Structural Engineering and Mechanics, Vol. 17, No. 1 (2004) 87-97
DOI: http://dx.doi.org/10.12989/sem.2004.17.1.087

Incompatible 3-node interpolation for gradient-dependent plasticity

G. Chen† and G. Baker‡

Faculty of Engineering & Surveying, The University of Southern Queensland, Toowoomba, QLD 4350, Australia

(Received May 15, 2003, Accepted October 9, 2003)

Abstract. In gradient-dependent plasticity theory, the yield strength depends on the Laplacian of an equivalent plastic strain measure (hardening parameter), and the consistency condition results in a differential equation with respect to the plastic multiplier. The plastic multiplier is then discretized in addition to the usual discretization of the displacements, and the consistency condition is solved simultaneously with the equilibrium equations. The disadvantage is that the plastic multiplier requires a Hermitian interpolation that has four degrees of freedom at each node. Instead of using a Hermitian interpolation, in this article, a 3-node incompatible (trigonometric) interpolation is proposed for the plastic multiplier. This incompatible interpolation uses only the function values of each node, but it is continuous across element boundaries and its second-order derivatives exist within the elements. It greatly reduces the degrees of freedom for a problem, and is shown through a numerical example on localization to yield good results.

Key words: gradient-dependent plasticity; incompatible element; trigonometric interpolation; strain localization.

1. Introduction

Classical continuum plasticity, which does not incorporate an internal length, suffers from pathological mesh dependence when strain-softening models are employed in numerical analyses. It leaves the size of the localization zone unspecified. In order to introduce a localization limiter (Belytschko and Lasry 1989, de Borst *et al.* 1993), Cosserat plasticity theory (Mühlhaus 1989, de Borst 1991, de Borst and Sluys 1991, de Borst 1993) and gradient-dependent plasticity theory (de Borst and Mühlhaus 1992, Pamin 1994, Meftah and Reynouard 1998) have been suggested as methods to incorporate an internal length.

In gradient-dependent plasticity theory, the yield strength depends not only on an equivalent plastic strain measure (hardening parameter), but also on the Laplacian thereof. The consistency condition results in a differential equation with respect to the plastic multiplier, instead of an algebraic equation as in conventional plasticity. Moreover, in gradient-dependent plasticity, the plastic multiplier cannot be decided at a local level (Gauss integration point) without referring to

† Research fellow
‡ Dean

other material points. In order to solve the consistency condition, de Borst and co-workers (de Borst and Mühlhaus 1992, Pamin 1994, de Borst *et al.* 1995) introduce the weak forms of the equilibrium equations and yield condition, and discretize the plastic multiplier as well as the displacements. The consistency condition is then solved simultaneously with the equilibrium equations.

The yield criterion depends on the Laplacian of the plastic strain measure that is related to (or equal to, for von Mises criterion) the plastic multiplier. Thus, in conventional formulation, a bending plate element is used to discretize the plastic multiplier to guarantee the existence of its second-order derivatives. At each node, four degrees of freedom (λ, λ_x, λ_y, λ_{xy}) are required, where for example $\lambda_x = \partial\lambda/\partial x$. Hence, it significantly increases the total degrees of freedom of the problem concerned. The interpolation (Hermitian interpolation) not only has a continuous function but also continuous first-order derivatives, which are superfluous. For both classical plasticity and gradient-dependent plasticity, the plastic multiplier is decided by different rules in elastic (equal to zero) and plastic zones, i.e., it is not a smooth function. Hence, it is not necessary to enforce the continuity of its derivatives.

Several improvements were proposed for gradient-dependent plasticity. Vardoulakis *et al.* (1992) approximated the plastic multiplier at a local level, alternatively, Zervos *et al.* (2001) proposed a unified theory that allows the formulation of the rate boundary value problem in terms of displacements only. Li and Cescotto (1996) presented a finite element scheme in which the Laplacian of the effective plastic strain at a quadrature point is evaluated by using the values of the effective plastic strains at neighboring quadrature point.

The aim of this paper is to propose an incompatible mode (shape functions) for the plastic multiplier, and to test the accuracy and efficiency of the formulation. The results of the one-dimension problems (de Borst and Mühlhaus 1992) shows that plastic strain has a bell-like shape across an element. Hence, in this paper, trigonometric “bell-shaped” functions are adopted as interpolation functions instead of Hermitian polynomials. Importantly, the proposed interpolation only involves the function values of each node, λ, and not the derivatives. It is continuous across element boundaries and both its first and second-order derivatives are continuous within the element.

The paper begins with a review of the formulation of gradient-dependent plasticity, together with its finite element discretization, highlighting the main features for consideration. The main contribution, namely the finite element implementation of the “bell-like” incompatible 3-node interpolation, follows. The performance of gradient-dependent plasticity with these elements is discussed through a classic localization problem.

2. Formulation of gradient-dependent theory

This section briefly states the standard formulation of gradient-dependent plasticity that was found in de Borst and Mühlhaus (1992), de Borst *et al.* (1995), and Pamin (1994).

Gradient-dependent plasticity theory is identified by the dependency of yield function on the Laplacian of the hardening parameter, i.e.,

$$f(\boldsymbol{\sigma}, k, \nabla^2 k) = 0 \tag{1}$$

where $\boldsymbol{\sigma}$ is stress tensor, k, the hardening parameter, and $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. This makes the consistency condition, $\dot{f} = 0$, a differential equation with respect to dk:

$$\boldsymbol{n}^T d\boldsymbol{\sigma} - h \cdot d\lambda + g \cdot \nabla^2(dk) = 0 \tag{2}$$

where $\boldsymbol{n} = \partial f/\partial\boldsymbol{\sigma}$, $h = -(\dot{k}/\dot{\lambda}) \cdot \partial f/\partial k$, $g = \partial f/\partial(\nabla^2 k)$, and λ is the plastic multiplier. Generally, the hardening parameter, k, and the plastic multiplier, λ, are related by $dk = \eta \cdot d\lambda$, with η being a constant depending on the yield function. For a von Mises yield function, it has been shown that $dk = d\lambda$ (Owen and Hinton 1980).

Eq. (2) results in a differential equation of plastic multiplier, $d\lambda$, which cannot be solved directly. de Borst *et al.* (1995) solved the equilibrium equations and the yield condition (or equivalently the consistency condition) simultaneously by the finite element method. For this purpose, it is necessary to employ a weak satisfaction of the yield condition and to discretize the plastic multiplier, in addition to the discretization of the displacement field. The weak satisfaction of the equilibrium equations is:

$$\int_V \delta\boldsymbol{u}^T(\boldsymbol{L}^T\boldsymbol{\sigma}_{j+1})dV = 0 \tag{3}$$

where the subscript $j + 1$ refers to the current iteration, and $\boldsymbol{L}^T\boldsymbol{\sigma}_{j+1} = 0$ represents equilibrium equations.

Moreover, unlike conventional plasticity, the yield criterion is satisfied in a distributed sense, that is:

$$\int_{V_\lambda} \delta\lambda f(\boldsymbol{\sigma}_{j+1}, \lambda_{j+1}, \nabla^2\lambda_{j+1})dV = 0 \tag{4}$$

where V_λ is the volume that has plastic strain developed in the current load step.

In the field Eq. (3) and yield condition (4), there appear the first-order derivatives of the displacements and the second-order derivatives of the plastic multiplier. Therefore, for the displacement field $\boldsymbol{u}$, the standard interpolation functions, which are assembled in $\boldsymbol{N}$, are used; for λ, a set of incompatible interpolation functions contained in $\boldsymbol{h}$ is proposed. That is

$$\boldsymbol{u} = \boldsymbol{N}\boldsymbol{a} \qquad \lambda = \boldsymbol{h}^T\boldsymbol{\Lambda} \tag{5}$$

where $\boldsymbol{a}$ is a nodal displacement vector, and $\boldsymbol{\Lambda}$ a vector of nodal values of the plastic multiplier. The discretization of strains has the form

$$\boldsymbol{\varepsilon} = \boldsymbol{B}\boldsymbol{a} \tag{6}$$

where $\boldsymbol{B} = \boldsymbol{L}\boldsymbol{N}$, and the discretization for the gradient and the Laplacian of the plastic multiplier are:

$$\nabla\lambda = \boldsymbol{q}^T\boldsymbol{\Lambda} \qquad \nabla^2\lambda = \boldsymbol{p}^T\boldsymbol{\Lambda} \tag{7}$$

where $\boldsymbol{q}^T = \nabla\boldsymbol{h}^T, \boldsymbol{p}^T = \nabla^2(\boldsymbol{h}^T)$.

Substitution of (5), (6) and (7) into Eqs. (3) and (4) results in the following set of algebraic equations for admissible variations of $\delta\boldsymbol{a}$ and $\delta\boldsymbol{\Lambda}$:

$$\begin{bmatrix} \boldsymbol{K}_{aa} & \boldsymbol{K}_{a\lambda} \\ \boldsymbol{K}_{\lambda a} & \boldsymbol{K}_{\lambda\lambda} \end{bmatrix} \begin{bmatrix} d\boldsymbol{a} \\ d\boldsymbol{\Lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_e + \boldsymbol{f}_a \\ \boldsymbol{f}_\lambda \end{bmatrix} \tag{8}$$

where the sub-matrix (de Borst *et al.* 1995) are defined as:

$$\boldsymbol{K}_{aa} = \int_V \boldsymbol{B}^T \boldsymbol{D}_e \boldsymbol{B} dV$$

$$\boldsymbol{K}_{a\lambda} = -\int_V \boldsymbol{B}^T \boldsymbol{D}_e \boldsymbol{n} \boldsymbol{h}^T dV \qquad \boldsymbol{K}_{\lambda a} = -\int_V \boldsymbol{h} \boldsymbol{n}^T \boldsymbol{D}_e \boldsymbol{B} dV$$

$$\boldsymbol{K}_{\lambda\lambda} = \int_V [(h + \boldsymbol{n}^T \boldsymbol{D}_e \boldsymbol{n}) \boldsymbol{h}\boldsymbol{h}^T - g \boldsymbol{h} \boldsymbol{p}^T] dV \tag{9}$$

and the force vectors are:

$$\boldsymbol{f}_e = \int_S \boldsymbol{N}^T \boldsymbol{t}_{j+1} dS \qquad \boldsymbol{f}_a = -\int_V \boldsymbol{B}^T \boldsymbol{\sigma}_j dV$$

$$\boldsymbol{f}_\lambda = \int_V f(\boldsymbol{\sigma}_j, \lambda_j, \nabla^2 \lambda_j) \boldsymbol{h} dV \tag{10}$$

where $\boldsymbol{t}$ is the tractions on the stress boundary.

Gradient plasticity lacks a stress update algorithm, the residual forces are introduced into the global iterations, so it requires many more iterations than classical plasticity.

3. Imcompatible 3-node elements

3.1 General properties

For gradient dependent plasticity theory, both the displacement field and the plastic multiplier need to be discretized. For the displacements, the interpolations are the same as in the normal elements. For the plastic multiplier, the yield function requires the existence of $\nabla^2 \lambda$. In order to guarantee the existence of $\nabla^2 \lambda$, de Borst *et al.* (1995) and Pamin (1994) use the C^1-continuous interpolation. Thus at each node, four degrees of freedom (λ, λ_x, λ_y, λ_{xy}) for the plastic multiplier are required in addition to the normal displacement interpolation.

Considering that λ is decided by different rules in the elastic zone (equal to zero) and in the plastic zone, it is not a smooth function and does not have continuous derivatives. Thus, only the existence of its second-order derivatives but not its continuity is needed to enforce within each element. In this article, the standard 3-node interpolations (Zienkiewicz and Taylor 2000) are used for displacements and geometrical coordinates, new shape functions are proposed for the interpolation of the plastic multiplier. The interpolations are defined in the local area coordinate system.

The area coordinate is defined with the aid of Fig. 1. For any point, P, within a triangular element 123, its three area coordinates are defined as:

$$L_1 = \frac{\Delta P23}{\Delta 123} \qquad L_2 = \frac{\Delta P31}{\Delta 123} \qquad L_3 = \frac{\Delta P12}{\Delta 123} = 1 - L_1 - L_2 \tag{11}$$

where $\Delta P23$, for example, is the area of the triangle $P23$.

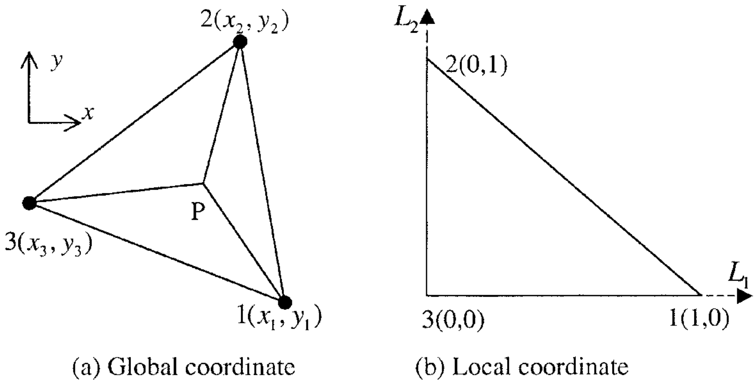

Fig. 1 Triangular element

Thus, for any point, after knowing its global coordinates (x, y), its area coordinates can be obtained. The transformation between the global coordinates and the area coordinates is linear, which can be written as

$$x = \sum_{i=1}^{3} x_i L_i \qquad y = \sum_{i=1}^{3} y_i L_i \tag{12}$$

The interpolations for the normal displacements and the plastic multiplier are expressed as:

$$u = \sum_{i=1}^{3} u_i L_i \qquad v = \sum_{i=1}^{3} v_i L_i$$

$$\lambda = \sum_{i=1}^{3} \lambda_i \tilde{N}_i \tag{13}$$

in which

$$\tilde{N}_1 = 1 - \sin^2\left(\frac{\pi}{2}L_2\right) - \sin^2\left(\frac{\pi}{2}L_3\right)$$

$$\tilde{N}_2 = 1 - \sin^2\left(\frac{\pi}{2}L_3\right) - \sin^2\left(\frac{\pi}{2}L_1\right)$$

$$\tilde{N}_3 = 1 - \sin^2\left(\frac{\pi}{2}L_1\right) - \sin^2\left(\frac{\pi}{2}L_2\right) \tag{14}$$

By applying $L_3 = 1 - L_1 - L_2$, they have the forms:

$$\tilde{N}_1(L_1, L_2) = 1 - \sin^2\left(\frac{\pi}{2}L_2\right) - \cos^2\left[\frac{\pi}{2}(L_1 + L_2)\right]$$

$$\tilde{N}_2(L_1, L_2) = 1 - \sin^2\left(\frac{\pi}{2}L_1\right) - \cos^2\left[\frac{\pi}{2}(L_1 + L_2)\right]$$

$$\tilde{N}_3(L_1, L_2) = 1 - \sin^2\left(\frac{\pi}{2}L_1\right) - \sin^2\left(\frac{\pi}{2}L_2\right) \tag{15}$$

It is easy to verify that $\tilde{N}_1(L_1, L_2) = 0$ on boundary 23 for which $L_1 = 0$, similarly $\tilde{N}_2(L_1, L_2) = 0$ on boundary 31 and $\tilde{N}_3(L_1, L_2) = 0$ on boundary 12. Hence, on each boundary λ is only related to the values of the two nodes of the concerned boundary. This property guarantees that λ is continuous across the element boundaries. Their first and second-order derivatives, which are required in the following context, are:

$$\begin{bmatrix} \dfrac{\partial \tilde{N}_1}{\partial L_1} & \dfrac{\partial \tilde{N}_2}{\partial L_1} & \dfrac{\partial \tilde{N}_3}{\partial L_1} \\ \dfrac{\partial \tilde{N}_1}{\partial L_2} & \dfrac{\partial \tilde{N}_2}{\partial L_2} & \dfrac{\partial \tilde{N}_3}{\partial L_2} \end{bmatrix} =$$

$$\frac{\pi}{2}\begin{bmatrix} \sin\pi(L_1 + L_2) & -\sin\pi L_1 + \sin\pi(L_1 + L_2) & -\sin\pi L_1 \\ -\sin\pi L_2 + \sin\pi(L_1 + L_2) & \sin\pi(L_1 + L_2) & -\sin\pi L_2 \end{bmatrix} \tag{16}$$

and

$$\begin{bmatrix} \dfrac{\partial^2 \tilde{N}_1}{\partial L_1^2} & \dfrac{\partial^2 \tilde{N}_2}{\partial L_1^2} & \dfrac{\partial^2 \tilde{N}_3}{\partial L_1^2} \\ \dfrac{\partial^2 \tilde{N}_1}{\partial L_1 \partial L_2} & \dfrac{\partial^2 \tilde{N}_2}{\partial L_1 \partial L_2} & \dfrac{\partial^2 \tilde{N}_3}{\partial L_1 \partial L_2} \\ \dfrac{\partial^2 \tilde{N}_1}{\partial L_2^2} & \dfrac{\partial^2 \tilde{N}_2}{\partial L_2^2} & \dfrac{\partial^2 \tilde{N}_3}{\partial L_2^2} \end{bmatrix} =$$

$$\frac{\pi^2}{2}\begin{bmatrix} \cos\pi(L_1 + L_2) & -\cos\pi L_1 + \cos\pi(L_1 + L_2) & -\cos\pi L_1 \\ \cos\pi(L_1 + L_2) & \cos\pi(L_1 + L_2) & 0 \\ -\cos\pi L_2 + \cos\pi(L_1 + L_2) & \cos\pi(L_1 + L_2) & -\cos\pi L_2 \end{bmatrix} \tag{17}$$

3.2 Laplacian

The yield criterion requires the Laplacian of λ in the global coordinate system, which can be obtained by finding the global derivatives in time-honored fashion using the Jacobian and local derivatives:

$$\begin{aligned} \frac{\partial \lambda}{\partial L_1} &= \frac{\partial \lambda}{\partial x}\frac{\partial x}{\partial L_1} + \frac{\partial \lambda}{\partial y}\frac{\partial y}{\partial L_1} \\ \frac{\partial \lambda}{\partial L_2} &= \frac{\partial \lambda}{\partial x}\frac{\partial x}{\partial L_2} + \frac{\partial \lambda}{\partial y}\frac{\partial y}{\partial L_2} \end{aligned} \tag{18}$$

whence,

$$\begin{Bmatrix} \dfrac{\partial\lambda}{\partial x} \\ \dfrac{\partial\lambda}{\partial y} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial L_1} & \dfrac{\partial y}{\partial L_1} \\ \dfrac{\partial x}{\partial L_2} & \dfrac{\partial y}{\partial L_2} \end{bmatrix}^{-1} \begin{Bmatrix} \dfrac{\partial\lambda}{\partial L_1} \\ \dfrac{\partial\lambda}{\partial L_2} \end{Bmatrix} \tag{19}$$

The real objective is to find the Laplacian from the second-order global derivatives. By differentiating the first equation of (18) with respect to L_1, the second equation of (18) with respect to L_2 and either with the cross derivative, three conditions are obtained from which the three second-order derivatives $(\partial^2\lambda/\partial x^2, \partial^2\lambda/\partial x\partial y, \partial^2\lambda/\partial y^2)$ can be solved. The result summarized in matrix format is:

$$\begin{Bmatrix} \dfrac{\partial^2\lambda}{\partial L_1^2} \\ \dfrac{\partial^2\lambda}{\partial L_1\partial L_2} \\ \dfrac{\partial^2\lambda}{\partial L_2^2} \end{Bmatrix} = \begin{bmatrix} \left(\dfrac{\partial x}{\partial L_1}\right)^2 & 2\left(\dfrac{\partial x}{\partial L_1}\dfrac{\partial y}{\partial L_1}\right) & \left(\dfrac{\partial y}{\partial L_1}\right)^2 \\ \left(\dfrac{\partial x}{\partial L_1}\dfrac{\partial x}{\partial L_2}\right) & \left(\dfrac{\partial x}{\partial L_1}\dfrac{\partial y}{\partial L_2} + \dfrac{\partial x}{\partial L_2}\dfrac{\partial y}{\partial L_1}\right) & \left(\dfrac{\partial y}{\partial L_1}\dfrac{\partial y}{\partial L_2}\right) \\ \left(\dfrac{\partial x}{\partial L_2}\right)^2 & 2\left(\dfrac{\partial x}{\partial L_2}\dfrac{\partial y}{\partial L_2}\right) & \left(\dfrac{\partial y}{\partial L_2}\right)^2 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial^2\lambda}{\partial x^2} \\ \dfrac{\partial^2\lambda}{\partial x\partial y} \\ \dfrac{\partial^2\lambda}{\partial y^2} \end{Bmatrix} + \begin{bmatrix} \dfrac{\partial^2 x}{\partial L_1^2} & \dfrac{\partial^2 y}{\partial L_1^2} \\ \dfrac{\partial^2 x}{\partial L_1\partial L_2} & \dfrac{\partial^2 y}{\partial L_1\partial L_2} \\ \dfrac{\partial^2 x}{\partial L_2^2} & \dfrac{\partial^2 y}{\partial L_2^2} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial\lambda}{\partial x} \\ \dfrac{\partial\lambda}{\partial y} \end{Bmatrix} \tag{20}$$

As shown in (12), the transformation between the global coordinates (x, y) and the local coordinates (L_1, L_2) is linear; the second derivatives of (x, y) with respect to (L_1, L_2) are all equal to zero. Hence, the global derivatives are expressed in terms of the local derivatives:

$$\begin{Bmatrix} \dfrac{\partial^2\lambda}{\partial x^2} \\ \dfrac{\partial^2\lambda}{\partial x\partial y} \\ \dfrac{\partial^2\lambda}{\partial y^2} \end{Bmatrix} = \begin{bmatrix} \left(\dfrac{\partial x}{\partial L_1}\right)^2 & 2\left(\dfrac{\partial x}{\partial L_1}\dfrac{\partial y}{\partial L_1}\right) & \left(\dfrac{\partial y}{\partial L_1}\right)^2 \\ 2\left(\dfrac{\partial x}{\partial L_1}\dfrac{\partial x}{\partial L_2}\right) & 2\left(\dfrac{\partial x}{\partial L_1}\dfrac{\partial y}{\partial L_2} + \dfrac{\partial x}{\partial L_2}\dfrac{\partial y}{\partial L_1}\right) & 2\left(\dfrac{\partial y}{\partial L_1}\dfrac{\partial y}{\partial L_2}\right) \\ \left(\dfrac{\partial x}{\partial L_2}\right)^2 & 2\left(\dfrac{\partial x}{\partial L_2}\dfrac{\partial y}{\partial L_2}\right) & \left(\dfrac{\partial y}{\partial L_2}\right)^2 \end{bmatrix}^{-1} \begin{Bmatrix} \dfrac{\partial^2\lambda}{\partial L_1^2} \\ 2\dfrac{\partial^2\lambda}{\partial L_1\partial L_2} \\ \dfrac{\partial^2\lambda}{\partial L_2^2} \end{Bmatrix} \tag{21}$$

The Jacobian is formulated in the usual way from (12)

$$\begin{bmatrix} \dfrac{\partial x}{\partial L_1} & \dfrac{\partial y}{\partial L_1} \\ \dfrac{\partial x}{\partial L_2} & \dfrac{\partial y}{\partial L_2} \end{bmatrix} = \begin{bmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{bmatrix} \tag{22}$$

whence

$$\begin{bmatrix} \left(\frac{\partial x}{\partial L_1}\right)^2 & 2\left(\frac{\partial x}{\partial L_1}\frac{\partial y}{\partial L_1}\right) & \left(\frac{\partial y}{\partial L_1}\right)^2 \\ 2\left(\frac{\partial x}{\partial L_1}\frac{\partial x}{\partial L_2}\right) & 2\left(\frac{\partial x}{\partial L_1}\frac{\partial y}{\partial L_2} + \frac{\partial x}{\partial L_2}\frac{\partial y}{\partial L_1}\right) & 2\left(\frac{\partial y}{\partial L_1}\frac{\partial y}{\partial L_2}\right) \\ \left(\frac{\partial x}{\partial L_2}\right)^2 & 2\left(\frac{\partial x}{\partial L_2}\frac{\partial y}{\partial L_2}\right) & \left(\frac{\partial y}{\partial L_2}\right)^2 \end{bmatrix} =$$

$$\begin{bmatrix} (x_1-x_3)^2 & 2(x_1-x_3)(y_1-y_3) & (y_1-y_3)^2 \\ 2(x_1-x_3)(x_2-x_3) & 2[(x_1-x_3)(y_2-y_3)+(x_2-x_3)(y_1-y_3)] & 2(y_1-y_3)(y_2-y_3) \\ (x_2-x_3)^2 & 2(x_2-x_3)(y_2-y_3) & (y_2-y_3)^2 \end{bmatrix} \tag{23}$$

The local derivatives of plastic multiplier are found from (13):

$$\begin{Bmatrix} \frac{\partial \lambda}{\partial L_1} \\ \frac{\partial \lambda}{\partial L_2} \end{Bmatrix} = \sum_{i=1}^{3} \lambda_i \begin{Bmatrix} \frac{\partial \tilde{N}_i}{\partial L_1} \\ \frac{\partial \tilde{N}_i}{\partial L_2} \end{Bmatrix} \qquad \begin{Bmatrix} \frac{\partial^2 \lambda}{\partial L_1^2} \\ \frac{\partial^2 \lambda}{\partial L_1 \partial L_2} \\ \frac{\partial^2 \lambda}{\partial L_2^2} \end{Bmatrix} = \sum_{i=1}^{3} \lambda_i \begin{Bmatrix} \frac{\partial^2 \tilde{N}_i}{\partial L_1^2} \\ \frac{\partial^2 \tilde{N}_i}{\partial L_1 \partial L_2} \\ \frac{\partial^2 \tilde{N}_i}{\partial L_2^2} \end{Bmatrix} \tag{24}$$

In general, having formed (23) and (24), at a given sampling point, Eq. (21) can be solved numerically for the second-order global derivatives only two of which are required to finally calculate the Laplacian:

$$\nabla^2 \lambda = \frac{\partial^2 \lambda}{\partial x^2} + \frac{\partial^2 \lambda}{\partial y^2} \tag{25}$$

4. Numerical test

Consider the example of a softening panel under compression analyzed in de Borst *et al.* (1995). For explicit comparison, the same properties are adopted: width B = 60 mm, height H = 120 mm, elastic shear modulus $G = 4{,}000$ N/mm^2, Poisson's ratio $\nu = 0.49$, yield stress $\sigma_y = 100.0$ N/mm^2, and constant softening modulus $h = -0.1G$. Pamin (1994) shows the gradient (length scale) constant to be $g = -l^2 h$, where l is the internal length scale. For this case, $g = 3{,}600$ N. A linear softening law and von Mises yield criterion are used.

Two meshes of crossed triangular elements, a 6×12 grid of 288 elements and a 12×24 grid of 1152 elements, are tested. An imperfection of 10% is introduced at either a corner or the center. The solution is convergent for all the cases. The load-displacement curves are shown in Fig. 2. Evidently, the responses with mesh refinement are very close using the gradient plasticity model.

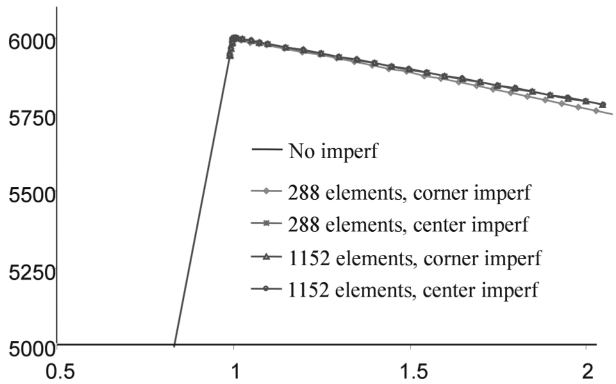

Fig. 2 Load-displacement curves

Fig. 3 shows the contour plots of the equivalent plastic multiplier. The cases with the imperfection at the right bottom corner are shown in (a) for the mesh of 288 elements and (b) for the mesh of 1152 elements respectively. The plastic strain develops in the direction of about 45^{o} starting from the imperfect elements, which is consistent with those given in de Borst *et al*. (1995). The refined mesh has a relatively narrow plastic strain zone. The cases with the imperfection in the center are shown in (c) for the mesh of 288 elements and (d) for the mesh of 1152 elements respectively. The plastic strain develops in "X" shape emitting from the imperfect elements. The refined mesh also has a relatively narrow plastic strain zone.

Fig. 4 shows the deformed shapes of the panel for the cases corresponding to Fig. 3.

Though the proposed interpolation for the plastic multiplier does not satisfy the "patch test" (Zienkiewicz and Taylor 2000), it yields excellent results. The interpolation is continuous everywhere, its first-order derivatives, however, are only continuous within the element but not

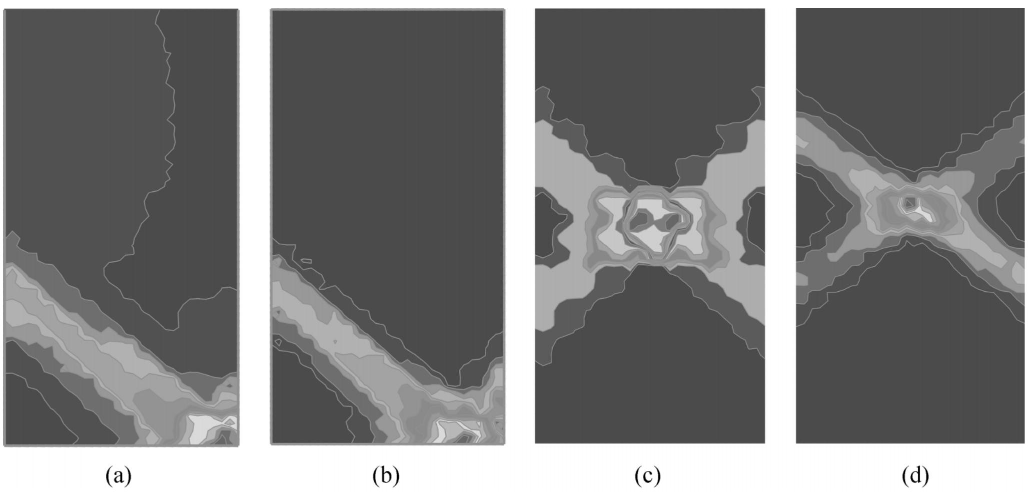

Fig. 3 Contour plots of the equivalent plastic strain (a) imperfection at a corner, 288 elements; (b) imperfection at a corner, 1152 elements; (c) imperfection in the center, 288 elements; (d) imperfection in the center, 1152 elements

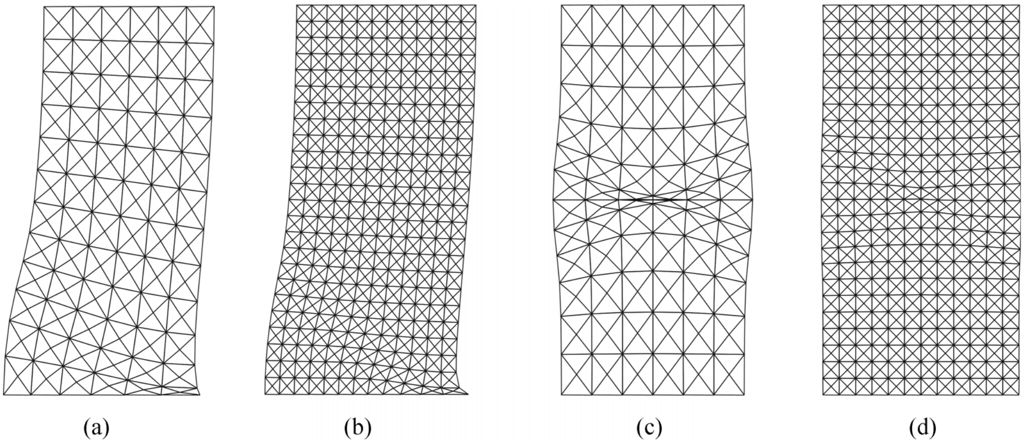

Fig. 4 Deformed shapes (a) imperfection at a corner, 288 elements; (b) imperfection at a corner, 1152 elements; (c) imperfection in the center, 288 elements; (d) imperfection in the center, 1152 elements

across element boundaries. Its second-order derivatives only exist within the elements. By recalling Eq. (4), it can be seen that the yield criterion, in which the second-order derivatives are involved, is satisfied in a distributed sense. Thus, the integral in Eq. (4) can be calculated even though the second-order derivatives are singular on element boundaries. The numerical results show that this interpolation possesses good convergence characteristics. It implies that the error in yield criterion calculation induced by the singularity of the second-order derivatives on element boundaries is smaller than that induced by the lack of stress mapping, which can be eliminated through the iteration process.

5. Conclusions

This paper has presented the formulation for gradient-dependent plasticity based on triangular elements using incompatible (trigonometric) functions to discretize the plastic multiplier. The Laplacian is formulated from the second-order derivatives of differentiable functions, and hence only three degrees of freedom per node are required, instead of the Hermitian polynomial approach requiring six. The accuracy and efficiency technique was assessed against a localization problem involving shear banding in compression.

Our motivation has been to provide a more efficient computational tool for gradient-dependent plasticity. It has been shown that the incompatible element has the following important characteristics:

1. The interpolation of the plastic multiplier uses only function values of nodes, so the degrees of freedom and computational time are largely reduced;
2. The interpolation guarantees the existence of the second-order derivatives within the elements. Thus the yield criterion, in which the second-order derivatives are required, can be evaluated;
3. The numerical results show that this interpolation has good convergence characteristics. It implies that the error in yield criterion calculation introduced by the non-existence of the second-order derivatives on element boundaries can be eliminated through the iteration process.

References

Belytschko, T. and Lasry, D. (1989), “A study of localization limiters for strain-softening in statics dynamics”, *Comput. Struct.*, **33**(3), 707-715.

de Borst, R. (1991), “Simulation of strain localization: A reappraisal of the Cosserat continuum”, *Engineering Computations (Swansea, Wales)*, **8**(4), 317-332.

de Borst, R. (1993), “Generalization of J_2-flow theory for polar continua”, *Comput. Meth. Appl. Mech. Eng.*, **103**(3), 347-362.

de Borst, R. and Mühlhaus, H.-B. (1992), “Gradient-dependent plasticity: Formulation and algorithmic aspects”, *Int. J. Numer. Meth. Eng.*, **35**(3), 521-539.

de Borst, R., Pamin, J. and Sluys, L.J. (1995), “Computation issues in gradient plasticity”, *Continuum Models for Materials with Microstructure*, H.B. Mühlhaus, ed., Wiley, Chichester, England, 159-200.

de Borst, R. and Sluys, L.J. (1991), “Localisation in a Cosserat continuum under static and dynamic loading conditions”, *Comput. Meth. Appl. Mech. Eng.*, **90**(1-3), 805-827.

de Borst, R., Sluys, L.J., Mühlhaus, H.B. and Pamin, J. (1993), “Fundamental issues in finite element analyses of localization of deformation”, *Engineering Computations (Swansea, Wales)*, **10**(2), 99-121.

Li, X. and Cescotto, S. (1996), “Finite element method for gradient plasticity at large strains”, *Int. J. Numer. Meth. Eng.*, **39**(4), 619-633.

Meftah, F. and Reynouard, J.M. (1998), “Multilayered mixed beam element in gradient plasticity for the analysis of localized failure modes”, *Mechanics of Cohesive-Frictional Materials*, **3**(4), 305-322.

Mühlhaus, H.B. (1989), “Application of Cosserat theory in numerical solutions of limit load problems”, *Ingenieur-Archiv*, **59**, 124-137.

Owen, D.R.J. and Hinton, E. (1980), *Finite Elements in Plasticity : Theory and Practice*, Pineridge Press, Swansea.

Pamin, J. (1994), “Gradient-dependent plasticity in numerical simulation of localization phenomena”, PhD Thesis, Dept. of Civil Eng., Delft University, Delft, Holland.

Vardoulakis, I., Shah, K.R. and Papanastasious, P. (1992), “Modelling of tool-rock shear interfaces using gradient-dependent flow theory of plasticity”, *Int. J. of Rock Mechanics and Mining Sciences & Geomechanics Abstracts*, **29**(6), 573-582.

Zervos, A., Papanastasiou, P. and Vardoulakis, I. (2001), “A finite element displacement formulation for gradient elastoplasticity”, *Int. J. Numer. Meth. Eng.*, **50**(6), 1369-1388.

Zienkiewicz, O.C. and Taylor, R.L. (2000), *The Finite Element Method*, Butterworth-Heinemann, Oxford; Boston.