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Damage assessment in periodic structures from measured natural frequencies by a sensitivity and transfer matrix-based method

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Abstract. This paper presents a damage assessment procedure applied to periodic spring mass systems using an eigenvalue sensitivity-based method. The damage is directly related to the stiffness reduction of the damage element. The natural frequencies of periodic structures with one single disorder are found by adopting the transfer matrix approach, consequently, the first order approximation of the natural frequencies with respect to the disordered stiffness in different elements is used to form the sensitivity matrix. The analysis shows that the sensitivity of natural frequencies to damage in different locations depends only on the mode number and the location of damage. The stiffness changes due to damage can be identified by solving a set of underdetermined equations based on the sensitivity matrix. The issues associated with many possible damage locations in large structural systems are addressed, and a means of improving the computational efficiency of damage detection while maintaining the accuracy for large periodic structures with limited available measured natural frequencies, is also introduced in this paper. The incomplete measurements and the effect of random error in terms of measurement noise in the natural frequencies are considered. Numerical results of a periodic spring-mass system of 20 degrees of freedom illustrate that the proposed method is simple and robust in locating single or multiple damages in a large periodic structure with a high computational efficiency.

Key words: damage assessment; periodic system; measured natural frequency; sensitivity analysis; transfer matrix.

1. Introduction

As any changes of stiffness result in changes of modal parameters, such as natural frequencies, mode shapes, etc., the location and content of damage in structures can be determined from the changes of modal characteristics (Idichandy and Ganapathy 1990, Kim and Stubbs 1995, Chang 1997). It is widely recognized that the natural frequencies are least contaminated by measurement noise and can generally be measured with good accuracy. In contrast, modal damping and mode shape estimates have error levels as much as 20 times worse than those in the corresponding natural

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frequency estimates (Farrac and Cone 1995). Furthermore, the natural frequencies can be effectively determined by measuring at only one point of the structure and are independent of the position chosen (Messina *et al*. 1996), thus, the method based on the measurement of natural frequencies is potentially very attractive.

The detection of damage in structures via techniques that examine changes in measured natural frequencies is a very important topic of research, several approaches are proposed. Cawley and Adams were among the first to identify the location and provide a rough estimate of structural damage by using an incomplete set of measured natural frequencies (Cawley and Adams 1979). All possible elements of a structure were checked individually to obtain the location of the damage, which makes its implementation impossible in general cases. However, the proposed method by Cawley and Adams cannot provide reliable results for the damage detection in larger structures and multiple damages. Stubbs *et al.* used continuum modelling of structures to the problem of damage detection of large space structures (Stubbs *et al.* 1990). Using the sensitivity relations, a system of equations related to natural frequencies is obtained. Since a continuum model is used, this method cannot be used for determining a specific position of a damage member. Hassiotis and Jeong used the first-order perturbation of the eigenvalue problem to yield the variation in the global stiffness matrix and the eigenvalues (Hassiotis and Jeong 1993, 1995). The result is a set of simultaneous equations that relate the changes in the eigenvalues to those of the element stiffnesses. An optimization algorithm is introduced to solve the set of equations and to identify both the location and magnitude of single or multiple damages, but numerical results from a 90 DOF frame showed that the approach can give good predictions if the number of damaged members is kept below three. Hassiotis introduced an identification algorithm that uses measurements of the Markov parameters in addition to natural frequency measurements to improve the identification of multiple damages (Hassiotis 2000). However, the parametric studies conducted with the new algorithm indicated that the number of measurements needed to identify damage depends on the number of members.

The effectiveness of some of these techniques has been verified on simple structures such as simply supported beams, cantilever beams and frame structures, but the damage detection of complex structures becomes difficult since there exist too many potential damage locations relative to the limited measured natural frequencies (Pabst and Hagedorn 1993). To overcome the problem of expensive computation and time consumption, a lot of efforts have been made. Bicanic and Chen generated a set of equations on the basis of the characteristic equations for the original and the damaged structures, and then utilized the direct iteration and the Gauss-Newton least-squares techniques to determine structural damage from only a limited number of measured natural frequencies (Bicanic and Chen 1997). Studies by Capecchi and Vestroni show that generally the problem can be dealt with in two stages: in the first the damaged zones are located; then in the second the degree of the damage is evaluated (Capecchi and Vestroni 1999). Messina, Williams and Contursi introduced an algorithm which can improve the computational efficiency to limit the damage to a subset of possible sites (Messina *et al*. 1998). The other techniques such as substructural method (Koh *et al*. 1991, Oreta and Tanabe 1994, Yun and Lee 1995) and submatrix scaling factor (Lim 1990, Yun and Bahng 2000) are employed to overcome the issues associated with many unknowns.

Engineering structures including multi-storey buildings, elevated guideways for high speed transportation vehicles ("Maglev" systems), multi-span bridges, multi-blade turbines and rotary compressors, chemical pipelines, stiffened plates and shells in aerospace and ship structures, the proposed space station structures and layered composite structures, can be considered as periodic systems. Accurate wave analysis for the free vibration of periodic structures not requiring the complete modeling of the structure is very appealing (Mead 1996), thus, the sensitivity analysis of the periodic structures can be greatly simplified by utilizing its periodic property, and the computational efficiency can be largely improved.

In this paper, the wave transfer matrix approach is adopted to analyze the free wave motion of large periodic structures with one single disorder (stiffness disorder), and to yield the relationship between the change of the element stiffness and the change of the natural frequencies. Consequently, the sensitivity analysis of natural frequencies to various element disorder of stiffness is conducted. Based on the sensitivity analysis, the location and size of damage of periodic systems can be found by optimally solving a set of simultaneous equations. In addition, a 2-stage detection approach based on the sensitivity analysis has been provided to improve the computational efficiency for the damage detection of large periodic structures. The accuracy and robustness of this proposed method are illustrated by detecting simulated damage of a periodic spring-mass system of 20 elements at one or more sites only based on the first 5 measured natural frequencies with and without noise injection.

2. Natural frequencies of periodic structures with one single disorder

Consider a disordered, fixed-free finite periodic system shown in Fig. 1. The system consists of: (1) the left subsystem with p elements, termed the "L-system"; (2) the disordered element, termed the "D-system"; and (3) the right subsystem with q elements, termed the "R-system". Consider the ith periodic element of the "L-system". Assuming harmonic oscillations, the displacements and internal forces at the right and left boundaries of this element are related by the transfer matrix

$$\begin{Bmatrix} r^L_{(i+1)L} \\ F^L_{(i+1)L} \end{Bmatrix} = \begin{bmatrix} t_{LL} & t_{LR} \\ t_{RL} & t_{RR} \end{bmatrix} \begin{Bmatrix} r^L_{iL} \\ F^L_{iL} \end{Bmatrix} \tag{1}$$

where t_{mn}, m, $n = L, R$ are the elements of the transfer matrix of the periodic element, respectively. The quantity t_{mn} is a function of the structural parameters and frequency. Displacement and force compatibility at the interface of periodic elements requires that $r^L_{(i+1)L} = r^L_{iR}$, and $F^L_{(i+1)L} = -F^L_{iR}$. Moreover, from the reciprocity, $t_{LL}t_{RR} - t_{LR}t_{RL} = 1$.

Based on Floquet theory, the displacement and force at the right and left boundaries of the ith periodic element of the "L-system" are related by

$$r^L_{iR} = r^L_{(i+1)L} = e^{-\mu} r^L_{iL}, \quad F^L_{iR} = -F^L_{(i+1)L} = -e^{-\mu} F^L_{iL} \tag{2}$$

where μ, termed the propagation constant, is determined by the transfer matrix (Faulkner and Hong 1985).

The state vector at the ith periodic element of the "L-system" is expressed in terms of the state vector at the first periodic element as

$$\begin{Bmatrix} r^L_{(i+1)L} \\ F^L_{(i+1)L} \end{Bmatrix} = \begin{bmatrix} T_{LL}(i) & T_{LR}(i) \\ T_{RL}(i) & T_{RR}(i) \end{bmatrix} \begin{Bmatrix} r^L_{1L} \\ F^L_{1L} \end{Bmatrix} \tag{3}$$

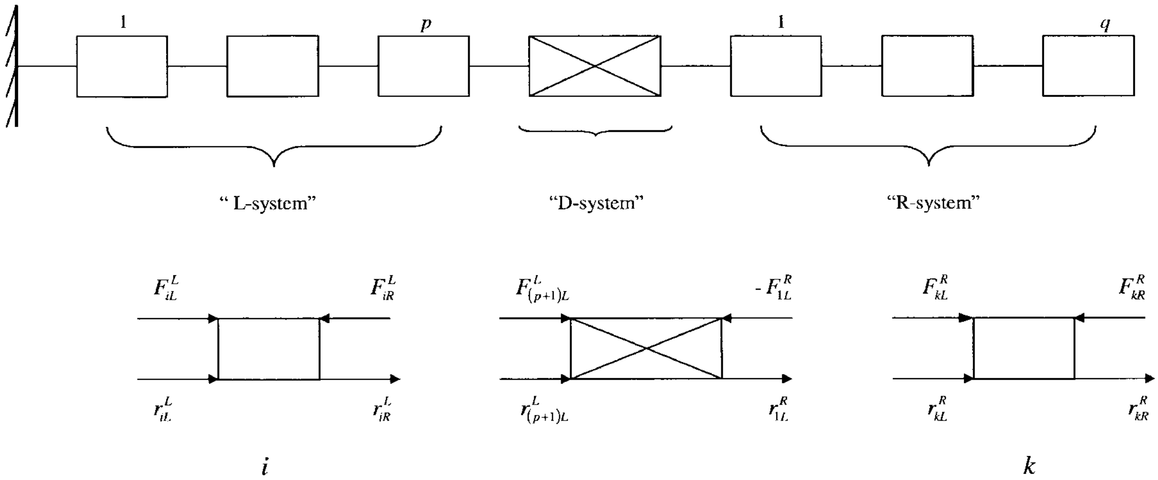

Fig. 1 The disordered, fixed-free finite periodic system

where the terms $T_{LL}(i)$, $T_{LR}(i)$, $T_{RL}(i)$ and $T_{RR}(i)$ are the functions of the elements of the transfer matrix of the "L-system" and the propagation constant (given by Faulkner and Hong 1985).

Referring to Fig. 1, the following relations hold for the displacements and internal forces at the interfaces between subsystems:

$$\begin{Bmatrix} r^L_{(p+1)L} \\ F^L_{(p+1)L} \end{Bmatrix} = \begin{bmatrix} T_{LL}(p) & T_{LR}(p) \\ T_{RL}(p) & T_{RR}(p) \end{bmatrix} \begin{Bmatrix} r^L_{1L} \\ F^L_{1L} \end{Bmatrix}, \qquad \begin{Bmatrix} r^R_{1L} \\ F^R_{1L} \end{Bmatrix} = \begin{bmatrix} t^D_{LL} & t^D_{LR} \\ t^D_{RL} & t^D_{RR} \end{bmatrix} \begin{Bmatrix} r^L_{(p+1)} \\ F^L_{(p+1)} \end{Bmatrix},$$

$$\begin{Bmatrix} r^R_{(q+1)L} \\ F^R_{(q+1)L} \end{Bmatrix} = \begin{bmatrix} T_{LL}(q) & T_{LR}(q) \\ T_{RL}(q) & T_{RR}(q) \end{bmatrix} \begin{Bmatrix} r^R_{1L} \\ F^R_{1L} \end{Bmatrix}. \tag{4a, b, c}$$

where t^D_{mn}, m, n = L, R, are components of the transfer matrix of the disorder element, and $T_{mn}(i)$, m, n = L, R, I = p, q, are determined as in Eq. (3). Combining relations (4a-c), the transfer matrix of the combined system is derived:

$$\begin{Bmatrix} r^R_{(q+1)L} \\ F^R_{(q+1)L} \end{Bmatrix} = \begin{bmatrix} T_{LL} & T_{LR} \\ T_{RL} & T_{RR} \end{bmatrix} \begin{Bmatrix} r^L_{1L} \\ F^L_{1L} \end{Bmatrix} \tag{5}$$

where T_{LL}, T_{LR}, T_{RL} and T_{RR} are expressed in terms of the elements of the transfer matrix of periodic elements and disorder element (Faulkner and Hong 1985).

For the fixed-free system shown in Fig. 1, the following relations hold

$$r^L_{1L} = 0, \quad \text{and} \quad F^R_{(q+1)L} = 0 \tag{6}$$

The natural frequencies of the fixed-free disordered periodic system follow the next equation by substitution of Eqs. (6) into (4)

$$T_{RR} = [T_{RL}(q)t^D_{LL} + T_{RR}(q)t^D_{RL}]T_{LR}(p) + [T_{RL}(q)t^D_{LR} + T_{RR}(q)t^D_{RR}]T_{RR}(p) = 0 \tag{7}$$

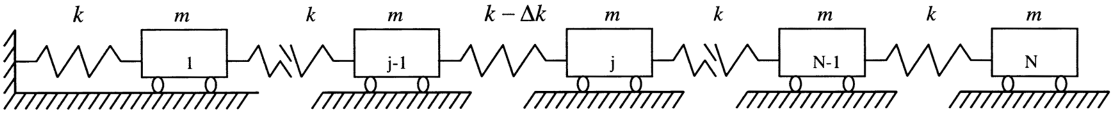

Fig. 2 The fixed-free periodic spring-mass system of N elements with a single disorder of stiffness in the jth element

3. Sensitivity analysis of natural frequencies to damage

The above analysis is quite general and can be applied to various types of finite periodic systems with a single disorder. However, the theory developed above will be applied here only to periodic spring mass systems with one disorder, and only a reduction of stiffness in the disorder element which represents the damage is considered. Typically, if the spring mass system represents the shearing model of a multi-storey building, the displacements and internal forces will be horizontal displacements and the shearing forces of each story.

Fig. 2 shows the N-element periodic spring-mass system with one fixed and one free ends. This finite periodic structure has one disorder in jth element, and only disorder of stiffness reduction is considered in this study. Corresponding to Fig. 1, here, $p = j - 1$, $q = N - j$, and

$$t_{LL} = 1, \quad t_{LR} = -\frac{1}{k}, \quad t_{RL} = m\omega^2, \quad t_{RR} = \frac{k - m\omega^2}{k} \tag{8a, b, c, d}$$

where m, k, are the mass and stiffness of the spring-mass element, respectively, and ω is vibration circular frequency.

The components of the transfer matrix of the disorder element (i.e., the jth element) are

$$t_{LL}^D = 1, \quad t_{LR}^D = -\frac{1}{k - \Delta k_j}, \quad t_{RL}^D = m\omega^2, \quad t_{RR}^D = 1 - \frac{m\omega^2}{k - \Delta k_j} \tag{9a, b, c, d}$$

where Δk_j is the change of stiffness of the jth element, a positive Δk_j indicates the reduction of stiffness, i.e., damage.

By substituting Eqs. (8) and (9) into Eq. (7), and after some manipulation, the characteristic equation for natural frequencies of the periodic spring mass system with a single disorder can be obtained

$$\cos\frac{\gamma}{2}\cos[(N+0.5)\gamma] + \frac{2\Delta k_j}{k}\sin\frac{\gamma}{2}\sin[(N-j+1)\gamma]\cos[(j-0.5)\gamma] = 0 \tag{10}$$

where $\gamma = i\mu$ is also the wave propagation constant.

Specially, substitution of $\Delta k_j = 0$ into Eq. (10) leads to the characteristic equation of natural frequencies of the periodic spring mass structure without any disorders

$$\cos\frac{\gamma}{2}\cos[(N+0.5)\gamma] = 0 \tag{11}$$

Substitution of Eqs. (8a,d) into the expression of the wave propagation constant gives the relation between γ and ω

$$\gamma = 2\arcsin\left(\frac{\omega}{2\omega_0}\right) \tag{12}$$

where $\omega_0 = \sqrt{\frac{k}{m}}$.

Furthermore, substitution of Eq. (12) into Eq. (11) gives the nth circular natural frequency of the finite periodic system of N elements without disorders

$$\omega_u^{(n)} = 2\sqrt{\frac{k}{m}}\sin\left[\frac{\pi}{2}\cdot\frac{2n-1}{2N+1}\right] \qquad (n = 1, 2, \ldots, N) \tag{13}$$

The corresponding propagation constant $\gamma_u^{(n)}$ at the nth natural frequency can be obtained by substituting Eq. (13) into Eq. (12)

$$\gamma_u^{(n)} = \frac{(2n-1)\pi}{2N+1} \tag{14}$$

The sensitivity of the nth natural frequency to the change of stiffness at all elements is given by Eq. (15) below.

$$\frac{\partial\omega_u^{(n)}}{\partial k} = \frac{1}{2}\cdot\frac{1}{k}\cdot\omega_u^{(n)}, \qquad \frac{\dfrac{\partial\omega_u^{(n)}}{\omega_u^{(n)}}}{\dfrac{\partial k}{k}} = \frac{1}{2} \tag{15a,b}$$

Eq. (15a) indicates that for a periodic system the sensitivity of natural frequency to the stiffness of element increases with the natural frequency. Eq. (15b) shows that the sensitivity of relative frequency to the relative change of stiffness is a constant (0.5), which does not depend on the physical properties of the system.

Let $\Delta\gamma_j^{(n)}$ be the change of $\gamma^{(n)}$ induced by the change Δk_j at jth element, the corresponding change $\Delta\omega_j^{(n)}$ of $\omega_u^{(n)}$ can be obtained by Eq. (12)

$$\Delta\omega_j^{(n)} = 2\omega_0\sin\left(\frac{\Delta\gamma_j^{(n)}}{2}\right) \tag{16}$$

By substituting $\gamma = \gamma_u^{(n)} + \Delta\gamma_j^{(n)}$ into Eq. (10), and after some manipulation, the first order approximation for the change $\Delta\gamma_j^{(n)}$ can be found

$$\Delta\gamma_j^{(n)} = -\frac{\Delta\alpha_j\cdot\sin\left(\dfrac{\gamma_u^{(n)}}{2}\right)\{1+\cos[(2j-1)\gamma_u^{(n)}]\}}{(N+0.5)\cos\left(\dfrac{\gamma_u^{(n)}}{2}\right)+\Delta\alpha_j\cdot(N+1.5-2j)\sin\left(\dfrac{\gamma_u^{(n)}}{2}\right)\sin[(2j-1)\gamma_u^{(n)}]} \tag{17}$$

where $\Delta\alpha_j = \frac{\Delta k_j}{k}$ is the relative change of stiffness at the jth element.

Moreover, the first order approximation of the change $\Delta\omega_j^{(n)}$ can be found by combination of Eqs. (16) and (17)

$$\Delta\omega_j^{(n)} = -\frac{0.5\cdot\omega_u^{(n)}\cdot\Delta\alpha_j\cdot\{1+\cos[(2j-1)\gamma_u^{(n)}]\}}{(N+0.5)\cos\left(\frac{\gamma_u^{(n)}}{2}\right)+\Delta\alpha_j(N+1.5-2j)\sin\left(\frac{\gamma_u^{(n)}}{2}\right)\sin[(2j-1)\gamma_u^{(n)}]} \tag{18}$$

Thus, the relative change of the *n*th circular frequency due to the change of stiffness in the *j*th element can be expressed as

$$\Delta\overline{\omega}_j^{(n)} = \frac{\Delta\omega_j^{(n)}}{\omega_u^{(n)}} = -\frac{0.5\cdot\Delta\alpha_j\cdot\{1+\cos[(2j-1)\gamma_u^{(n)}]\}}{(N+0.5)\cos\left(\frac{\gamma_u^{(n)}}{2}\right)+\Delta\alpha_j(N+1.5-2j)\sin\left(\frac{\gamma_u^{(n)}}{2}\right)\sin[(2j-1)\gamma_u^{(n)}]} \tag{19}$$

Furthermore, the sensitivity of the *n*th relative natural frequency to the relative change of stiffness in the *j*th element is given by Eq. (20) below

$$\overline{S}_j^{(n)} = \frac{\partial\overline{\omega}_j^{(n)}}{\partial\alpha_j} = \lim_{\Delta\alpha_j\to 0}\frac{\Delta\overline{\omega}_j^{(n)}}{\Delta\alpha_j} = \frac{1+\cos\left[\frac{(2j-1)(2n-1)}{2N+1}\cdot\pi\right]}{(2N+1)\cdot\cos\left[\frac{2n-1}{2N+1}\cdot\frac{\pi}{2}\right]} \tag{20}$$

Eq. (20) tells us that for a periodic spring-mass system, the sensitivity of relative natural frequency to damage depends only on the number of the element of the periodic system (*N*), the natural frequency number (*n*) and the location of damage (j), it does not depend on the structural parameters such as the stiffness and mass of the periodic system.

From Eq. (20), for given *n* and *N*, the sensitivity of a specified natural frequency varies with the location of damage. The maximum value of sensitivity appears when, $\cos\left[\frac{(2j-1)(2n-1)}{2N+1}\cdot\pi\right] = 1$ i.e.,

$$\overline{S}_{\max}^{(n)} = \mathrm{Max}\{\overline{S}_1^{(n)}\ \ldots\ \overline{S}_j^{(n)}\ \ldots\ \overline{S}_N^{(n)}\} = \frac{2}{(2N+1)\cos\left[\frac{2n-1}{2N+1}\cdot\frac{\pi}{2}\right]}, \tag{21a}$$

and the locations of damage where the maximum value occurs are

$$j \approx \frac{m(2N+1)}{2n-1} + 0.5 \qquad (1 \le j \le N) \tag{21b}$$

where *m* is a non-negative integer.

Eq. (21a) shows that for a given system (i.e., *N* is given) the maximum value, $\overline{S}_{\max}^{(n)}$, of the *n*th natural frequency's sensitivity increases with the natural frequency number (*n*). Among all the maximum values $\overline{S}_{\max}^{(n)}$ ($n = 1, N$), the maximum and minimum values, and the corresponding locations of damage can be obtained by substitution of $n = 1$ and $n = N$ into Eq. (21a, b), respectively

$$(\overline{S}_{\max})_{\min} = \mathrm{Min}\{\overline{S}^{(1)}_{\max}\ \ldots\ \overline{S}^{(i)}_{\max}\ \ldots\ \overline{S}^{(N)}_{\max}\} = \overline{S}^{(1)}_{\max} \approx \frac{2}{(2N+1)}, \qquad (N \gg 1,\ j=1) \qquad (22a)$$

and

$$(\overline{S}_{\max})_{\max} = \mathrm{Max}\{\overline{S}^{(1)}_{\max}\ \ldots\ \overline{S}^{(i)}_{\max}\ \ldots\ \overline{S}^{(N)}_{\max}\} = \overline{S}^{(N)}_{\max} \leq \frac{2}{\pi} = 0.6369, \qquad (N \gg 1) \qquad (22b)$$

$$\text{when } j = \begin{cases} \dfrac{N}{2}+1 & (N \text{ is even}) \\ \dfrac{N+1}{2} & (N \text{ is odd}) \end{cases}.$$

Table 1 shows relation between the value, $(\overline{S}_{\max})_{\max}$, of periodic systems with different element numbers (N) and the corresponding frequency number (n) and location of damage (j). The case only considering the sensitivity of the first five natural frequencies is also shown in Table 1. The results in Table 1 indicate that the most sensitive natural frequency is the highest frequency within a specified range of frequency. The maximum value, $(\overline{S}_{\max})_{\max}$, increases and approaches towards the upper bound (0.6369) with the number of element when the sensitivity of all natural frequencies is considered; it decreases as the number of element increases when only the sensitivities of a couple of natural frequencies are considered. For example, compared with the 10-element system, the sensitivity of the first five frequencies of the 20-element system to damage becomes weaker.

Fig. 3 show the five locations with the highest damage sensitivities for each of the first five natural frequencies of different periodic systems. From the four graphs of this figure, the first five elements are most sensitive for the first natural frequency of all the four systems, and the sensitivity decreases as the element number increases. Further observation from the figure indicates that the

Table 1 The maximum value, $(\overline{S}_{\max})_{\max}$ of sensitivity of natural frequencies to damage and the corresponding locations (n and j)

Number of element (N)		1	2	3	4	5	6	7	8	9	10
Sensitivity of all the natural frequencies	$(\overline{S}_{\max})_{\max}$	0.57735	0.615537	0.625898	0.630142	0.632287	0.633519	0.634291	0.634807	0.635169	0.635432
	n	1	2	3	4	5	6	7	8	9	10
	j	1	2	2	3	3	4	4	5	5	6
Sensitivity of the first five natural frequencies	$(\overline{S}_{\max})_{\max}$	0.57735	0.615537	0.625898	0.630142	0.632287	0.326239	0.205179	0.173143	0.142099	0.115782
	n	1	2	3	4	5	5	5	5	5	5
	j	1	2	2	3	3	5	2,4,7	8	9	3,5,10

Number of element (N)		12	14	16	18	20	22	24	26	28	30
Sensitivity of all the natural frequencies	$(\overline{S}_{\max})_{\max}$	0.635782	0.635997	0.636139	0.636237	0.636308	0.636361	0.636402	0.636433	0.636459	0.636479
	n	12	14	16	18	20	22	24	26	28	30
	j	7	8	9	10	11	12	13	14	15	16
Sensitivity of the first five natural frequencies	$(\overline{S}_{\max})_{\max}$	0.094376	0.077830	0.065278	0.058150	0.051755	0.042269	0.042532	0.039085	0.035949	0.033665
	n	5	5	5	5	5	5	5	5	5	5
	j	6	7	4,8,15	17	5	1,5,10,15	6	24	1,13,26	14

elements with the highest sensitivities for a specified natural frequency are limited to some areas, and the number of the areas is the same as the frequency number. For example, the damage of elements near the two ends of the periodic system produces larger changes in the second frequency than the other elements; similarly, there are three areas where the damage causes larger changes in the third natural frequency.

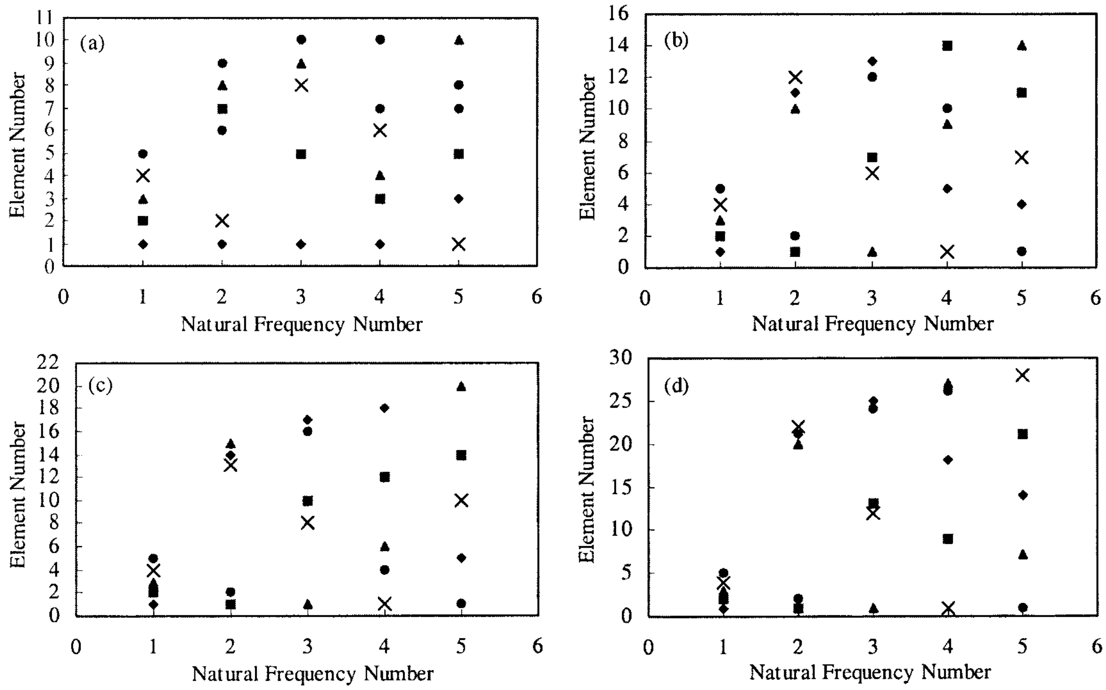

Fig. 3 The 5 locations with the highest damage sensitivities for the first five natural frequencies of the periodic spring-mass systems with different numbers of element: (a) $N = 10$; (b) $N = 15$; (c) $N = 20$; (d) $N = 30$. ◆ the highest; ■ the second highest; ▲ the third highest; × the fourth highest; ● the fifth highest

4. Damage detection based on the sensitivity analysis

The stiffness decrease factor $\Delta\alpha_j$ for the jth element is used to evaluate the degree of damage such that $\Delta\alpha_j = 0$ for no damage and $\Delta\alpha_j = 1$ for complete loss of the jth element (100% damage). For any combination of size and location of damage at one or more sites, it is assumed that the corresponding reductions in the natural frequencies can be written using a linear combination of the sensitivities in the form:

$$\begin{aligned}\delta\overline{\omega}^{(1)} &= \frac{\partial\overline{\omega}^{(1)}}{\partial\alpha_1}\delta\alpha_1 + \frac{\partial\overline{\omega}^{(1)}}{\partial\alpha_2}\delta\alpha_2 + \ldots + \frac{\partial\overline{\omega}^{(1)}}{\partial\alpha_N}\delta\alpha_N \\ &\cdots\cdots \\ \delta\overline{\omega}^{(o)} &= \frac{\partial\overline{\omega}^{(o)}}{\partial\alpha_1}\delta\alpha_1 + \frac{\partial\overline{\omega}^{(o)}}{\partial\alpha_2}\delta\alpha_2 + \ldots + \frac{\partial\overline{\omega}^{(o)}}{\partial\alpha_N}\delta\alpha_N\end{aligned}$$

$$\text{or} \quad \{\delta\overline{\omega}\} = \begin{bmatrix} \dfrac{\partial\overline{\omega}^{(1)}}{\partial\alpha_1} & \cdots & \dfrac{\partial\overline{\omega}^{(1)}}{\partial\alpha_N} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial\overline{\omega}^{(o)}}{\partial\alpha_1} & \cdots & \dfrac{\partial\overline{\omega}^{(o)}}{\partial\alpha_N} \end{bmatrix} \{\delta\alpha\} \tag{23}$$

or

$$\{\delta\overline{\omega}\} = \{S\}\{\delta\alpha\}$$

The set of simultaneous equations in Eq. (23) relate the change of the stiffness of each element, $\{\delta\alpha\}$, to the changes in the natural circular frequency of the structure, $\{\delta\overline{\omega}\}$. In this problem, it is assumed that there are o natural frequencies of the damaged structure available through measurements. The solution of Eq. (23) yields the corresponding changes in the element stiffness. Theoretically, if the number of available changes in frequency, o, is equal to N, a solution may be determined uniquely. However, only a small number of natural frequencies can usually be measured. Hence, the number of the measured changes is less than the number of elements, ($o < N$), which renders the equations underdetermined. They can be solved uniquely only after the introduction of an optimality criterion.

In this study, the optimization problem can be stated as that the best approximations are those which minimize the next vector norm:

$$g = \|[S]\{\delta\alpha\} - \{\delta\overline{\omega}\}\| \tag{24a}$$

Since a positive change in the stiffness can never be produced by damage, the inequality constraint given in Eq. (24b) is introduced.

$$\{\delta\alpha\} \le 0 \tag{24b}$$

In principle, all elements in the structure could be considered as potential damage sites. For a large and complex periodic structure, the optimum solution procedure is computationally expensive. However, the next section discusses a method of reducing computational effort by excluding elements that are unlikely to be damaged.

5. Analytical example

To evaluate the performance and robustness of the proposed method in locating light or severe damage, a periodic spring mass system of 20 elements shown in Fig. 2 is investigated. The mass and stiffness of each spring-mass element are 2.0×10^5 kg and 3.3636×10^8 N/m, respectively. 10 simulated damage states (See Table 2) are used to assess the effectiveness of this approach when the measured modes are incomplete with or without noise.

Without loss of generality, frequency changes for the first 5 modes are used in the following damage localization procedure. In order to simulate an experimental analysis, random noise was added to the values of the natural frequencies. Since Messina *et al.* suggest a standard error of

Table 2 Damage scenarios for the periodic spring mass system of 20 elements

Mode No.	Frequency change $\Delta\omega_i/\omega_i$ in percentage due to damage									
	Case 1		Case 2		Case 3		Case 4		Case 5	
	Element	Damage	Element	Damage	Element	Damage	Element	Damage	Element	Damage
	1	5%	10	5%	20	5%	1	40%	10	40%
1	–0.26		–0.14		0.00		–3.14		–1.78	
2	–0.25		–0.09		–0.01		–3.04		–1.07	
3	–0.25		–0.20		–0.04		–2.87		–2.42	
4	–0.24		–0.04		–0.07		–2.63		–0.45	
5	–0.22		–0.24		–0.11		–2.37		–2.85	
	Case 6		Cases 7		Cases 8		Case 9		Case 10	
	Element	Damage	Element	Damage	Element	Damage	Element	Damage	Element	Damage
	20	40%	1	5%	1	5%	10	5%	1	40%
			10	40%	20	40%	20	40%	10	5%
									20	5%
1	–0.02		–2.02 (–1.71)		–0.28 (–0.23)		–0.16 (–0.25)		–3.27 (–3.12)	
2	–0.17		–1.34 (–1.35)		–0.42 (–0.43)		–0.26 (–0.34)		–3.15 (–3.26)	
3	–0.49		–2.62 (–2.50)		–0.73 (–0.29)		–0.69 (–0.57)		–3.06 (–3.24)	
4	–0.98		–0.72 (–0.77)		–1.20 (–1.04)		–1.00 (–0.99)		–2.76 (–2.91)	
5	–1.60		–3.04 (–2.94)		–1.81 (–2.01)		–1.84 (–1.88)		–2.66 (–2.65)	

±0.15% as a benchmark figure for natural frequencies measured in the laboratory with the impulse hammer technique (Messina *et al*. 1996), the natural frequencies perturbed randomly by 1% were used reasonably as the "measured" ones with noise injection in this study.

First, the first five natural frequencies of the system before damage are obtained by solving Eq. (13), consequently, the sensitivity matrix, $[S]$, is formed through calculating all the sensitivity terms, $\partial\bar{\omega}^{(n)}/\partial\alpha_j$ $(n = 1, 2, \ldots, 5; j = 1, 2, \ldots, N)$, by using Eq. (20). Then a known reduction in stiffness which is referred to as here the actual damage is induced in one or more elements of the periodic system for the 10 cases listed in Table 2. The difference in the first five natural frequencies between the undamaged and damaged models is calculated and shown in Table 2. In Table 2, Cases 1 to 6 do not consider the effect of noise, while Cases 7 to 10 take into account the contamination with noise where the values in the parenthesis are corresponding differences with noise injection. Assuming that this difference is the only known data, the decrease in stiffness is back-calculated by solving the optimization problem of Eqs. (24a,b). The results of this calculation will be referred as the predicted damage in this study.

In the first damage detection, all the 20 elements are considered as the possible damaged elements when searching the solutions of Eq. (24). 10 cases of 5% and 40% stiffness decreases in one or multiple elements are considered and the results of detection for the 10 cases are given by predicted damage 1 shown in Figs. 4-6. The results for damage of single element with stiffness decrease 5% in elements 1, 10 and 20 are shown in the first, second, and last graphs of Fig. 4, respectively. It can be seen that both the location and the magnitude of the light damage were predicted correctly in all

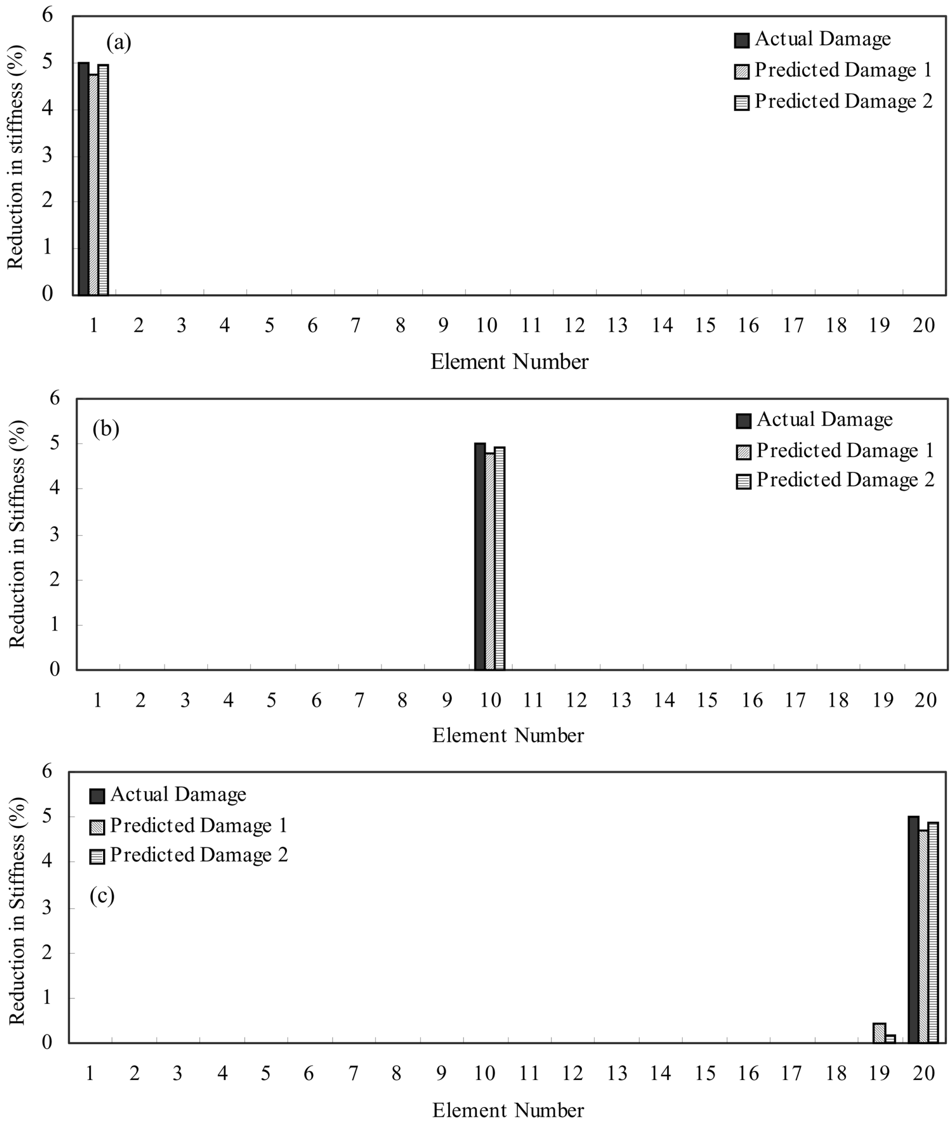

Fig. 4 Detection of light damage of the 20-element periodic spring mass structure without noise injection: (a) Case 1; (b) Case 2; (c) Case 3

cases except that additional element 19 in case 3 was identified as damage element. This is because a damage of element 19 influences the natural frequencies which change largely in the nearly same manner as the corresponding actual damage element 20. However, the identified damage in element 19 is relatively smaller than the identified one of element 20.

The process was repeated for detection of severe damage. The stiffness of the damaged elements 1, 10 and 20 was reduced by 40%. The results for the cases of single severe damage are shown in the three graphs of Fig. 5. It can be observed that the location of the damage in all cases can be correctly identified, but the magnitude of damage was over-predicted. This is because the

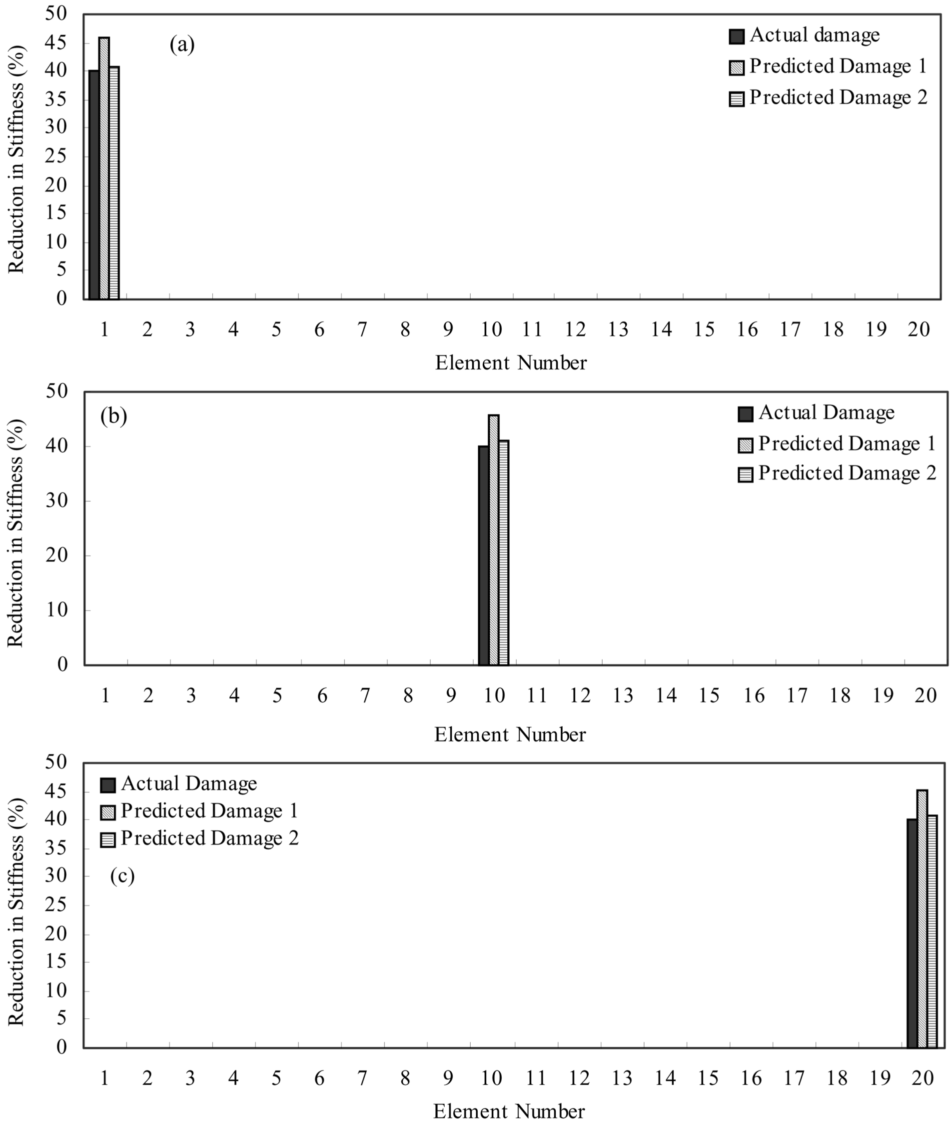

Fig. 5 Detection of severe damage of the 20-element periodic spring mass structure without noise injection: (a) Case 4; (b) Case 5; (c) Case 6

relationship between frequency changes and damage in the cases of severe damage is non-linear but the sensitivity coefficients were derived around the original structure. To combat the significant errors in the size estimates for severe damage, a second order approximation will be conducted in the further research.

Detection of multiple damages is shown in Fig. 6. In the first graph elements 1 and 10 are shown damaged by 5% and 40%, respectively. Stiffness reduction 5% in element 1 and reduction 40% in element 20 are shown in the second graph. The third graph shows element 10 damaged by 5% and element 20 damaged by 40%. The case that element 1 is lightly damaged and the elements 10 and

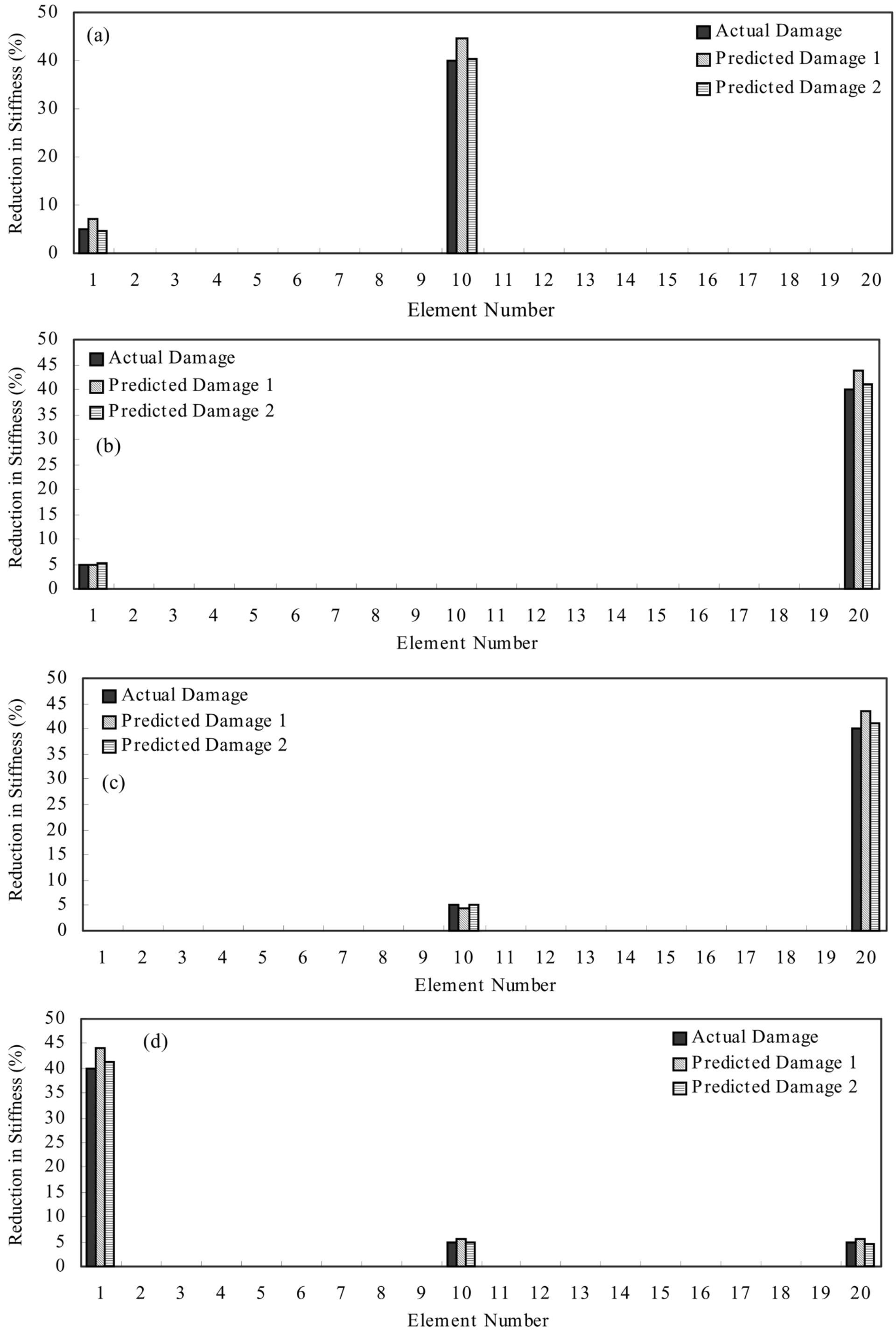

Fig. 6 Detection of multiple damages of the 20-element periodic spring mass structure without noise injection: (a) Case 7; (b) Case 8; (c) Case 9; (d) Case 10

Table 3 The probable damage locations of the 20-element periodic spring mass system identified by the two-stage detection method

Cases	1, 4, 10	2, 5	3, 6	7	9	8
No. of Element	1, 2, 10, 14, 15, 18, 20	1, 2, 3, 5, 10, 14, 17, 20	10, 12, 14 17, 20	1, 2, 3, 5, 10, 14, 15, 17, 20	1, 9, 10, 12 14, 17, 20	1, 9, 10, 12 14, 17, 20; (1, 5, 6, 12, 14, 15, 18, 20)

20 are damaged severely is shown in the last graph. As seen, both the location and the magnitude of the light damage were predicted reasonably well in all cases, but for the elements with severe damage, the location can be correctly identified while the content was over-predicted.

It is well recognized that the search for the optimum solution from Eq. (24) can be computationally expensive if the structure is complex and the number of the potential damaged sites is large, and an erroneous solution may even appear if the amount of available measured frequencies is not sufficient. In other words, significant time savings and improvement in the accuracy of the solution can be achieved if it is possible to limit the search to a sub-set of possible damage sites.

The results from sensitivity analysis shown in Fig. 3 indicate that a specified natural frequency is more sensitive to damage at some locations than at others, in other words, damage at a particular location induces larger changes in some natural frequencies than in others. That is, if only the large differences in the natural frequencies are used, the damaged locations which take the greatest influence on the natural frequencies can be considered as the potential damaged elements in the first stage. It might be expected that this approach could identify the damage site correctly for a single-damage case. In multiple-damage situation, it is particularly true that the use of in-sufficient modes whose frequencies change largely risks losing one or more damaged sites. However, if sufficient modes are considered, the list of possible damage sites can be large enough to minimize this risk. In the second stage, the damage severities of the identified elements are estimated by solving the optimization problem of Eq. (24).

Similarly, 10 cases listed in Table 2 are used to illustrate the benefits of the two-stage detection approach developed herein. Table 3 shows the probable damage locations identified from the first stage, the values in the parenthesis represent the corresponding probable locations by using the data with noise. From the table, it can be seen that the search lists were reduced to not more than nine locations, in some cases only to five locations. The magnitudes of the damage calculated from Eq. (24) corresponding to the 10 cases are shown in Figs. 4-6 by the predicted damage 2. In each case of light damage (stiffness reduction 5%), the correct damage locations were found with good indications of the size of the stiffness reduction. For the cases of severe damage (stiffness reduction 40%), there is a little deterioration in accuracy of the damage size prediction while the identification of damage locations is accurate. However, compared with the damage detection 1, the degree of the accuracy of the damage size detection by the two-stage detection method has been improved largely.

Fig. 7 shows the comparison of results of damage detection with and without noise injection for cases 7-10 by the proposed 2-stage method. It can be seen that even with the addition of noise both the magnitude and location of multiple damages can be identified reasonably well. The errors of the identified magnitude of damage are increased due to the presence of measurement noise. However, this effect is small when the 2-stage detection method is used and the accuracy of detection is acceptable.

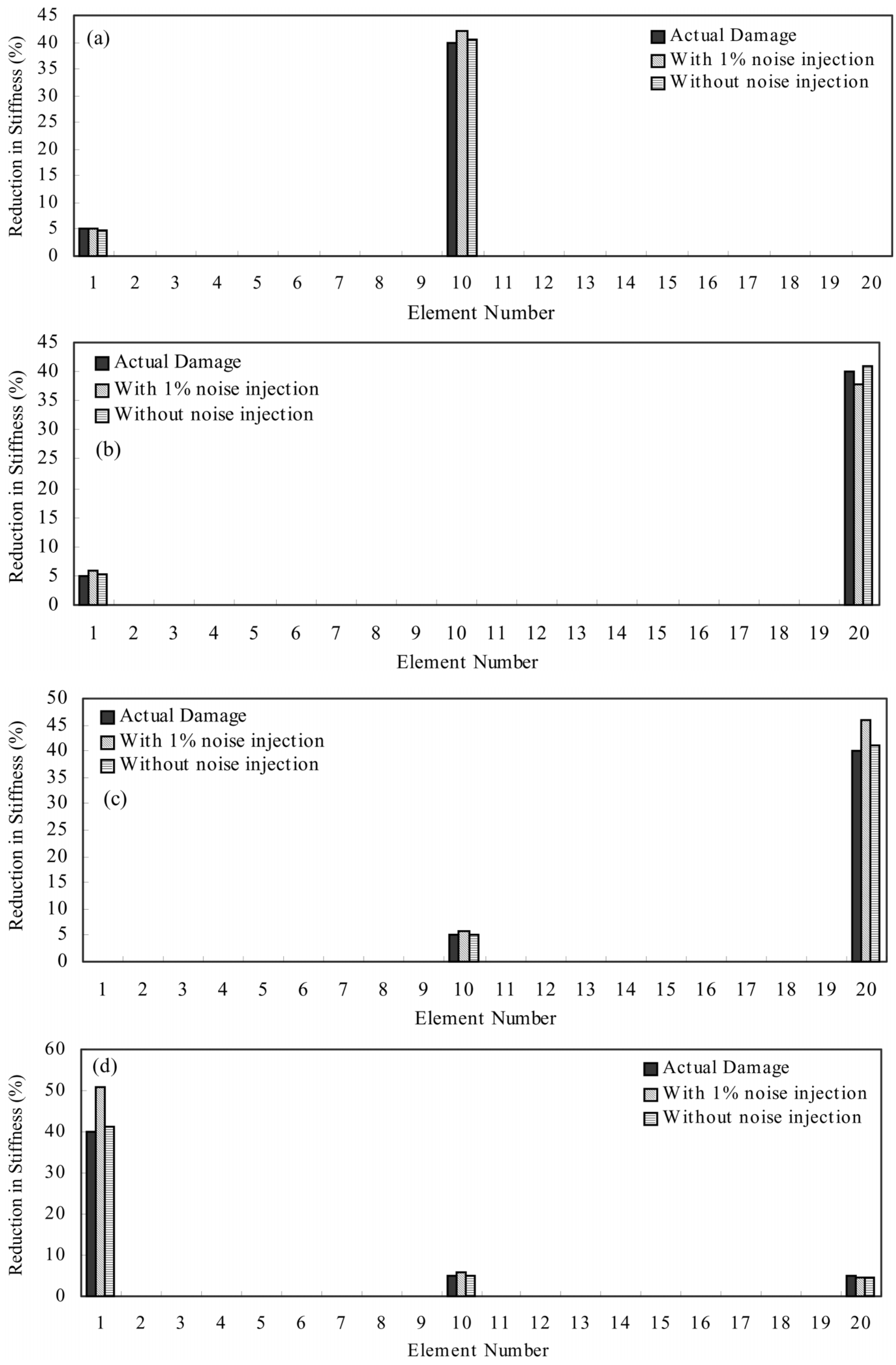

Fig. 7 Detection of multiple damages of the 20-element periodic spring mass structure without and with noise injection by the 2-stage detection method: (a) Case 7; (b) Case 8; (9) Case 9; (10) Case 10

6. Conclusions

The natural frequencies of finite periodic systems with one single disorder were found by adopting the wave transfer matrix method, consequently, the sensitivity of natural frequencies to the stiffness reduction of fixed-free finite spring-mass periodic structures were obtained and presented as a set of undetermined equations. By assuming that damage induces only the loss of structural stiffness, the locations and size of light and severe damages at one or more sites can be determined by solving the optimization problem of these equations. The proposed approach can not only obtain the accurate expressions for the sensitivity of natural frequencies to stiffness, but also simplify the computation by making use of the structural periodic property. Especially, for periodic spring mass systems, the sensitivity matrix only relating to the number of periodic element does not depend on the structural parameters such as stiffness and mass. That is, the identification of the location of damage based on the sensitivity analysis depends only on the number of periodic element and the difference in the natural frequencies between the undamaged and damaged states, it does not require any prior information such as the stiffness and mass on the undamaged structures. This makes the proposed method pretty attractive in practice.

This proposed method was applied to the identification of damage in a periodic spring mass structure of 20 elements. In the application, the changes in the first five natural frequencies were used as the only known variables. Light damage, in the order of 5% decrease in the stiffness, was identified accurately. The location of severe damage, 40% decrease in the stiffness, was also identified correctly, but the magnitude of such severe damage was over-predicted. Since only a few of measured natural frequencies are needed, the implementation of practical measurement is possible. The effect of measured noise on damage detection is also considered by this numerical example. Numerical results indicate that measurement noise affects the damage detection result, however, the detection can be achieved with an acceptable accuracy.

For large periodic systems, a two-stage detection approach has also been proposed to improve the computation efficiency and the degree of accuracy of detection. In the first stage, based on the sensitivity analysis, the elements which are more likely than others to produce large changes in these frequencies whose difference between the undamaged and damaged states is more significant are selected as the probable damage sites. Then, the damage content of the selected elements was estimated by solving the optimal problem of the sensitivity equations in the second stage. Since the number of the parameters to be estimated is kept reasonably small, the result of detection and computation efficiency can be improved, and the number of the required measurement data can be minimized. Numerical analysis of the 20-element periodic spring-mass system shows the feasibility and effectiveness of the proposed 2-stage detection approach.

It must be emphasized that the system considered here is rather simple compared to most real periodic engineering structures. Some matters such as detection of damage in a zone with low sensitivity by the 2-stage method, the general effectiveness of the proposed optimal criterion should be investigated before this proposed approach becomes a truly variable method of damage detection in periodic structures. However, the simplicity of the system allows a number of physical insights to be made, and many of the present results may be of use for guiding the direction of further studies on more complex periodic or near periodic structures.

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