer $\mathbf{u}^{t+\Delta t}$. On the other hand, if $\mathbf{C} \neq \mathbf{0}$ is a diagonal matrix, i.e., if Rayleigh damping (Bathe 1996) is assumed, displacements at time $(t + \Delta t)$ are still computed in an explicit way.

The computational implementation of the fourth order method follows the same steps of the central difference method. Note that both methods are appropriate for parallel computing codes.

The fourth order difference is fourth-order accurate. This is verified by initially expanding $\mathbf{u}$ in the vicinity of $(t + \Delta t)$ and $(t - \Delta t)$ by Taylor series, see expressions (7) and (8) below:

$$\mathbf{u}^{t+\Delta t} = \mathbf{u}^{t} + \Delta t \dot{\mathbf{u}}^{t} + \frac{\Delta t^2}{2}\ddot{\mathbf{u}}^{t} + \frac{\Delta t^3}{6}\dddot{\mathbf{u}}^{t} + \frac{\Delta t^4}{24}\mathbf{u}^{(iv)t} + \frac{\Delta t^5}{120}\mathbf{u}^{(v)t} + \frac{\Delta t^6}{720}\mathbf{u}^{(vi)t} + O(\Delta t^7) \tag{7}$$

and

$$\mathbf{u}^{t-\Delta t} = \mathbf{u}^{t} - \Delta t \dot{\mathbf{u}}^{t} + \frac{\Delta t^2}{2}\ddot{\mathbf{u}}^{t} - \frac{\Delta t^3}{6}\dddot{\mathbf{u}}^{t} + \frac{\Delta t^4}{24}\mathbf{u}^{(iv)t} - \frac{\Delta t^5}{120}\mathbf{u}^{(v)t} + \frac{\Delta t^6}{720}\mathbf{u}^{(vi)t} - O(\Delta t^7) \tag{8}$$

Subtracting Eq. (8) from Eq. (7) and solving to $\dot{\mathbf{u}}^{t}$ leads to:

$$\dot{\mathbf{u}}^{t} = \frac{\mathbf{u}^{t+\Delta t} - \mathbf{u}^{t-\Delta t}}{2\Delta t} - \frac{\Delta t^2}{6}\dddot{\mathbf{u}}^{t} - \frac{\Delta t^4}{120}\mathbf{u}^{(v)t} - \dots \tag{9}$$

Adding Eqs. (7) and (8) and solving to $\ddot{\mathbf{u}}^{t}$ leads to:

$$\ddot{\mathbf{u}}^{t} = \frac{\mathbf{u}^{t+\Delta t} - 2\mathbf{u}^{t} + \mathbf{u}^{t-\Delta t}}{\Delta t^2} - \frac{\Delta t^2}{12}\mathbf{u}^{(iv)t} - \frac{\Delta t^4}{360}\mathbf{u}^{(vi)t} - \dots \tag{10}$$

The comparison of expressions (9) and (10) with expressions (2) and (3), respectively, shows that only the terms containing Δt^4 and higher powers have been omitted from expressions (2) and (3). Consequently, the fourth order formulae, expressions (2) and (3), are fourth-order accurate.

It is important to note that the second term on the right-hand-side of Eq. (2) is precisely the third order derivative of the interpolated variable, that is:

$$\dddot{\mathbf{u}}^{t} = \frac{1}{2\Delta t^3}(3\mathbf{u}^{t+\Delta t} - 10\mathbf{u}^{t} + 12\mathbf{u}^{t-\Delta t} - 6\mathbf{u}^{t-2\Delta t} + \mathbf{u}^{t-3\Delta t}) \tag{11}$$

Additionally, the second term on the right-hand-side of Eq. (3) is precisely the fourth order derivative of the interpolated variable, that is:

$$\mathbf{u}^{(iv)t} = \frac{1}{\Delta t^4}(\mathbf{u}^{t+\Delta t} - 4\mathbf{u}^{t} + 6\mathbf{u}^{t-\Delta t} - 4\mathbf{u}^{t-2\Delta t} + \mathbf{u}^{t-3\Delta t}) \tag{12}$$

3. Stability analysis

In order to have an accurate dynamic response of the structure, all system equilibrium equations, represented by Eq. (1), must be integrated to high precision. The correct choice of the time-step length fulfils a capital role to achieve the desired accuracy in the numerical results, making them reliable. For this reason, it is necessary to search for a value of the time-step such that the fourth order method is stable. With this aim, initially the property of orthogonality of the eigenvectors is invoked, and only one typical row taken from the assembled system of equations (if **u** is the transformed vector), represented by Eq. (1), needs to be examined. This row is written as (note that no reference has been made to the row number or to the time interval):

$$\ddot{u} + 2\xi\omega\dot{u} + \omega^2 u = r \tag{13}$$

where ω is the natural frequency and ξ is the damping ratio, given by:

$$\xi = c/2m\omega \tag{14}$$

Next, the operator form (Bathe 1996, Weaver and Jonhston 1987) of the procedure is written as follows (for each row):

$$\hat{\mathbf{u}}^{t+\Delta t} = \mathbf{A}\hat{\mathbf{u}}^{t} + \mathbf{l}r^{t} \tag{15}$$

In Eq. (15) **A** represents the amplification matrix and the load operator is represented by the vector **l**. Vectors storing the solution quantities (only displacements for the present formulation) are represented by $\hat{\mathbf{u}}^{t+\Delta t}$ and $\hat{\mathbf{u}}^{t}$ and the load at time t is represented by $\mathbf{r}^t$. After substituting Eqs. (2) and (3) in Eq. (13), one has:

$$\begin{Bmatrix} \mathbf{u}^{t+\Delta t} \\ \mathbf{u}^{t} \\ \mathbf{u}^{t-\Delta t} \\ \mathbf{u}^{t-2\Delta t} \end{Bmatrix} = \mathbf{A}\begin{Bmatrix} \mathbf{u}^{t} \\ \mathbf{u}^{t-\Delta t} \\ \mathbf{u}^{t-2\Delta t} \\ \mathbf{u}^{t-3\Delta t} \end{Bmatrix} + \mathbf{l}r^{t} \tag{16}$$

Matrix **A** and vector **l** are given by:

$$\mathbf{A} = \begin{bmatrix} \left(\frac{5}{3}\beta - \frac{5}{3}\Delta t\kappa - \omega^2\Delta t^2\beta\right) & \left(-\frac{1}{2}\beta + 3\Delta t\kappa\right) & \left(-\frac{1}{3}\beta - \Delta t\kappa\right) & \left(\frac{1}{12}\beta + \frac{1}{6}\Delta t\kappa\right) \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{17}$$

and

$$\mathbf{l} = \begin{Bmatrix} \Delta t^2\beta \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{18}$$

In expressions (17) and (18), one has:

$$\beta = \frac{1}{(11/12) + (1/2)\xi\omega\Delta t};\kappa = \xi\omega\beta \tag{19}$$

If there is no loading, Eq. (13) simplifies to:

$$\hat{\mathbf{u}}^{t+\Delta t} = \mathbf{A}\hat{\mathbf{u}}^{t} \tag{20}$$

for free vibrational response.

Stability requires that the spectral radius of **A**, defined as:

$$\rho(\mathbf{A}) = \max|\lambda_i|, \quad {}_{i = 1, 2, \ldots} \tag{21}$$

is smaller than unity. In other words, stability depends on the eigenvalues (λ_i) of the approximation operator. The eigenvalues, by their turn, are obtained from the eigenvalue problem:

$$\mathbf{Au} = \lambda\mathbf{u} \tag{22}$$

By assuming no damping effects ($\xi = 0$), the eigenvalue problem corresponding to the fourth order method is written as:

$$\begin{bmatrix} \left(\frac{5}{3}\beta - \omega^2\Delta t^2\beta\right) & -\frac{1}{2}\beta & -\frac{1}{3}\beta & \frac{1}{12}\beta \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\mathbf{u} = \lambda\mathbf{u} \tag{23}$$

The characteristic polynomial of **A** arises from the condition below:

$$\begin{bmatrix} \left(\frac{5}{3}\beta - \omega^2\Delta t^2\beta - \lambda\right) & -\frac{1}{2}\beta & -\frac{1}{3}\beta & \frac{1}{12}\beta \\ 1 & -\lambda & 0 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 1 & -\lambda \end{bmatrix} = 0 \tag{24}$$

After expanding the determinant in Eq. (24), the following equation is obtained:

$$11\lambda^4 + (12\omega^2\Delta t^2 - 20)\lambda^3 + 6\lambda^2 + 4\lambda - 1 = 0 \tag{25}$$

The root of the characteristic polynomial (Eq. (25)) that satisfies the requirement expressed by Eq. (21) implies in the following condition:

$$\Delta t \le \frac{2}{\omega}\sqrt{\frac{2}{3}} \tag{26}$$

Hence, in order to guarantee stability, one must impose the following condition:

$$\Delta t_{crf} \le \frac{2}{\omega_{\max}}\sqrt{\frac{2}{3}} \tag{27}$$

In expression (27) $\omega_{\max}$ is the highest natural frequency of the problem.

The fourth order method is thus conditionally stable, as mentioned at the beginning of this work, and its critical time-step is 20 per cent lower than the critical time-step predicted by the central difference method (Bathe 1996, Cook *et al*. 1989, Weaver and Johnston 1987) ($\Delta t_{cr} = 2/\omega_{\max}$). However, as will be demonstrated, this fact is compensated by the high level of accuracy provided by the fourth order method in the numerical results.

4. D-BEM formulation

Before presenting the D-BEM formulation, a discussion concerning the use of static fundamental solution must be carried out. Naturally, part of the elegance of the BEM approach is lost when, in a time-domain analysis, the fundamental solution employed is not a time-dependent one. Besides, as the static fundamental solution does not fulfil the Sommerfeld radiation condition, this implies that infinite and semi-infinite domains cannot be treated by boundary discretizations only. However, in spite of these unfavorable aspects, a very simple approach is obtained if one holds the static fundamental solution and the domain discretization in the formulation. On the other hand, this formulation presents some advantage in elastoplastic analysis, since the part of the domain where inelastic variables are expected to occur needs to be discretized, see References (Carrer and Telles 1992, Hatzigeorgiou and Beskos 2001). The domain discretization can be avoided if an alternative approach, known by dual reciprocity (DR-BEM), is employed. However, it should be pointed out the dependence of the solution on the number of internal points (Partridge *et al*. 1992, Kontoni and Beskos 1993).

As mentioned earlier, both the D-BEM and DR-BEM formulations employ the Houbolt method as the time-marching scheme. Other time-marching schemes, such as the Newmark scheme (1959), largely employed in FEM approaches due to its stability and accuracy (see the discussion concerned with these topics in Bathe (1989)), fail when employed in D and DR-BEM formulations. Here appears another contribution of the present work, i.e., to present an alternative time-marching scheme that can be employed in the D-BEM (and perhaps DR-BEM) formulation, producing sufficiently accurate results.

A brief summary of the D-BEM formulation is given next. The starting integral equation is written as follows:

$$C_{ik}(\xi)u_k(\xi) = \int_\Gamma u^*_{ik}(\xi,X)p_k(X)d\Gamma(X) - \int_\Gamma p^*_{ik}(\xi,X)u_k(X)d\Gamma(X) - \rho\int_\Omega u^*_{ik}(\xi,X)\ddot{u}_k(X)d\Omega(X) \quad (28)$$

In Eq. (28) Γ is the boundary and Ω is the domain of the problem, X and ξ are collocation points referred to as field and source points, and ρ is the constant mass density. The fundamental solution $u^*_{ik}(\xi, X)$, also called Kelvin fundamental solution (Telles 1983), corresponds to a displacement at the field point X in k direction due to a unity load applied at the source point ξ in the i direction. The same interpretation is valid for the traction $p^*_{ik}(\xi, X)$. For plane strain problems, one has (note that for 2-D problems, $i, j = x, y$):

$$u^*_{ik}(\xi, X) = \frac{1}{8\pi G(1-\nu)}[(3-4\nu)\ln(1/r)\delta_{ik} + r_{,i}r_{,k}] \quad (29)$$

$$p^*_{ik}(\xi, X) = -\frac{1}{4\pi(1-\nu)r}\left[\frac{\partial r}{\partial n}((1-2\nu)\delta_{ik} + 2r_{,i}r_{,k}) - (1-2\nu)(r_{,i}n_k - r_{,k}n_i)\right] \quad (30)$$

where $r = r(\xi, X)$ is the distance between the source and field points and δ_{ik} is the Kronecker delta. In expression (29) G is the shear modulus, and ν is the Poisson ratio. In expression (30) n stands for the direction of the normal outward vector (n_k is its component in the k direction). Additionally:

$$r = (r_i r_i)^{1/2}$$

$$r_i = x_i(X) - x_i(\xi)$$

$$r_{,i} = \frac{\partial r}{\partial x_i(X)} = \frac{r_i}{r} \tag{31}$$

Plane strain expressions are valid for plane stress if the Poisson coefficient ν is replaced by $\bar{\nu} = \nu/(1+\nu)$.

The coefficient $C_{ik}(\xi)$ is obtained by geometric considerations. The reader is referred to Hartmann (1980) for additional details concerning its computation.

For computational purposes, the boundary was discretized with linear elements, whereas for the domain discretization triangular linear cells were employed. The application of the discretized version of Eq. (28) to all boundary nodes and internal points, see Reference (Carrer and Telles 1992), generates the following system of equations:

$$\mathbf{Hu} = \mathbf{Gp} - \mathbf{M\ddot{u}} \tag{32}$$

Due to the nature of the method, not only displacements but also tractions are unknowns to be determined at each time interval. For this reason, Eq. (32) should be written for the time $(t + \Delta t)$; this implies in adopting an approximation for the acceleration at the time $(t + \Delta t)$. This approximation is written below:

$$\ddot{\mathbf{u}}^{t+\Delta t} = \frac{1}{12\Delta t^2}(35\mathbf{u}^{t+\Delta t} - 104\mathbf{u}^{t} + 114\mathbf{u}^{t-\Delta t} - 56\mathbf{u}^{t-2\Delta t} + 11\mathbf{u}^{t-3\Delta t}) \tag{33}$$

However, this procedure did not provide the expected good results, as demonstrated in Fig. 8 from the first example and in Fig. 15 from the second example. Then, another effort was then made and consisted in writing Eq. (32) with a little lack of equilibrium, i.e., whereas the variables **u** and **p** are written at $(t + \Delta t)$, the acceleration is substituted by its approximation at the time t. Naturally, if sufficiently small time steps are adopted it is expected that the lack of equilibrium becomes smaller and acceptable. Taking this into account, after the substitution of Eq. (3) into Eq. (32) the following expression arises:

$$\mathbf{Hu}^{t+\Delta t} = \mathbf{Gp}^{t+\Delta t} - \frac{1}{\Delta t^2}\mathbf{M}\left(\frac{11}{12}\mathbf{u}^{t+\Delta t} - \frac{5}{3}\mathbf{u}^{t} + \frac{1}{2}\mathbf{u}^{t-\Delta t} + \frac{1}{3}\mathbf{u}^{t-2\Delta t} - \frac{1}{12}\mathbf{u}^{t-3\Delta t}\right) \tag{34}$$

After rearranging the equation, one has:

$$\left(\Delta t^2\mathbf{H} + \frac{11}{12}\mathbf{M}\right)\mathbf{u}^{t+\Delta t} = \Delta t^2\mathbf{Gp}^{t+\Delta t} - \mathbf{M}\left(-\frac{5}{3}\mathbf{u}^{t} + \frac{1}{2}\mathbf{u}^{t-\Delta t} + \frac{1}{3}\mathbf{u}^{t-2\Delta t} - \frac{1}{12}\mathbf{u}^{t-3\Delta t}\right) \tag{35}$$

In a simplified notation, one has:

$$\bar{\mathbf{H}}\mathbf{u}^{t+\Delta t} = \bar{\mathbf{G}}\mathbf{p}^{t+\Delta t} + \mathbf{q}^{h} \tag{36}$$

The imposition of the boundary conditions leads to the following system:

$$\mathbf{B}\mathbf{y}^{t+\Delta t} = \mathbf{q}^{t+\Delta t} + \mathbf{q}^{h} \tag{37}$$

in which the vector $\mathbf{y}^{t+\Delta t}$ stores the unknown variables (displacements and tractions) and the vector $\mathbf{q}^{t+\Delta t}$ stores the contributions of the boundary conditions. The time-history contribution is furnished by vector $\mathbf{q}^{h}$.

5. Examples

In the examples presented in this section, the accuracy of the fourth order method is verified. For the D-BEM, the dimensionless parameter (Mansur 1983) $\beta_{\Delta t} = \frac{c_d \Delta t}{l}$ provides a good indication of the time-step length (c_d is the primary wave propagation velocity and l is the maximum element length used in the boundary discretization). FEM analyses employed $\Delta t = 0.06$. For the D-BEM analyses, $\beta_{\Delta t} = (2/15)$. In the first example, the FEM solution furnished by the Newmark scheme (with $\beta = 0.25$ and $\gamma = 0.5$) will be assumed as the analytical one. In the second example, the numerical results are compared with the available analytical solution.

5.1 Deep beam simply supported

This plane stress example consists of a simply supported deep beam submitted to a Heaviside-type uniform load $p_y = \bar{p}H(t-0)$, as depicted in Fig. 1. FEM mesh for half of the beam consists of 128 four-node isoparametric elements and is depicted in Fig. 2. D-BEM analysis employed 48 linear elements and 256 triangular linear cells for the boundary and domain discretizations, respectively, see Fig. 3. Although not indicated in Figs. 2 and 3, the essential boundary conditions were taken into account as follows: the displacements on the left side were restricted in the vertical direction; on the right side, in order to simulate the symmetry of the problem, the displacements were restricted in the horizontal direction. The material parameters are: $E = 100.0$, $\nu = 1/3$ and $\rho = 1.5$. FEM vertical displacement results at point $A(0,0)$ depicted in Fig. 2, provided by the fourth order method, are compared with the corresponding ones provided by the Newmark (Bathe 1996, Cook 1989, Newmark 1959) method in Fig. 4. In Fig. 5 the comparison is made between the results

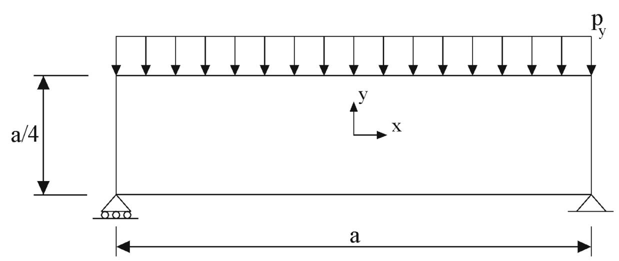


Fig. 1 Deep beam simply supported: geometry and boundary conditions

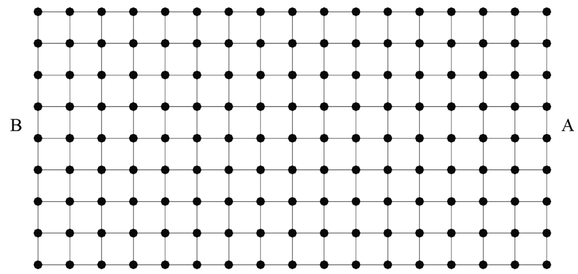


Fig. 2 FEM mesh

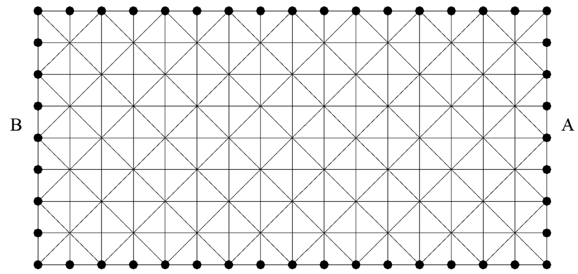


Fig. 3 D-BEM mesh: boundary elements and internal cells

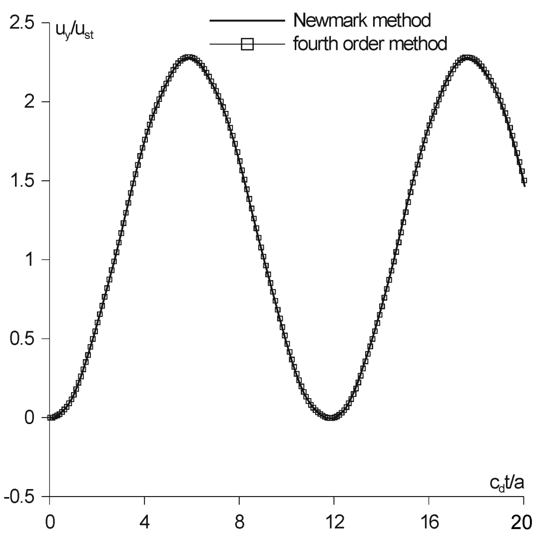


Fig. 4 Deep beam: displacement component u_y at node A(0, 0) for FEM analyses with Newmark and fourth order methods

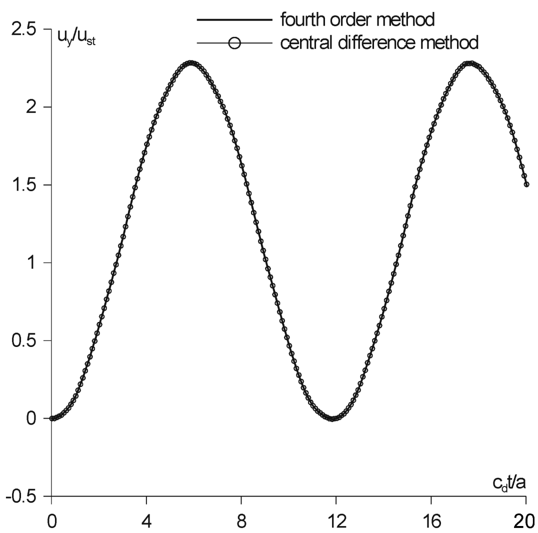


Fig. 5 Deep beam: displacement component u_y at node A(0, 0) for FEM analyses with fourth order and central difference methods

provided by the fourth order and central difference methods. A good agreement is observed between the results in Figs. 4 and 5. The next step is to show the D-BEM results. It is instructive to show how the Newmark scheme (with the parameters $\beta = 0.25$ and $\gamma = 0.5$) fails completely when used in a D-BEM formulation. This is done in Fig. 6. The fourth order method, on the order hand, provided the reliable results presented in Fig. 7, demonstrating its accuracy and applicability with the D-BEM

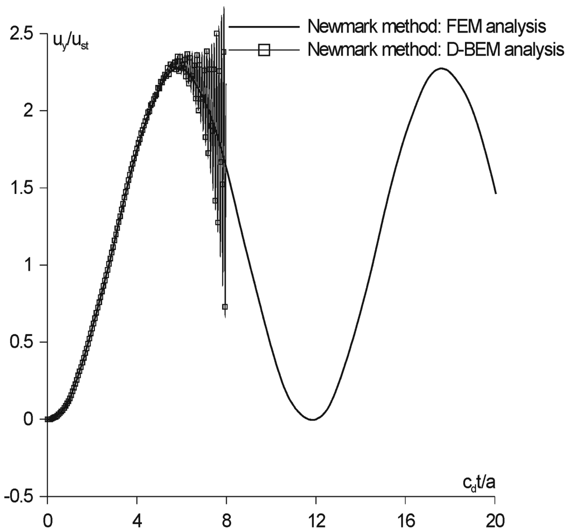


Fig. 6 Deep beam: displacement component u_y at node A(0, 0) for FEM and D-BEM analyses with Newmark method

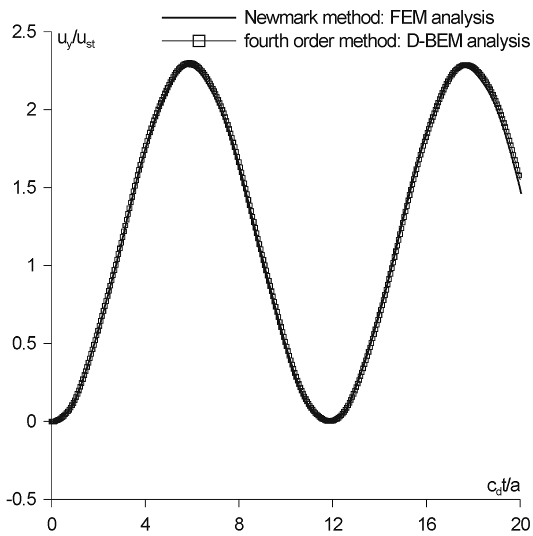


Fig. 7 Deep beam: displacement component u_y at node A(0, 0) for FEM (Newmark) and D-BEM (fourth order method) analyses

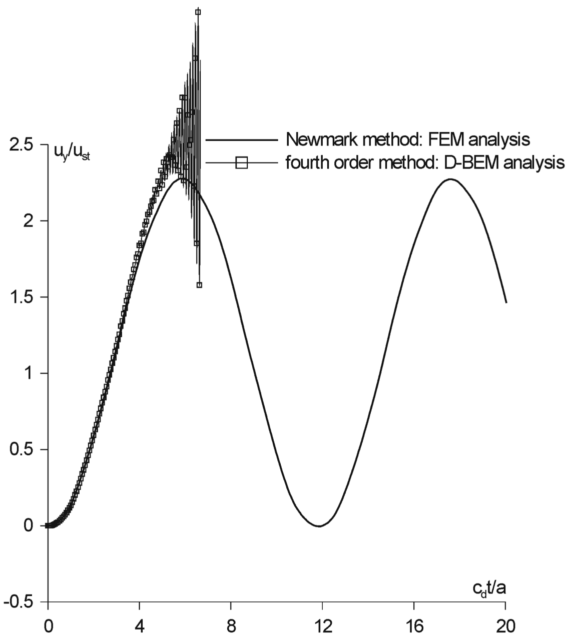


Fig. 8 Deep beam: displacement component u_y at node A(0, 0) for FEM (Newmark) and D-BEM (fourth order method with the acceleration approximated at $t + \Delta t$) analyses

formulation. As mentioned previously, the implementation of the approximation described by Eq. (33) fails completely, producing the results depicted in Fig. 8 (obtained with $\beta_{\Delta t} = 1/3$). In Figs. 4-8, u_{st} is the static vertical displacement, computed according to the classical beam theory; $u_{st} = 0.024$

5.2 One-dimensional rod

This example simulates a one-dimensional rod under the Heaviside-type forcing function: $p_x = \bar{p}H(t-0)$. The boundary conditions and the geometry are depicted in Fig. 9. FEM and D-BEM meshes are the same of the previous example. In order to simulate the one-dimensional problem, the Poisson coefficient is taken as null. For the other material parameters, the following values were adopted: $E = 100.0$ and $\rho = 1.0$.

FEM results corresponding to the horizontal displacement at point $A(a, a/4)$, furnished by the fourth order and central difference methods, are compared with the analytical solution in Figs. 10(a) and 10(b). The numerical answers are fairly good, demonstrating the accuracy of both integration schemes. This conclusion is no longer valid for the results corresponding to the traction component p_x at point $B(0, a/4)$: the results provided by the fourth order method are more stable than the corresponding ones provided by the central difference method for the critical time-step, as can be

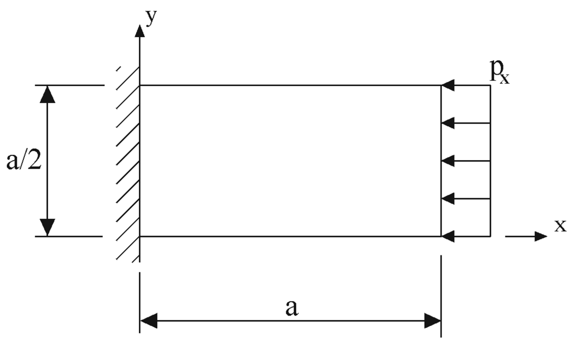


Fig. 9 One-dimensional rod: geometry and boundary conditions

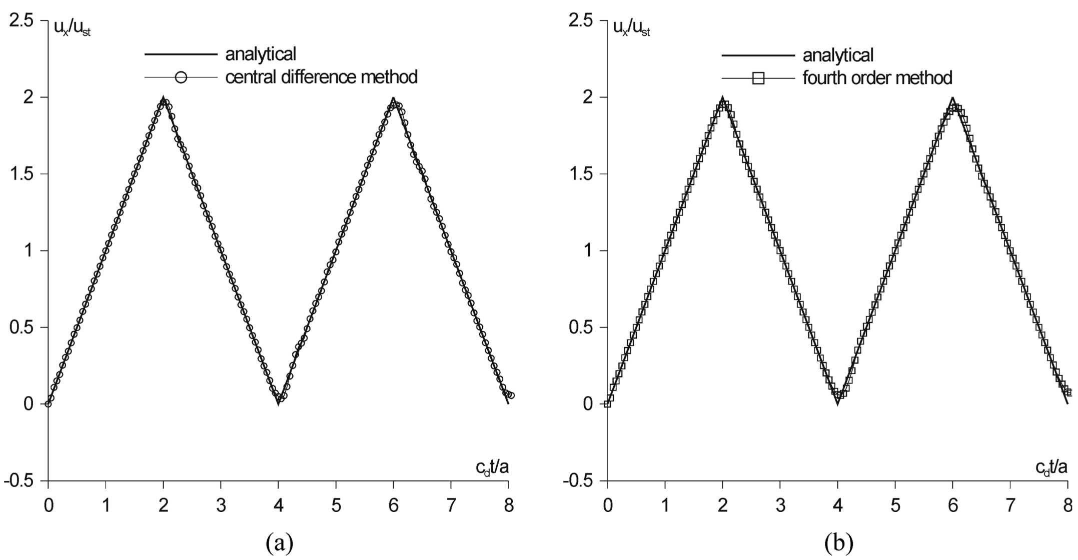
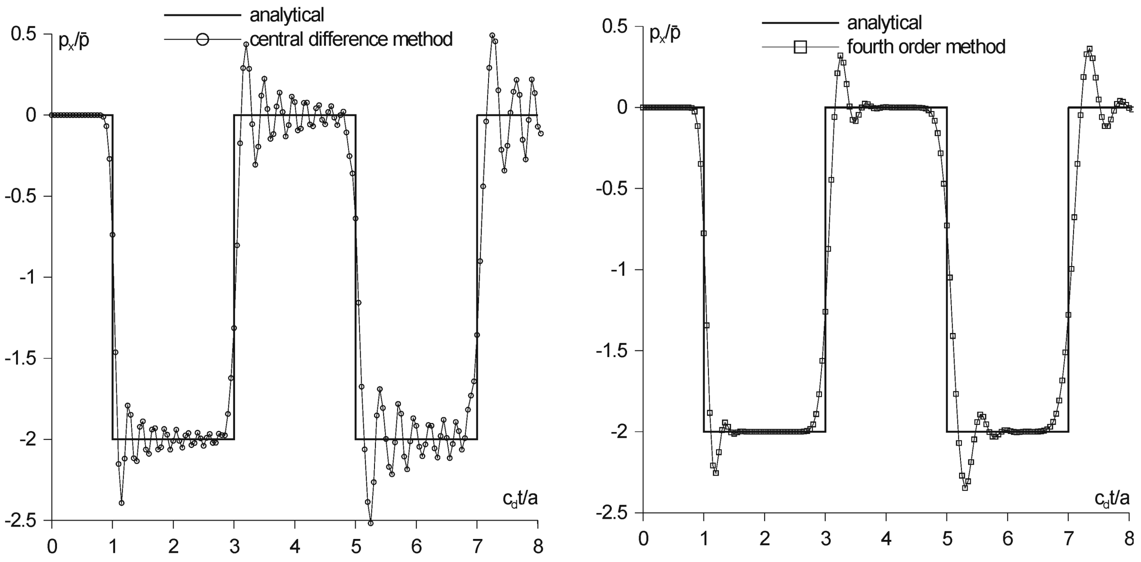


Fig. 10 One-dimensional rod: displacement component u_x at node $A(a, a/4)$ for FEM analyses with: (a) central difference method and (b) fourth order method



(a) (b)

Fig. 11 One-dimensional rod: traction component p_x at node $B(0, a/4)$ for FEM analyses with: (a) central difference method and (b) fourth order method

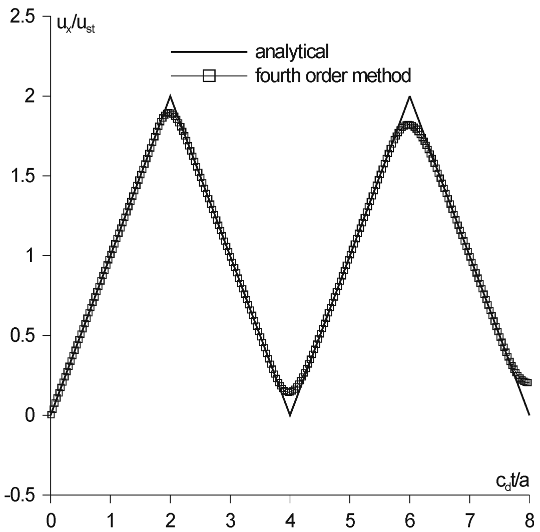


Fig. 12 One-dimensional rod: displacement component u_x at node $A(a, a/4)$ for D-BEM analysis with fourth order method

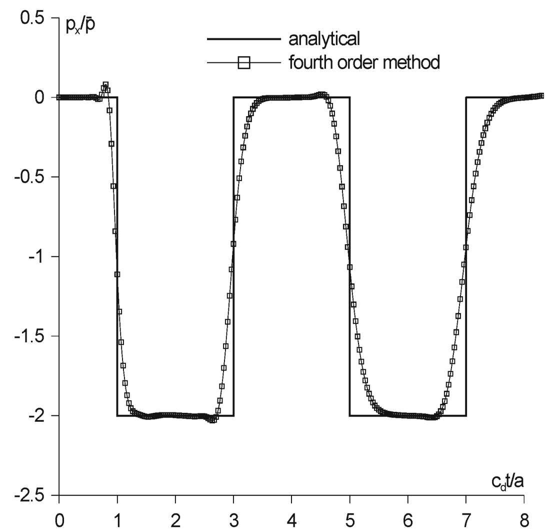


Fig. 13 One-dimensional rod: traction component p_x at node $B(0, a/4)$ for D-BEM analysis with fourth order method

observed in Figs. 11(a) and 11(b).

The D-BEM results, Fig. 12 for displacement u_x and Fig. 13 for traction p_x, are very good. A numerical damping is observed in Figs. 12 and 13. This feature is also observed in analyses carried out with the Houbolt scheme (Carrer and Telles 1992) (see Fig. 14, which presents Houbolt results

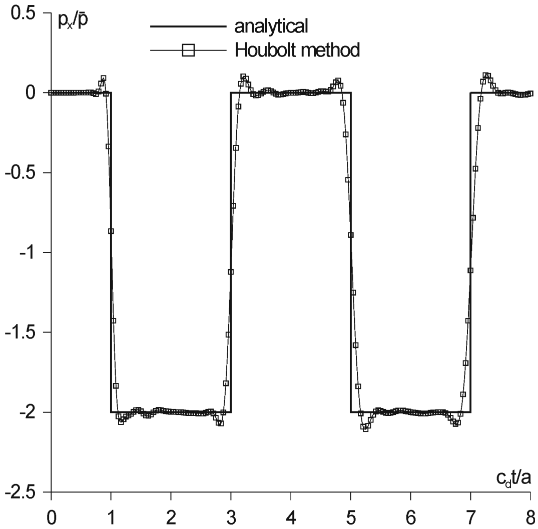


Fig. 14 One-dimensional rod: traction component p_x at node $B(0, a/4)$ for D-BEM analysis with Houbolt method

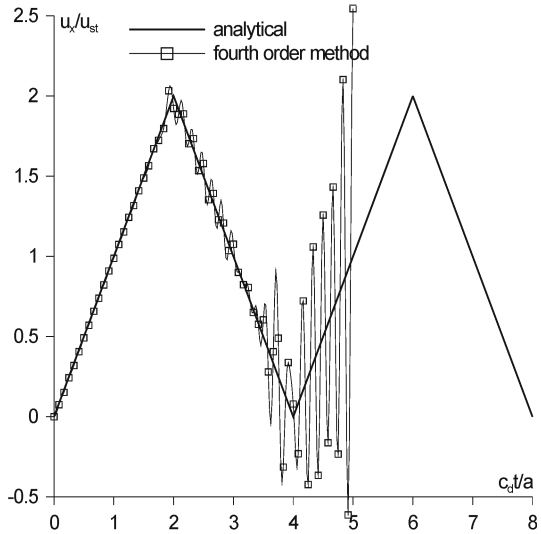


Fig. 15 One-dimensional rod: displacement component u_x at node $A(a, a/4)$ for D-BEM analysis with fourth order method with the acceleration approximated at $t + \Delta t$

obtained with $\beta_{\Delta t} = 1/3$). Fig. 15 presents the results corresponding to the implementation of the approximation described by Eq. (33) ($\beta_{\Delta t} = 1/3$). Once more, unreliable results were obtained.

In Figs. 10, 12 and 15, u_{st} is the static horizontal displacement; $u_{st} = 0.12$.

6. Conclusions

This work presents a direct integration method for step-by-step elastodynamic analysis. The approximations are based on fourth order Lagrange interpolation of the displacements, and the approach is referred to along the text as fourth order method. The FEM explicit version of the method was successfully implemented: this is the first objective of this work, i.e., to present an explicit integration method that produces more accurate results than those provided by the central difference method. The method is also applicable to the D-BEM approach, providing reliable results. This is the second objective of this work, i.e., to present an alternative to the Houbolt integration scheme that could be adopted in elastodynamic analysis by the D-BEM approach. It is the author's opinion that further research work concerning the fourth order method can still be done in the finite elements and in the boundary elements areas. The implementation of the implicit version of the method for the FEM is one possibility. Another possibility is the application of the method to the DR-BEM formulation.

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