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Feedback control of intelligent structures with uncertainties and its robustness analysis

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Abstract. Variations in system parameters due to uncertainties of parameters may result in system performance deterioration and create system internal stability problems. Uncertainties in structural modeling of structures are often considered to ensure that the control system is robust with respect to response errors. So the uncertain concept plays an important role in the analysis and design of the engineering structures. In this paper, the active control of the intelligent structures with the uncertainties is studied and a new method for analyzing the robustness of systems with the uncertainties is presented. Firstly, the system with uncertain parameters is considered as the perturbation of the system with deterministic parameters. Secondly, the feedback control law is designed on the basis of deterministic system. Thirdly, perturbation analysis and robustness analysis of intelligent structures with uncertainties are discussed when the feedback control law is applied to the original system and perturbed system. Combining the convex model of uncertainties with the finite element method, the analysis theory of the robustness of intelligent structures with the uncertainties can be developed. The description and computation of the robustness of intelligent structures with uncertain parameters is obtained. Finally, a numerical example of the application of the present method is given to show the validity of the method.

Key words: robust; uncertainty; feedback control; perturbation; intelligent structure.

1. Introduction

Analysis and design methods for the engineering structures are generally on the basis of deterministic parameters. Because of the inaccuracy of measurement, errors in manufacture, etc., in practice, there are often uncertainties such as material and geometric properties, external forces, and

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boundary conditions. In most situations, the effect of uncertainties to the dynamic stabilities is small, but the combination of those uncertainties can lead to large and unexpected excursion of the response that may cause structural instability, which may lead to a drastic reduction in accuracy and precision of operation, especially in multicomponent systems.

Many techniques are developed to solve the uncertain problems, such as a random vector methods (Conlreras 1980), fuzzy set methods (Ibrahim 1987) and the probabilistic approaches (Chen *et al*. 1992, Contracts 1980, Chen 1992). The probabilistic approaches are not able to deliver the reliable results at the required precision without sufficient experiment data to validate the assumptions made regarding the joint probability densities of the random variables. For the fuzzy model, uncertainties in the fuzzy statistics still exist such as the fuzzy statistical errors or uncertainties in the fuzzy statistics, and the choice of subjection degree function has the artificial uncertainties (Liu and Cheng 1989). Recently, modeling of uncertainties in parameters have drawn interest both from the system control robustness analysis and from the structural failure measure field. Ben-Haim and Elishakoff (1990) presented an unknown-but-bounded imperfection model, later the study of dynamic response and failure of structures under pulsed parametric loading were discussed by Lindberg (1991). An unknown-but-bounded models of uncertainties in parameters has also been used in determining the robustness of a control system by Shi and Gao (1987). Ibbini and Alawneh (1998) presented two different approaches to improve the robustness of the resulting closed-loop system based on the source of eigenvalues perturbation. The free parameters are adjusted to minimize the closed-loop system condition number which is well known as a robustness measure with respect to system parameter variation (Karbassi and Bell 1993). Minimizing the system entropy as a measure of an upper bound on H_∞ norm can be used to improve the robustness of the closed-loop system (Kautsky *et al.* 1985) when external disturbance simulated as white noise input presented. Chen and Cao (2000) only discussed determining locations of the piezoelectric sensor/actuator for vibration control of intelligent structures, feedback control was not considered. Cao *et al.* (2001) only studied feedback control and robustness of intelligent structures with deterministic parameters, but not discussed robustness of intelligent structures with uncertainties. The robustness of a closed-loop system is one of the most important concerns of control system designers. Variations in system parameters due to uncertainties of parameters may result in system performance deterioration and even in system internal stability. Uncertainties in structural modeling are often considered to ensure that the control system is robust with respect to response errors. So the uncertain concept plays an important role in the analysis and design of the engineering structures.

In this paper, the active control of the intelligent structures with the uncertainties is studied and a new method for analyzing the robustness of systems with the uncertainties is presented. First, the system with uncertain parameters is considered as the perturbation of the system with deterministic parameters. Second, the feedback control law is designed on the basis of deterministic system. Third, perturbation analysis and robustness analysis of intelligent structures with uncertainties are discussed when the feedback control law is applied to the original system and perturbed system. Combining the convex model of uncertainties with the finite element method, the analysis theory of the robustness of intelligent structures with the uncertainties can be developed. The expressions for computing the upper and lower bounds of the robustness of intelligent structures with the uncertainties are obtained. Finally, a numerical example of the application of the present method to show the validity of the method is given.

2. Motion equations of intelligent structures

The vibration control equations of intelligent structures with distributed sensors and actuators are given as follows (Chen *et al.* 1999)

$$\begin{cases} \boldsymbol{M}\ddot{\boldsymbol{q}}(t) + \boldsymbol{C}\dot{\boldsymbol{q}}(t) + \boldsymbol{K}\boldsymbol{q}(t) = \boldsymbol{B}_0\boldsymbol{F}(t) \\ \boldsymbol{V}_s = \boldsymbol{D}_0\boldsymbol{q}(t) \end{cases} \tag{1}$$

where $\boldsymbol{M}$, $\boldsymbol{K}$ and $\boldsymbol{C} \in R^{n\times n}$ are the mass, stiffness and damping matrices of intelligent structures with uncertain parameters, respectively; $\boldsymbol{q} \in R^n$ is a displacement vector; $\boldsymbol{F}(t) \in R^p$ is a control force vector; $\boldsymbol{B}_0 \in R^{n\times p}$ is the controllable matrix determined by placements of actuators; $\boldsymbol{V}_s \in R^p$ is the output vector of sensors; $\boldsymbol{D}_0 \in R^{p\times n}$ is the observable matrix determined by placements of sensors.

Suppose the uncertain parameters are denoted by δ_{mj}, δ_{kj} and δ_{cj} ($j = 1, 2, ..., N$), and the mass, stiffness and damping matrices of systems with uncertainties can be expressed as

$$\begin{cases} \boldsymbol{M} = \boldsymbol{M}_0 + \sum_{j=1}^{N} \delta_{mj}\boldsymbol{M}_j \\ \boldsymbol{K} = \boldsymbol{K}_0 + \sum_{j=1}^{N} \delta_{kj}\boldsymbol{K}_j \\ \boldsymbol{C} = \boldsymbol{C}_0 + \sum_{j=1}^{N} \delta_{cj}\boldsymbol{C}_j \end{cases} \tag{2}$$

where $\boldsymbol{M}_0$, $\boldsymbol{K}_0$ and $\boldsymbol{C}_0 \in R^{n\times n}$ are the mass, stiffness and damping matrices of structures with deterministic parameters; $\boldsymbol{M}_j$, $\boldsymbol{K}_j$ and $\boldsymbol{C}_j$ are changes of the jth element mass, stiffness and damping matrices of the intelligent structures with uncertainties, respectively. δ_{mj}, δ_{kj} and δ_{cj} ($j = 1, 2, ..., N$) are their uncertain parameters and N is the total number of the elements with uncertainties. Using the following notations

$$\begin{cases} \sum_{j=1}^{N} \delta_{mj}\boldsymbol{M}_j = \boldsymbol{M}_u \\ \sum_{j=1}^{N} \delta_{kj}\boldsymbol{K}_j = \boldsymbol{K}_u \\ \sum_{j=1}^{N} \delta_{cj}\boldsymbol{C}_j = \boldsymbol{C}_u \end{cases} \tag{3}$$

then Eq. (2) can be expressed as

$$\begin{cases} \boldsymbol{M} = \boldsymbol{M}_0 + \boldsymbol{M}_u \\ \boldsymbol{K} = \boldsymbol{K}_0 + \boldsymbol{K}_u \\ \boldsymbol{C} = \boldsymbol{C}_0 + \boldsymbol{C}_u \end{cases} \tag{4}$$

As can be seen from Eq. (4) matrices of systems with uncertain parameters are equal to matrices of systems with deterministic parameters and changes corresponding to matrices. Thus the system with uncertain parameters can be considered as the perturbation of the system with deterministic parameters. The vibration control equations of intelligent structures with deterministic parameters are written as follows:

$$\begin{cases} \boldsymbol{M}_0\ddot{\boldsymbol{q}}(t) + \boldsymbol{C}_0\dot{\boldsymbol{q}}(t) + \boldsymbol{K}_0\boldsymbol{q}(t) = \boldsymbol{B}_0\boldsymbol{F}(t) \\ \boldsymbol{V}_s = \boldsymbol{D}_0\boldsymbol{q}(t) \end{cases} \tag{5}$$

The eigenproblem corresponding to Eq. (5) is

$$\boldsymbol{K}_0\boldsymbol{\Phi}_0 = \lambda_0\boldsymbol{M}_0\boldsymbol{\Phi}_0 \tag{6}$$

where $\lambda_0 = \mathrm{diag}(\omega_{01}^2, \omega_{02}^2, \ldots, \omega_{0n}^2) \in R^{n\times n}$ is a diagonal of eigenvalues matrix; $\boldsymbol{\Phi}_0 = [\phi_{01}, \phi_{02}, ..., \phi_{0n}] \in R^{n\times n}$ is a modal matrix. ϕ_0 and λ_0 satisfy following equation

$$\begin{cases} \boldsymbol{K}_0\phi_{0i} = \omega_{0i}^2\boldsymbol{M}_0\phi_{0i} \\ \phi_{0i}^T\boldsymbol{M}_0\phi_{0i} = 1 \end{cases} \tag{7}$$

Transforming Eq. (5) into the modal coordinates through the coordinate transformation

$$\boldsymbol{q}(t) = \boldsymbol{\Phi}_0\eta(t)$$

yields

$$\ddot{\eta}(t) + \boldsymbol{Z}\dot{\eta}(t) + \lambda_0\eta(t) = \boldsymbol{BF}(t) \tag{9}$$

$$\boldsymbol{V}_s = \boldsymbol{D}\eta(t) \tag{10}$$

where $\boldsymbol{Z} = \boldsymbol{\Phi}_0^T\boldsymbol{C}_0\boldsymbol{\Phi}_0 \in R^{n\times n}$; $\boldsymbol{B} = \boldsymbol{\Phi}_0^T\boldsymbol{B}_0 \in R^{n\times p}$, $\boldsymbol{D} = \boldsymbol{D}_0\boldsymbol{\Phi}_0 \in R^{p\times n}$. It can be seen that Eq. (9) does not illustrate the relation between the feedback control force. ($\boldsymbol{F}(t)$) and controllability and that Eq. (10) does not illustrate the relation between $\boldsymbol{V}_s$ and observability. In order to reveal the relation between $\boldsymbol{F}(t)$ and controllability and between $\boldsymbol{V}_s$ and observability, the singular value decomposition of $\boldsymbol{B}$ is taken, the following equation can be obtained

$$\boldsymbol{B} = \boldsymbol{U}_0\sum\boldsymbol{V}_0^T \tag{11}$$

where $\boldsymbol{U}_0$ and $\boldsymbol{V}_0$ are left and right singular vectors of $\boldsymbol{B}$, respectively; $\boldsymbol{U}_0 = R^{n\times n}$, $\boldsymbol{V}_0 = R^{p\times p}$, $\boldsymbol{U}_0^T\boldsymbol{U}_0 = \boldsymbol{I}_n$, $\boldsymbol{V}_0^T\boldsymbol{V}_0 = \boldsymbol{I}_p$; $\Sigma = \begin{bmatrix} \Sigma_0 & 0 \\ 0 & 0 \end{bmatrix}_{n\times p}$, $\Sigma_0 = \mathrm{diag}(\sigma_{a1}, \sigma_{a2}, \ldots, \sigma_{a\bar{a}})$, in which $\bar{a}$ is the number of controllable modes. σ_{ai} is a measure of the controllability of the ith mode, $\sigma_{ai} > 0$. We assume that $\bar{a} = p$ here. Similarly, the singular value decomposition of $\boldsymbol{D}$ can be obtained

$$\boldsymbol{D} = \bar{\boldsymbol{V}}_0\Sigma'\bar{\boldsymbol{U}}_0^T \tag{12}$$

where $\Sigma' = \begin{bmatrix} \Sigma_0' & 0 \\ 0 & 0 \end{bmatrix}_{p\times n}^T$, $\Sigma_0' = \mathrm{diag}(\sigma_{s1}, \sigma_{s2}, \ldots, \sigma_{s\bar{s}})$, in which $\bar{s}$ is the number of observable modes. σ_{si} is a measure of the observability of the ith mode, $\sigma_{si} > 0$. We assume that $\bar{s} = p$ here. If the sensor is not only used for measuring the motion but also for controlling and suppressing the vibration of intelligent structures as the actuator, then $\bar{\boldsymbol{U}}_0^T \boldsymbol{U}_0 = \boldsymbol{I}_n$ and $\bar{\boldsymbol{V}}_0^T \boldsymbol{V}_0 = \boldsymbol{I}_p$. The modal transformation, $\eta(t) = \boldsymbol{U}_0 x(t)$, can be used, Eqs. (9) and (10) can be changed into the following forms

$$\ddot{\boldsymbol{x}}(t) + 2\omega_0 \xi_0 \dot{\boldsymbol{x}}(t) + \omega_0^2 \boldsymbol{x}(t) = \Sigma_0 \boldsymbol{f}(t) \tag{13}$$

$$\boldsymbol{V}_f = \Sigma_0' \boldsymbol{x}(t) \tag{14}$$

where $\boldsymbol{V}_f = \boldsymbol{V}_0^T \boldsymbol{V}_s$, $\boldsymbol{f}(t) = \boldsymbol{V}_0^T \boldsymbol{F}(t)$.

3. The feedback control of intelligent structures with uncertainties

For a direct output feedback control system, the control force $\boldsymbol{f}(t)$ of Eq. (13) is assumed to be the following form

$$\boldsymbol{f}(t) = -\boldsymbol{G}_1 \boldsymbol{V}_f - \boldsymbol{G}_2 \dot{\boldsymbol{V}}_f = -\boldsymbol{G}_1 \Sigma_0' \boldsymbol{x}(t) - \boldsymbol{G}_2 \Sigma_0' \dot{\boldsymbol{x}}(t) \tag{15}$$

where $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ are the feedback gain matrices of displacement and velocity, $\in R^{p \times p}$. Coefficients of $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ are to be determined. From Eq. (15), it can be seen that $f(t)$ is proportional to Σ_0' when $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, $\boldsymbol{x}(t)$ and $\dot{x}(t)$ are given. The greater σ_{si} is, the smaller the feedback gain is required to produce the same control effect on x_i. It is assumed that $\boldsymbol{G}_1 = \mathrm{diag}(g_{11}, g_{12}, ..., g_{1p})$ and that $\boldsymbol{G}_2 = \mathrm{diag}(g_{21}, g_{22}, ..., g_{2p})$. Substituting Eq. (15) into Eq. (13), then Eq. (13) can be expressed as

$$\ddot{\boldsymbol{x}}(t) + (2\omega_0 \xi_0 + \Sigma_0 \boldsymbol{G}_2 \Sigma_0')\dot{\boldsymbol{x}}(t) + (\omega_0^2 + \Sigma_0 \boldsymbol{G}_1 \Sigma_0')\boldsymbol{x}(t) = 0 \tag{16}$$

Eq. (16) is decoupled, so we have

$$\ddot{\boldsymbol{x}}_i(t) + (2\omega_{0i}\xi_{0i} + \sigma_{ai}\sigma_{si}g_{2i})\dot{\boldsymbol{x}}_i(t) + (\omega_{0i}^2 + \sigma_{ai}\sigma_{si}g_{1i})\boldsymbol{x}_i(t) = 0 \tag{17}$$

The key factor of control and suppression vibration is the damping factor. It is assumed that the poles of the closed-loop system are $S_{0j}(= -\alpha_j \pm i\beta_j)$, $i = 1, 2, \ldots, p$, $\alpha_j > 0$. Then we have

$$\begin{cases} 2\omega_{0i}\xi_{0i} + \sigma_{ai}\sigma_{si}g_{2i} = 2\alpha_i \\ \omega_{0i}^2 + \sigma_{ai}\sigma_{si}g_{1i} = \beta_i^2 + \alpha_i^2 \end{cases}, \qquad i = 1, 2, \ldots, p \tag{18}$$

From Eq. (18) we obtain

$$\begin{cases} g_{1i} = (\beta_i^2 + \alpha_i^2 - \omega_{0i}^2)/(\sigma_{ai}\sigma_{si}) \\ g_{2i} = 2(\alpha_i - \omega_{0i}\xi_{0i})/(\sigma_{ai}\sigma_{si}) \end{cases} \tag{19}$$

Then the control force $\boldsymbol{F}(t)$ in Eq. (1) is

$$\boldsymbol{F}(t) = -\boldsymbol{V}_0 \boldsymbol{G}_1 \Sigma_0' \boldsymbol{U}_0^T \boldsymbol{\Phi}_0^T \boldsymbol{q}(t) - \boldsymbol{V}_0 \boldsymbol{G}_2 \Sigma_0' \boldsymbol{U}_0^T \boldsymbol{\Phi}_0^T \dot{\boldsymbol{q}}(t) \tag{20}$$

4. Perturbation analysis of intelligent structures with uncertainties

When the control force $\boldsymbol{F}(t)$ of Eq. (20) is applied to Eq. (5), we have

$$\boldsymbol{M}_0\ddot{\boldsymbol{q}}(t) + \underline{\boldsymbol{C}}_0\dot{\boldsymbol{q}}(t) + \underline{\boldsymbol{K}}_0\boldsymbol{q}(t) = 0 \tag{21}$$

where $\underline{\boldsymbol{C}}_0 = \boldsymbol{C}_0 + \boldsymbol{\Phi}_0\boldsymbol{U}_0\Sigma_0\boldsymbol{G}_2\Sigma_0'\boldsymbol{U}_0^T\boldsymbol{\Phi}_0^T$; $\underline{\boldsymbol{K}}_0 = \boldsymbol{K}_0 + \boldsymbol{\Phi}_0\boldsymbol{U}_0\Sigma_0\boldsymbol{G}_1\Sigma_0'\boldsymbol{U}_0^T\boldsymbol{\Phi}_0^T$. The eigenvalue problem with deterministic parameters corresponding to Eq. (21) is

$$(\boldsymbol{M}_0\boldsymbol{P}_0^2 + \underline{\boldsymbol{C}}_0\boldsymbol{P}_0 + \underline{\boldsymbol{K}}_0)\boldsymbol{\Phi}_0 = 0 \tag{22}$$

where $\boldsymbol{P}_0 \in R^{n\times n}$ is a complex diagonal matrix. Let us introduce a state vector

$$\boldsymbol{u}_0 = \begin{Bmatrix} \boldsymbol{P}_0\boldsymbol{\Phi}_0 \\ \boldsymbol{\Phi}_0 \end{Bmatrix} \tag{23}$$

where $\boldsymbol{u}_0$ is the eigenvectors. Hence Eq. (22) becomes

$$(\boldsymbol{A}_0\boldsymbol{S}_0 + \boldsymbol{E}_0)\boldsymbol{u}_0 = 0 \tag{24}$$

where

$$\boldsymbol{A}_0 = \begin{bmatrix} 0 & \boldsymbol{M}_0 \\ \boldsymbol{M}_0 & \underline{\boldsymbol{C}}_0 \end{bmatrix}; \qquad \boldsymbol{E}_0 = \begin{bmatrix} -\boldsymbol{M}_0 & 0 \\ 0 & \underline{\boldsymbol{K}}_0 \end{bmatrix}; \qquad \boldsymbol{S}_0 = \begin{bmatrix} P_0 & 0 \\ 0 & \boldsymbol{P}_0 \end{bmatrix} \tag{25}$$

where $\boldsymbol{S}_0 \in R^{2n\times 2n}$ is a complex diagonal matrix. If the uncertain parameters ($\boldsymbol{M}_u$, $\boldsymbol{C}_u$ and $\boldsymbol{K}_u$) are small, the corresponding state equation with uncertain parameters like Eq. (24) is as follows:

$$(\boldsymbol{A}\boldsymbol{S} + \boldsymbol{E})\boldsymbol{u} = 0 \tag{26}$$

where

$$\begin{cases} \boldsymbol{A} = \boldsymbol{A}_0 + \varepsilon\boldsymbol{A}_1 \\ \boldsymbol{E} = \boldsymbol{E}_0 + \varepsilon\boldsymbol{E}_1 \end{cases} \tag{27}$$

and

$$\boldsymbol{A}_1 = \begin{bmatrix} 0 & \boldsymbol{M}_u \\ \boldsymbol{M}_u & \boldsymbol{C}_u \end{bmatrix}, \qquad \boldsymbol{E}_1 = \begin{bmatrix} -\boldsymbol{M}_u & 0 \\ 0 & \boldsymbol{K}_u \end{bmatrix} \tag{28}$$

According to the perturbation theory (Chen 1992), the eigenvalue and eigenvector can be expressed as the power series in ε, that is

$$\boldsymbol{S} = \boldsymbol{S}_0 + \varepsilon\boldsymbol{S}_1 + \varepsilon^2\boldsymbol{S}_2 + \ldots \tag{29}$$

$$\boldsymbol{u} = \boldsymbol{u}_0 + \varepsilon\boldsymbol{u}_1 + \varepsilon^2\boldsymbol{u}_2 + \ldots \tag{30}$$

where $S_{0j} = -\alpha_j \pm i\beta_j$. According to the matrix perturbation theory (Chen 1992), the 1st order perturbation of eigenvalues structures with uncertain parameters is

$$S_{1i} = -\boldsymbol{u}_{0i}^T(\boldsymbol{E}_1 + S_{0i}\boldsymbol{A}_1)\boldsymbol{u}_{0i} \tag{31}$$

where S_{0i} is the ith eigenvalue of the deterministic structure. $\boldsymbol{\Phi}_{0i}$ is the corresponding ith mode. Substituting Eq. (28) into Eq. (31), Eq. (31) can be changed as follows:

$$S_{1i} = -S_{0i}^2\boldsymbol{\Phi}_{0i}^T\boldsymbol{M}_u\boldsymbol{\Phi}_{0i} - S_{0i}\boldsymbol{\Phi}_{0i}^T\boldsymbol{C}_u\boldsymbol{\Phi}_{0i} - \boldsymbol{\Phi}_{0i}^T\boldsymbol{K}_u\boldsymbol{\Phi}_{0i} \tag{32}$$

Substituting Eq. (3) into (32), yields

$$\left.\begin{aligned} S_{1i} &= -S_{0i}^2\boldsymbol{\Phi}_{0i}^T\left(\sum_{j=1}^{N}\delta_{mj}\boldsymbol{M}_j\right)\boldsymbol{\Phi}_{0i} - S_{0i}\boldsymbol{\Phi}_{0i}^T\left(\sum_{j=1}^{N}\delta_{cj}\boldsymbol{C}_j\right)\boldsymbol{\Phi}_{0i} - \boldsymbol{\Phi}_{0i}^T\left(\sum_{j=1}^{N}\delta_{kj}\boldsymbol{K}_j\right)\boldsymbol{\Phi}_{0i} \\ &= -S_{0i}^2\sum_{j=1}^{N}[\delta_{mj}(\overline{\boldsymbol{\Phi}}_{0ij}^T\boldsymbol{M}_j\overline{\boldsymbol{\Phi}}_{0ij})] - S_{0i}\sum_{j=1}^{N}[\delta_{cj}(\overline{\boldsymbol{\Phi}}_{0ij}^T\boldsymbol{C}_j\overline{\boldsymbol{\Phi}}_{0ij})] - \sum_{j=1}^{N}[\delta_{kj}(\overline{\boldsymbol{\Phi}}_{0ij}^T\boldsymbol{K}_j\overline{\boldsymbol{\Phi}}_{0ij})] \end{aligned}\right\} \tag{33}$$

where $\overline{\boldsymbol{\Phi}}_{0ij}$ is the element mode of the jth element in the ith mode. Using the following notations

$$\left.\begin{aligned} \overline{\boldsymbol{\Phi}}_{0ij}^T\boldsymbol{M}_j\overline{\boldsymbol{\Phi}}_{0ij} &= M_j^{(i)} \\ \overline{\boldsymbol{\Phi}}_{0ij}^T\boldsymbol{C}_j\overline{\boldsymbol{\Phi}}_{0ij} &= C_j^{(i)} \\ \overline{\boldsymbol{\Phi}}_{0ij}^T\boldsymbol{K}_j\overline{\boldsymbol{\Phi}}_{0ij} &= K_j^{(i)} \end{aligned}\right\} \tag{34}$$

Eq. (33) becomes

$$S_{1i} = -S_{0i}^2\sum_{j=1}^{N}\delta_{mj}M_j^{(i)} - S_{0i}\sum_{j=1}^{N}\delta_{cj}C_j^{(i)} - \sum_{j=1}^{N}\delta_{kj}K_j^{(i)}, \qquad i = 1, 2, \ldots \tag{35}$$

Let

$$\left.\begin{aligned} \overline{\boldsymbol{M}} &= [M_1^{(i)}, M_2^{(i)}, \ldots, M_N^{(i)}]^T \\ \overline{\boldsymbol{C}} &= [C_1^{(i)}, C_2^{(i)}, \ldots, C_N^{(i)}]^T \\ \overline{\boldsymbol{K}} &= [K_1^{(i)}, K_2^{(i)}, \ldots, K_N^{(i)}]^T \end{aligned}\right\} \tag{36}$$

Eq. (35) can be expressed as

$$S_{1i} = -S_{0i}^2\delta_m^T\overline{\boldsymbol{M}} - S_{0i}\delta_c^T\overline{\boldsymbol{C}} - \delta_k^T\overline{\boldsymbol{K}} \tag{37}$$

The ith eigenvalue of the perturbed system with uncertainties can be expressed as

$$S^{(i)} = S_{0i} - S_{0i}^2\delta_m^T\overline{\boldsymbol{M}} - S_{0i}\delta_c^T\overline{\boldsymbol{C}} - \delta_k^T\overline{\boldsymbol{K}} \tag{38}$$

5. Robustness analysis of intelligent structures with uncertainties

As mentioned above, the uncertainty is not equal to the randomness, and the probability model is not the only way to illustrate uncertainties. The uncertainties of the convex model theory can be described by the non-random bounded models (Ben-Haim and Elishakoff 1990), δ_{mj}, δ_{kj} and δ_{cj} ($j = 1, 2, ..., N$) satisfy the following constraint conditions

$$\begin{cases} \delta_m^T \Omega_1 \delta_m \le \delta_1^2 \\ \delta_c^T \Omega_2 \delta_c \le \delta_2^2 \\ \delta_k^T \Omega_3 \delta_k \le \delta_3^2 \end{cases} \tag{39}$$

where Ω_1, Ω_2 and Ω_3 are the N-dimension symmetric positive weighted matrices; δ_1, δ_2 and δ_3 are the given positive real numbers. It can be seen from the above discussion that if Eq. (38) takes the extremum where the uncertain parameters δ_m, δ_c and δ_k satisfy the constraint condition Eq. (39), the upper and lower bounds of robustness of intelligent structures with uncertain parameters can be obtained. The Lagarange Multiplier Method can be used to seek the upper and lower bounds, we construct a function H_i

$$H_i = S^{(i)} + t_1(\delta_m^T \Omega_1 \delta_m - \delta_1^2) + t_2(\delta_c^T \Omega_2 \delta_c - \delta_2^2) + t_3(\delta_k^T \Omega_3 \delta_k - \delta_3^2) \tag{40}$$

where t_1, t_2 and t_3 are the Lagarange multipliers, respectively. Let the partial derivatives of Eq. (40) with respect to δ_m, δ_c and δ_k be zero, yield

$$\left.\begin{aligned} \frac{\partial H_i}{\partial \delta_m} &= -S_{0i}^2 \overline{\boldsymbol{M}} + 2t_1 \Omega_1 \delta_m = 0 \\ \frac{\partial H_i}{\partial \delta_c} &= -S_{0i} \overline{\boldsymbol{C}} + 2t_2 \Omega_2 \delta_c = 0 \\ \frac{\partial H_i}{\partial \delta_k} &= -\overline{\boldsymbol{K}} + 2t_3 \Omega_3 \delta_k = 0 \end{aligned}\right\} \tag{41}$$

From Eq. (41), we get

$$\left.\begin{aligned} \delta_m &= \frac{S_{0i}^2 \Omega_1^{-1} \overline{\boldsymbol{M}}}{2t_1} \\ \delta_c &= \frac{S_{0i} \Omega_2^{-1} \overline{\boldsymbol{C}}}{2t_2} \\ \delta_k &= \frac{\Omega_3^{-1} \overline{\boldsymbol{K}}}{2t_3} \end{aligned}\right\} \tag{42}$$

Substituting Eq. (42) into the constraint condition (39), we have

$$\left.\begin{aligned} t_1 &= \pm\frac{S_{0i}^2\sqrt{\overline{\boldsymbol{M}}^T\Omega_1^{-1}\overline{\boldsymbol{M}}}}{2\delta_1} \\ t_2 &= \pm\frac{S_{0i}\sqrt{\overline{\boldsymbol{C}}^T\Omega_2^{-1}\overline{\boldsymbol{C}}}}{2\delta_2} \\ t_3 &= \pm\frac{\sqrt{\overline{\boldsymbol{K}}^T\Omega_3^{-1}\overline{\boldsymbol{K}}}}{2\delta_3} \end{aligned}\right\} \tag{43}$$

Substituting Eq. (43) into Eq. (42) yield

$$\left.\begin{aligned} \delta_m &= \pm\frac{\delta_1\Omega_1^{-1}\overline{\boldsymbol{M}}}{\sqrt{\overline{\boldsymbol{M}}^T\Omega_1^{-1}\overline{\boldsymbol{M}}}} \\ \delta_c &= \pm\frac{\delta_2\Omega_2^{-1}\overline{\boldsymbol{C}}}{\sqrt{\overline{\boldsymbol{C}}^T\Omega_2^{-1}\overline{\boldsymbol{C}}}} \\ \delta_k &= \pm\frac{\delta_3\Omega_3^{-1}\overline{\boldsymbol{K}}}{\sqrt{\overline{\boldsymbol{K}}^T\Omega_3^{-1}\overline{\boldsymbol{K}}}} \end{aligned}\right\} \tag{44}$$

Substituting Eq. (44) into (37), the expression for the extremum of the 1st perturbation of eigenvalues can be obtained

$$S_{1i} = \pm S_{0i}^2\delta_1\sqrt{\overline{\boldsymbol{M}}^T\Omega_1^{-1}\overline{\boldsymbol{M}}} \pm S_{0i}\delta_2\sqrt{\overline{\boldsymbol{C}}^T\Omega_2^{-1}\overline{\boldsymbol{C}}} \pm \delta_3\sqrt{\overline{\boldsymbol{K}}^T\Omega_3^{-1}\overline{\boldsymbol{K}}} \tag{45}$$

Substituting $S_{0i} = -\alpha_i \pm i\beta_i$ into Eq. (45), the following equation can be obtained

$$S_{1i} = S_{1R}^{(i)} + S_{1I}^{(i)}i \tag{46}$$

where

$$\left.\begin{aligned} S_{1R}^{(i)} &= \pm[(\beta_i^2-\alpha_i^2)\delta_1\sqrt{\overline{\boldsymbol{M}}^T\Omega_1^{-1}\overline{\boldsymbol{M}}} + \alpha_i\delta_2\sqrt{\overline{\boldsymbol{C}}^T\Omega_2^{-1}\overline{\boldsymbol{C}}} - \delta_3\sqrt{\overline{\boldsymbol{K}}^T\Omega_3^{-1}\overline{\boldsymbol{K}}}] \\ S_{1I}^{(i)} &= \pm(2\alpha_i\beta_i\delta_1\sqrt{\overline{\boldsymbol{M}}^T\Omega_1^{-1}\overline{\boldsymbol{M}}} - \beta_i\delta_2\sqrt{\overline{\boldsymbol{C}}^T\Omega_2^{-1}\overline{\boldsymbol{C}}})i \end{aligned}\right\} \tag{47}$$

It is well known that the change of the *j*th damping factor of the closed-loop system with uncertainties only concerns with the real part of $S_1^{(i)}$. We define $(S_{1R}^{(i)})_{\min} = \min(S_{1R}^{(i)})$, $(S_{1R}^{(i)})_{\max} = \max(S_{1R}^{(i)})$, then

$$\left.\begin{aligned} (S_{1R}^{(i)})_{\min} &= -\left|(\beta_i^2-\alpha_i^2)\delta_1\sqrt{\overline{\boldsymbol{M}}^T\Omega_1^{-1}\overline{\boldsymbol{M}}} + \alpha_i\delta_2\sqrt{\overline{\boldsymbol{C}}^T\Omega_2^{-1}\overline{\boldsymbol{C}}} - \delta_3\sqrt{\overline{\boldsymbol{K}}^T\Omega_3^{-1}\overline{\boldsymbol{K}}}\right| \\ (S_{1R}^{(i)})_{\max} &= \left|(\beta_i^2-\alpha_i^2)\delta_1\sqrt{\overline{\boldsymbol{M}}^T\Omega_1^{-1}\overline{\boldsymbol{M}}} + \alpha_i\delta_2\sqrt{\overline{\boldsymbol{C}}^T\Omega_2^{-1}\overline{\boldsymbol{C}}} - \delta_3\sqrt{\overline{\boldsymbol{K}}^T\Omega_3^{-1}\overline{\boldsymbol{K}}}\right| \end{aligned}\right\} \tag{48}$$

Then the jth modal damping factor (η_j) of the closed-loop system of intelligent structures with uncertainties satisfies the following condition

$$0 < (S_{1R}^{(i)})_{\min} \le \eta_j \le (S_{1R}^{(i)})_{\max} \tag{49}$$

It can be seen from Eq. (49) that if α_j is suitably selected, intelligent structures will have enough dynamic stability tolerance.

6. Numerical example

The numerical example of a cantilever beam is given to illustrate the application of the method presented in this paper. The cantilever beam with *S/As*(shown in Fig. 1), made of the graphite/epoxy materials, is sandwich with piezoelectric polymeric PVDF layers which are bonded on both the upper and bottom surfaces of the beam. The PVDF layers are assumed to be perfectly bonded to the surfaces of beam and modelled as thin films. Piezoelectric polymeric PVDF layers are each 0.5 mm thick. The mass density of the PVDF is equal to 1680 Kg/m^3, and Young's modulus $E_1 = E_2 = 0.20\text{E} + 10$ N/m^2, and Poison ratio $\nu_{12} = \nu_{21} = 0.28$. Dielectric constants in coulomb per square meter, with Z direction (thickness) being the poling direction, are $\Gamma_{11} = \Gamma_{22} = \Gamma_{33} = 0.1062\text{E-}9$ and piezoelectric constants $e_{31} = e_{32} = 0.046$, $e_{33} = e_{24} = e_{15} = 0.0$. The mass density of the graphite/epoxy materials equals 2680 Kg/m^3, and Young's modulus $E_1 = E_2 = 0.80\text{E+}10$ N/m^2, and Poison ratio $\nu_{12} = \nu_{21} = 0.29$.

The finite element method of the cantilever beam is utilized to model the system in Fig. 1, which consists of 32 elements with 133 nodes. The eigenvalue analysis of original determinate structures is carried out with the finite element method, the eigenvalues of the first ten lower modes are shown in Table 1. In Table 1, ω_0 denotes the eigenvalues of the original open-loop system of denotes the eigenvalues of determinate structures. It is assumed that α_j is equal to 0.6 and $\beta_j = \omega_{0j}$. It can be assumed that the 4th, the 10th, the 20th and the 26th elements are uncertain. Then whole stiffness and mass matrices of the beam with uncertainties can be expressed as

$$\left.\begin{aligned} \boldsymbol{M} &= \boldsymbol{M}_0 + \delta_{m4}\boldsymbol{M}_4 + \delta_{m10}\boldsymbol{M}_{10} + \delta_{m20}\boldsymbol{M}_{20} + \delta_{m26}\boldsymbol{M}_{26} \\ \boldsymbol{K} &= \boldsymbol{K}_0 + \delta_{k4}\boldsymbol{K}_4 + \delta_{k10}\boldsymbol{K}_{10} + \delta_{k20}\boldsymbol{K}_{20} + \delta_{k26}\boldsymbol{K}_{26} \end{aligned}\right\} \tag{50}$$

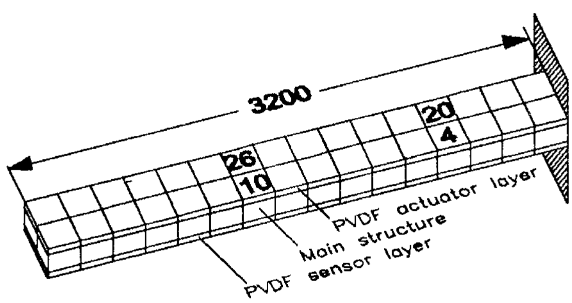

Fig. 1 The cantilever beam with uncertain parameters

Table 1 The first lower eigenvalues of determinate structures

Mode No	Eigenvalues(ω_0)
1	0.86338238902092E+01
2	0.54036695336482E+02
3	0.13218764044069E+03
4	0.15117686023797E+03
5	0.29606045562640E+03
6	0.39952573826196E+03
7	0.48920254320226E+03
8	0.67561320365956E+03
9	0.73069077978485E+03
10	0.96587764277874E+03

In the computations, we assume that $\delta_1 = \delta_2 = \delta_3 = \delta = 0.01$, $\boldsymbol{\Omega}_1 = \boldsymbol{\Omega}_2 = \boldsymbol{\Omega}_3 = \boldsymbol{I}_{4\times4}$, let $\delta_{cj} = 0.0$. In this case, the constraint condition (39) can be changed as

$$\delta_{m4}^2 + \delta_{m10}^2 + \delta_{m20}^2 + \delta_{m26}^2 = \delta_1^2; \qquad \delta_{k4}^2 + \delta_{k10}^2 + \delta_{k26}^2 + \delta_{k26}^2 = \delta_3^2; \tag{51}$$

where δ_{mj}, δ_{kj} ($j = 1, 2, ..., 4$) are the uncertain parameters. Two cases can be considered as the perturbation of the system with deterministic parameters. Firstly, assumed that the mass and stiffness of each element mentioned above is changed into ±1% to those of the determining system at the same time. Then the robustness of intelligent structures with uncertain parameters is obtained, shown in Fig. 2; secondly, assumed that the mass and stiffness of each element mentioned above is changed into ±10% to those of the determining system at the same time. Then the robustness of intelligent structures with uncertain parameters is obtained, shown in Fig. 3. Similarly, if uncertain

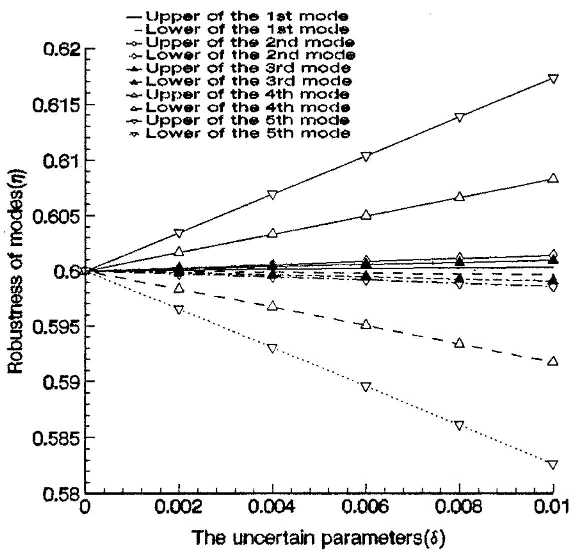

Fig. 2 The upper and lower bounds of robustness of different modes

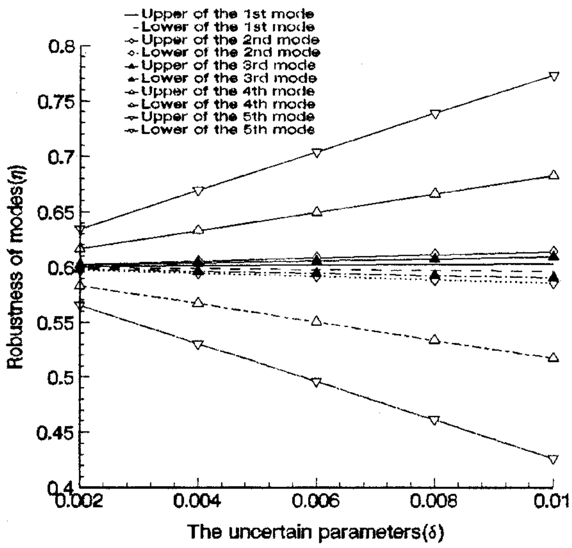

Fig. 3 The upper and lower bounds of robustness of different modes

Table 2 Robustness of intelligent structures with uncertainties ($\alpha_j = 0.4$ and changes as ±5%)

		δ					
order	bounds	0.000	0.002	0.004	0.006	0.008	0.010
1	upper	0.400000	0.400336	0.400673	0.401009	0.401345	0.401682
	lower	0.400000	0.399664	0.399327	0.398991	0.398655	0.398318
2	upper	0.400000	0.401384	0.402767	0.404151	0.405535	0.406918
	lower	0.400000	0.398616	0.397233	0.395849	0.394465	0.393082
3	upper	0.400000	0.400928	0.401856	0.402785	0.403713	0.404641
	lower	0.400000	0.399072	0.398144	0.397215	0.396287	0.395359
4	upper	0.400000	0.408250	0.416500	0.424750	0.433000	0.441251
	lower	0.400000	0.391750	0.383500	0.375250	0.367000	0.358749
5	upper	0.400000	0.417338	0.434677	0.452015	0.469354	0.486692
	lower	0.400000	0.382662	0.365323	0.347985	0.330646	0.313308
6	upper	0.400000	0.405982	0.411964	0.417946	0.423928	0.429910
	lower	0.400000	0.394018	0.388036	0.382054	0.376072	0.370090
7	upper	0.400000	0.409315	0.418630	0.427945	0.437261	0.446576
	lower	0.400000	0.390685	0.381370	0.372055	0.362739	0.353424
8	upper	0.400000	0.479140	0.558281	0.637421	0.716562	0.795702
	lower	0.400000	0.320860	0.241719	0.162579	0.083438	0.004298
9	upper	0.400000	0.402240	0.404481	0.406721	0.408961	0.411202
	lower	0.400000	0.397760	0.395519	0.393279	0.391039	0.388798
10	upper	0.400000	0.430346	0.460692	0.491038	0.521385	0.551731
	lower	0.400000	0.369654	0.339308	0.308962	0.278615	0.248269

Table 3 Robustness of intelligent structures with uncertainties($\alpha_j = 0.4$ and changes as ±10%)

		δ					
order	bounds	0.000	0.002	0.004	0.006	0.008	0.010
1	upper	0.400000	0.400673	0.401345	0.402018	0.402690	0.403363
	lower	0.400000	0.399327	0.398655	0.397982	0.397310	0.396637
2	upper	0.400000	0.402767	0.405535	0.408302	0.411070	0.413837
	lower	0.400000	0.397233	0.394465	0.391698	0.388930	0.386163
3	upper	0.400000	0.401856	0.403713	0.405569	0.407426	0.409282
	lower	0.400000	0.398144	0.396287	0.394431	0.392574	0.390718
4	upper	0.400000	0.416500	0.433000	0.449501	0.466001	0.482501
	lower	0.400000	0.383500	0.367000	0.350499	0.333999	0.317499
5	upper	0.400000	0.434677	0.469354	0.504030	0.538707	0.573384
	lower	0.400000	0.365323	0.330646	0.295970	0.261293	0.226616
6	upper	0.400000	0.411964	0.423928	0.435892	0.447856	0.459820
	lower	0.400000	0.388036	0.376072	0.364108	0.352144	0.340180
7	upper	0.400000	0.418630	0.437261	0.455891	0.474521	0.493151
	lower	0.400000	0.381370	0.362739	0.344109	0.325479	0.306849

parameters of intelligent structures are changed, the robustness of systems can be obtained with the method presented in this paper.

Similarly, if let α = 0.4, robustness of intelligent structures with uncertain parameters can be obtained. Here we only consider two cases as follows: firstly, assume that the mass and stiffness of each element mentioned above is changed into ±5% to those of the determining system at the same time, the robustness of intelligent structures with uncertain parameters is obtained, shown in Table 2; secondly, assume that the mass and stiffness of each element mentioned above is changed into ±10% to those of the determining system at the same time, the robustness of intelligent structures with uncertain parameters is obtained, shown in Table 3.

It can be seen from Figs. 2, 3 and Tables 2, 3 that if α_j is suitably given, robustness of intelligent structures with uncertain parameters can be obtained. Then the dynamic stability tolerance of intelligent structures with uncertain parameters can be keep when α_j is suitably selected.

7. Conclusions

In this paper, the active control of the intelligent structures with the uncertainties is studied and a new method for analyzing the robustness of systems with the uncertainties is presented when as following main steps are done. Firstly, the system with uncertain parameters is considered as the perturbation of the system with deterministic parameters. Secondly, the feedback control law is designed on the basis of deterministic system. Thirdly, perturbation analysis and robustness analysis of intelligent structures with uncertainties are discussed when the feedback control law is applied to the original system and perturbed system. Combining the convex model of uncertainties with the finite element method, the analysis theory of the robustness of intelligent structures with the uncertainties can be developed. The description and computation of the robustness of intelligent structures with uncertain parameters is obtained. The numerical results prove that the present method is effective.

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