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Non-conforming modes for improvement of finite element performance

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Abstract. This paper presents an efficiency of various non-conforming (NC) modes in development of a series of new finite elements with the special emphasis on 4-node quadrilateral elements. The NC modes have been used as a key scheme to improve the behaviors of various types of new finite elements, i.e., Mindlin plate bending elements, membrane elements with drilling degrees of freedom, flat shell elements. The NC modes are classified into three groups according to the 'correction constants' of 'Direct Modification Method'. The first group is 'basic NC modes', which have been widely used by a number of researchers in the finite element communities. The basic NC modes are effective to improve the behaviors of regular shaped elements. The second group is 'hierarchical NC modes' which improve the behaviors of distorted elements effectively. The last group is 'higher order NC modes' which improve the behaviors of plate-bending elements. When the basic NC modes are combined with hierarchical or higher order NC modes, the elements become insensitive to mesh distortions. When the membrane component of a flat shell has 'hierarchical NC modes', the membrane locking can be suppressed. A number of numerical tests are carried out to show the positive effect of aforementioned various NC modes incorporated into various types of finite elements.

Key words: basic non-conforming modes; Direct Modification Method; hierarchical non-conforming modes; higher order non-conforming modes; membrane locking.

1. Introduction

Efforts to develop more efficient and accurate finite elements have been continued ever since the first introduction of isoparametric element (Iron 1966) and considerable progress has been made in recent years. The research in the non-conforming finite elements has also aroused the interest of a number of researchers in the finite element communities. Wilson *et al.* (1973) and Choi and Schnobrich (1975) are the first contributors for the area. A theoretical investigation was given by Lesaint (1976). When this type of elements took the shape of an arbitrary quadrilateral, the erratic behavior was noted and the element was found to fail the patch test.

Taylor *et al.* (1976) proposed a remedy that in computing strains due to non-conforming modes the Jacobian matrix is replaced with the constant Jacobian matrix computed at the center of each

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element. Wilson and Ibrahimbegovic (1990) proposed ‘B-bar method’ in which a constant correction matrix is added to the strain-displacement matrix.

Recently, Choi *et al.* (2001) proposed ‘Direct Modification Method’ in which the derivatives of various types of NC modes are corrected using correction constants that are analytically calculated. Choi and Lee (2002a) suggested an extended use of the Direct Modification Method in which the NC modes are directly corrected using correction constants that is intended to eliminate the variation of strain energy due to the additional NC displacement modes. Lee (2002) classified various non-conforming (NC) modes into ‘basic NC modes’, ‘hierarchical NC modes’, and ‘higher order NC modes’ according to the correction constants of Direct Modification Method.

This paper presents the efficiency of various NC modes, which were successfully incorporated into the latest development of a series of high performance elements, e.g., non-conforming Mindlin plate bending elements (Choi and Lee 2002a), non-conforming membrane elements with drilling degrees of freedom (Choi *et al.* 2002d), and non-conforming flat shell elements (Choi and Lee 2002b). Numerical tests carried out in this paper reveal that the various NC modes incorporated into the formulation of various elements play an important role to improve the element behavior.

2. Non-conforming modes

The accuracy attainable by the standard and conventional isoparametric element tends to be rather low when subjected to bending as the element deforms in a shear mode. An approach to improve the basic behavior of 2-D isoparametric element by eliminating the excessive shear strain through the addition of non-conforming displacement modes was first adopted by Wilson *et al.* (1973). Choi and Schnobrich (1975) successfully extended this concept to the plate/shell problems.

2.1 Basic concepts

The additional displacement modes are of the same form as what is missing in the original displacement approximation and thus the actual displacement field can be better approximated by the addition of NC modes (Fig. 1). The basic concept of this approach is to restore the true deformation of elements by adding additional deformation modes which is called non-conforming modes as the addition of deformation mode may cause the violation of compatibility along the inter-element boundary.

The total displacement field of any element with additional displacement modes can be expressed as

$$\boldsymbol{u} = \sum N_j \boldsymbol{u}_j + \sum \overline{N}_j \overline{\boldsymbol{u}}_j \tag{1}$$

in which $\overline{N}_j$ are the additional non-conforming modes and $\overline{\boldsymbol{u}}_j$ are the additional unknowns corresponding to the additional displacement modes. It is noted that the additional degrees of freedom $\overline{\boldsymbol{u}}_j$ are interpreted as the amplitudes of the added displacement modes rather than physical displacements at nodes.

The additional unknowns can be determined by minimizing the strain energy in an element. The resulting stiffness matrix has been enlarged over the original element matrix because of the unknowns corresponding to the additional modes. Since there are no loads corresponding to the internal degrees of freedom $\overline{\boldsymbol{u}}_j$ the load-deflection equations may be partitioned as

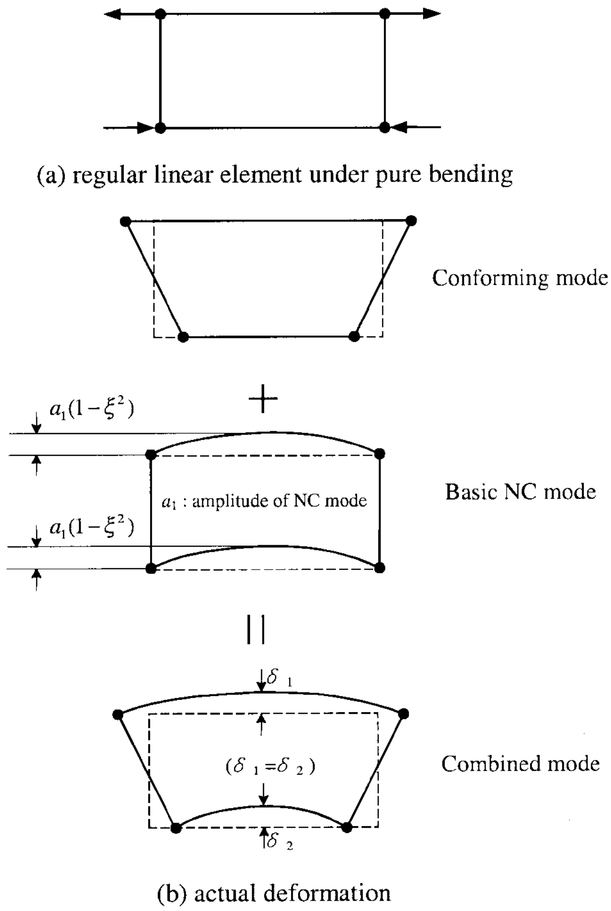

Fig. 1 Regular linear elements under pure bending

$$\begin{bmatrix} \boldsymbol{K}_{CC} & \boldsymbol{K}_{CN} \\ \boldsymbol{K}_{CN}^T & \boldsymbol{K}_{NN} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \bar{\boldsymbol{u}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{0} \end{bmatrix} \tag{2}$$

where the subscript ‘C’ denotes conforming whereas ‘N’ denotes non-conforming part. The enlarged element stiffness matrix in Eq. (2) can be condensed back to the original size of stiffness matrix of the ordinary conforming element through the static condensation as

$$\boldsymbol{K}^e = \boldsymbol{K}_{CC} - \boldsymbol{K}_{CN}\boldsymbol{K}_{NN}^{-1}\boldsymbol{K}_{CN}^T \tag{3}$$

More details of this approach can be found in the references by Choi and his coworkers (Choi and Schnobrich 1975, Choi and Kim 1989, Kim and Choi 1992, Choi *et al.* 1999a, 1999b, 2001, 2002d, Choi and Lee 2002a, 2002b, 2002c).

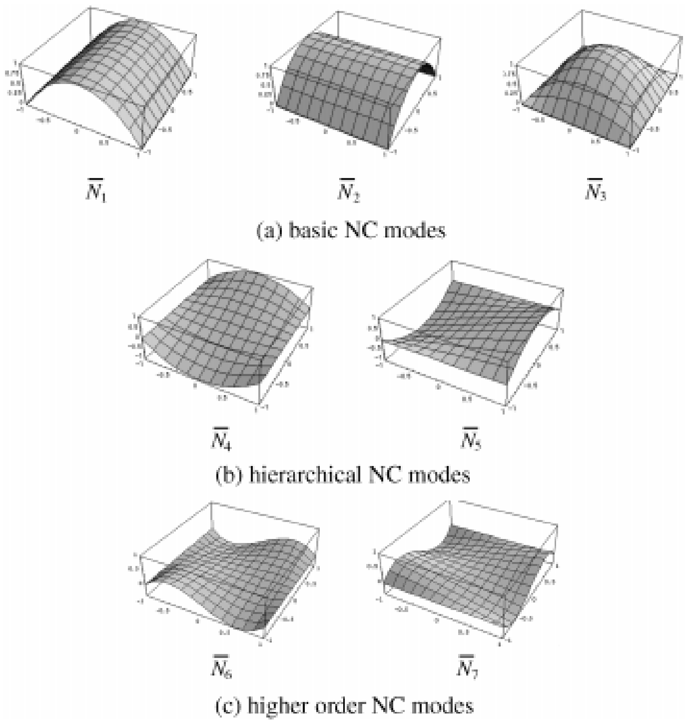

Fig. 2 Non-conforming modes

The possible non-conforming shape functions for qudarilateral elements are given in Eq. (4) and shown graphically in Fig. 2.

$$\overline{N}_1 = 1-\xi^2, \qquad \overline{N}_2 = 1-\eta^2, \qquad \overline{N}_3 = (1-\xi^2)(1-\eta^2) \qquad \text{(Basic NC modes)} \tag{4a}$$

$$\overline{N}_4 = (1-\xi^2)\eta, \qquad \overline{N}_5 = (1-\eta^2)\xi \qquad \text{(Hierarchical NC modes)} \tag{4b}$$

$$\overline{N}_6 = (1-\xi^2)\xi\eta, \qquad \overline{N}_7 = (1-\eta^2)\xi\eta \qquad \text{(Higher order NC modes)} \tag{4c}$$

2.2 Grouping NC modes by direct modification method

Recently, Choi *et al.* (2001) suggested 'Direct Modification Method' which set the non-conforming modes free from patch test failure with relatively small computing efforts. For the sake of better understanding, some important equations are repeated here as follows.

$$(\overline{N}_{j,x})^* = \frac{|\boldsymbol{J}(0,0)|}{|\boldsymbol{J}(\xi,\eta)|}\sum_{\alpha=1}^{2}\left\{J_{1\alpha}^{-1}(0,0)\left(\frac{\partial \overline{N}_j}{\partial \xi_\alpha}+c_{j\alpha}\right)\right\} \tag{5a}$$

$$(\overline{N}_{j,y})^* = \frac{|\boldsymbol{J}(0,0)|}{|\boldsymbol{J}(\xi,\eta)|}\sum_{\alpha=1}^{2}\left\{J_{2\alpha}^{-1}(0,0)\left(\frac{\partial \overline{N}_j}{\partial \xi_\alpha}+c_{j\alpha}\right)\right\} \tag{5b}$$

Table 1 Correction constants for various non-conforming modes

Non-conforming modes $\overline{N}_j$		Correction constants: Derivatives of NC modes $c_{j\xi}$	$c_{j\eta}$	NC modes d_j	Remarks
$\overline{N}_1$	$1-\xi^2$	0	0	–2/3	Basic modes ($c_{j\alpha}=0$, $d_j \neq 0$)
$\overline{N}_2$	$1-\eta^2$	0	0	–2/3	
$\overline{N}_3$	$(1-\xi^2)(1-\eta^2)$	0	0	–4/9	
$\overline{N}_4$	$(1-\xi^2)\eta$	0	–2/3	0	Hierarchical modes ($c_{j\alpha} \neq 0$, $d_j = 0$)
$\overline{N}_5$	$(1-\eta^2)\xi$	–2/3	0	0	
$\overline{N}_6$	$(1-\xi^2)\xi\eta$	0	0	0	Higher order modes ($c_{j\alpha}=0$, $d_j = 0$)
$\overline{N}_7$	$(1-\eta^2)\xi\eta$	0	0	0	

where the correction constants for derivatives of NC modes $c_{j\alpha}$ are defined as Eq. (6) and can be calculated analytically as tabulated in Table 1.

$$c_{j\alpha} = -\frac{1}{4}\int_{-1}^{1}\int_{-1}^{1}\frac{\partial\overline{N}_j}{\partial\xi_\alpha}d\xi d\eta \tag{6}$$

Furthermore, the extended use of Direct Modification Method was proposed by Choi and Lee (2002a) when non-conforming modes are used in rotational fields of plate part. In their study, it is required to add a correction constants d_j to the non-conforming mode $\overline{N}_j$ directly as

$$\int_\Omega \overline{N}_j^* d\Omega = \int\frac{1}{|J(\xi,\eta)|}(\overline{N}_j + d_j)d\Omega = \int_{-1}^{1}\int_{-1}^{1}(\overline{N}_j + d_j)d\xi d\eta = 0 \tag{7}$$

$$d_j = -\frac{1}{4}\int_{-1}^{1}\int_{-1}^{1}\overline{N}_j d\xi d\eta \tag{8}$$

where the correction constants for NC modes d_j in Eq. (8) can be calculated analytically obtaining constants as tabulated in Table 1. More details of this method can be found in the published literatures (Choi *et al.* 2001, Choi and Lee 2002a). According to the value of correction constants of Direct Modification Method in Table 1, various NC modes can be grouped into three categories (Lee 2002); 1) the non-conforming modes with non-zero correction constants for NC modes d_j (Eq. (8)) are defined as 'Basic NC modes' (Eq. (4a)), 2) the non-conforming modes with zero correction constants for NC modes d_j and non-zero correction constants for derivatives of NC modes $c_{j\alpha}$ (Eq. (6)) are defined as 'Hierarchical NC modes' (Eq. (4b)), 3) while those of which the correction constants are all zero and do not need any corrections at all are defined as 'Higher order NC modes' (Eq. (4c)) (Lee 2002).

3. Effectiveness of NC modes

The improved behaviors of the various quadrilateral elements discussed in this paper (see Appendices) are originated mainly from the addition of adequate NC modes and supplementarily from other remedial schemes, e.g., selective/modified integration technique and substitution of shear strain matrix. NC modes included in the formulation of various elements discussed in this paper play an important role to improve element behaviors.

3.1 Basic NC modes

As shown in Fig. 1, the actual deformation of isoparametric type elements under a pure bending can be restored by the addition of quadratic non-conforming mode $a_1(1-\xi^2)$ to the conforming mode which eliminates the excessive shear strain. It is noted here that the basic NC modes, $(1-\xi^2)$ and $(1-\eta^2)$, are sufficient to restore the actual in-plane deformation of rectangular elements.

The analysis for a thick cantilever beam subjected to a parabolically distributed force, as shown in

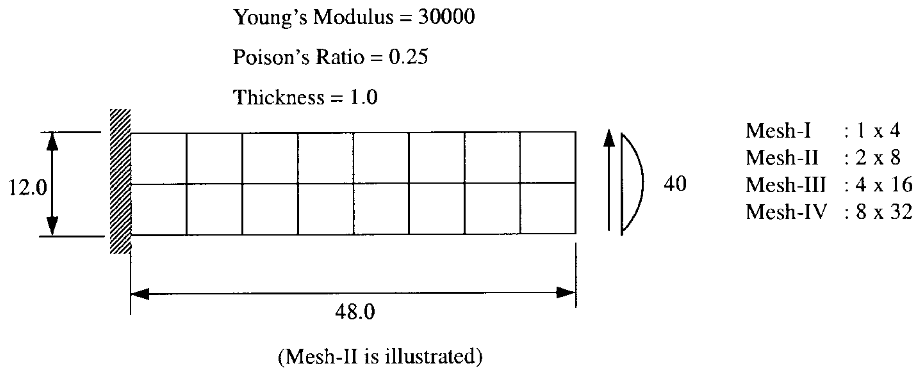

Fig. 3 Thick cantilever beam (regular mesh)

Table 2 Results of cantilever beam : regular mesh (NMD4-series)

Elements	Tip displacements				Normalized Values			
	Mesh-I	Mesh-II	Mesh-III	Mesh-IV	Mesh-I	Mesh-II	Mesh-III	Mesh-IV
CMD4	0.2263	0.3090	0.3424	0.3523	0.636	0.868	0.962	0.990
NMD4-I†	0.3491	0.3515	0.3543	0.3555	0.981	0.988	0.996	0.999
NMD4-II†*	0.3491	0.3517	0.3544	0.3555	0.981	0.988	0.996	0.999
NMD4-III†*	0.3491	0.3517	0.3544	0.3555	0.981	0.988	0.996	0.999
NMD4-IV‡*	0.3491	0.3515	0.3543	0.3555	0.981	0.988	0.996	0.999
Allman (1988)	0.3026	0.3394	0.3512	N/A	0.850	0.954	0.987	N/A
Ibrahimbegovic *et al.* (1990)	0.3445	0.3504	0.3543	N/A	0.968	0.985	0.996	N/A
Choi *et al.* (1999)	0.3445	0.3502	0.3539	N/A	0.968	0.984	0.995	N/A
Frey (1989)	0.3283	0.3460	0.3529	N/A	0.923	0.972	0.992	N/A
Sabir (1985)	0.3281	0.3454	0.3527	N/A	0.922	0.971	0.991	N/A
Reference value	0.3558				1.000			

†Basic NC modes without bubble modes (1,2)
‡Basic NC modes with bubble modes (1,2,3)
*Hierarchical NC modes (4,5)

Table 3 Results of cantilever beam : distorted mesh (NMD4-series)

Elements	Tip displacements			Normalized Values			Table 2 (Mesh-I)
	Point A	Point B	Avg.	Point A	Point B	Avg.	
CMD4	0.1986	0.1991	0.1989	0.558	0.560	0.559	0.636
NMD4-I†	0.3206	0.3237	0.3222	0.901	0.910	0.905	0.981
NMD4-II$^{\dagger *}$	0.3285	0.3311	0.3298	0.923	0.930	0.927	0.981
NMD4-III$^{\dagger *}$	0.3285	0.3310	0.3298	0.923	0.930	0.927	0.981
NMD4-IV$^{\ddagger *}$	0.3214	0.3243	0.3228	0.903	0.911	0.907	0.981
Ibrahimbegovic *et al.* (1990)	N/A	N/A	0.3065	N/A	N/A	0.861	
MacNeal and Harder (1988)	N/A	N/A	0.2978	N/A	N/A	0.837	
Groenwold and Stander (1995)	N/A	N/A	0.3086	N/A	N/A	0.867	
Reference value		0.3558			1.000		

†Basic NC modes without bubble modes (1,2)
‡Basic NC modes with bubble modes (1,2,3)
*Hierarchical NC modes (4,5)

Fig. 3, was carried out to examine the effects of basic NC modes for the regular mesh. The material properties are $E = 30000$, and $\nu = 0.25$. The elastic solution (Timoshenko and Goodier 1951) for the tip displacement is 0.3558 for the properties selected. The fixed boundary condition is idealized by constraining all the degrees of freedom. The results obtained are compared with other elements (Choi *et al.* 1999a, Allman 1988, Ibrahimbegovic *et al.* 1990, Frey 1989, Sabir 1985) (Table 2). All the non-conforming elements, NMD4-I to -VI (Table 7), produce significantly improved solution comparing with those obtained by compatible element (CMD4) and also show good convergence. The NMD4-II to -IV elements have more additional NC modes (hierarchical NC modes) than NMD4-I element but behave similarly. This reveals that in the case of regular mesh the effect of basic NC modes is dominant in the improvement of element behaviors but that of hierarchical NC modes is not so significant. When the distorted elements are used as tested in the next sub-section (Fig. 5), the improvement achieved by the basic NC modes is not sufficient as the NMD4-II and -III elements, which have hierarchical NC modes in addition to basic NC modes, produce more improved solutions than NMD4-I element (Table 3) which has basic NC modes only. It is noted here that the elements of Ibrahimbegovic *et al.* (1990) and Choi *et al.* (1999a) require 3×3 integration whereas NMD4-series elements discussed in this paper require 2×2 integration for evaluating the conforming part of stiffness matrix.

3.2 Hierarchical NC modes

In the practical use, it is desirable to have an element that is insensitive to the mesh distortions and thus able to produce solutions without significant losses in accuracy even though the distorted elements are used. Fig. 4 shows the distorted linear elements under pure bending. Apart from the case of regular element shown in Fig. 1, the actual deformation can be restored only when hierarchical NC modes are additionally used together with basic NC modes. Sticking to this point, Choi *et al.* (2001) suggested that a more general configuration of deformation can be described for non-conforming elements by the addition of a set of hierarchical NC modes ($\bar{N}_4$ and $\bar{N}_5$) to the set of basic NC modes.

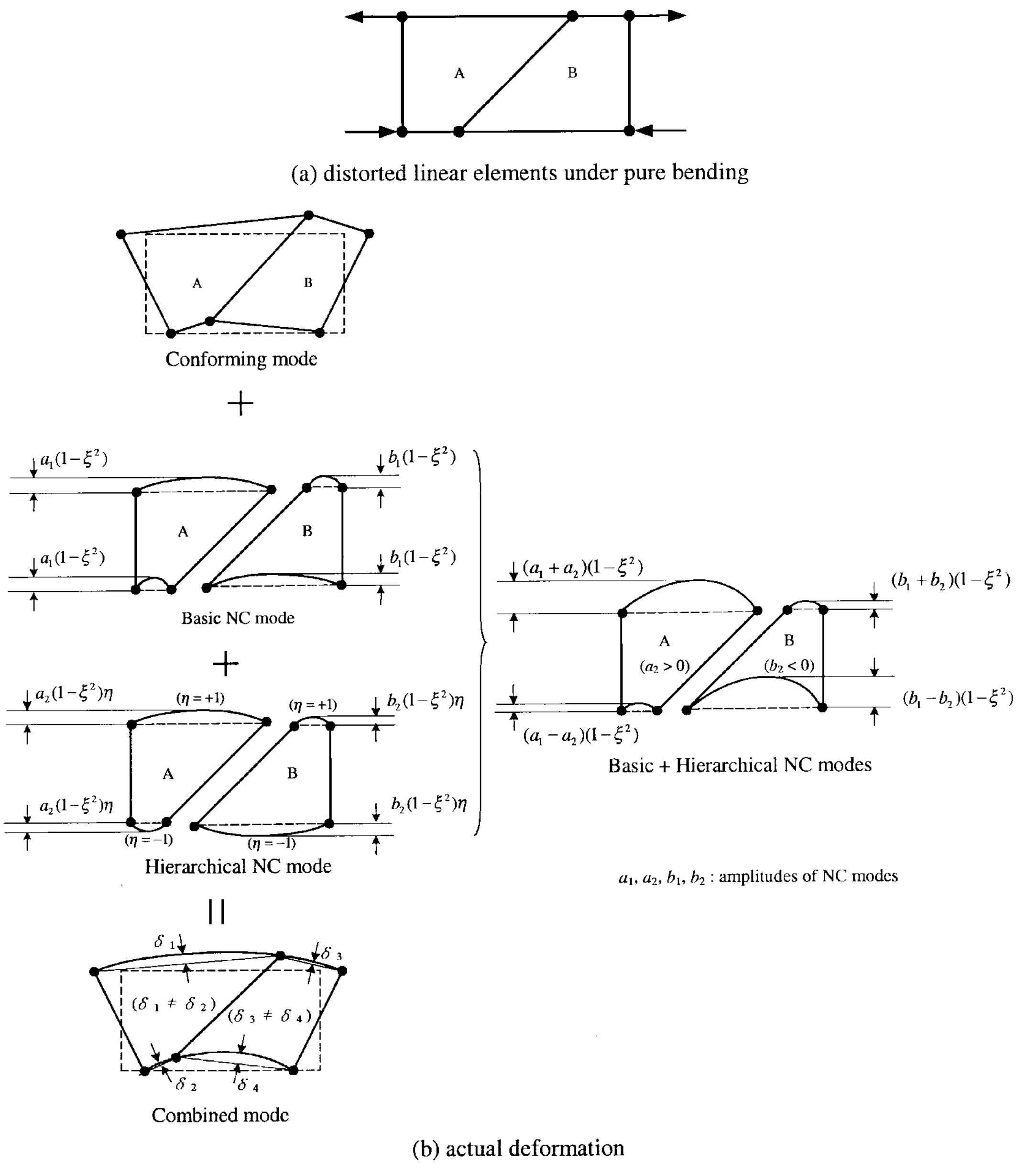

Fig. 4 Distorted linear elements under pure bending

The first test is a thick cantilever beam subjected to a parabolically distributed force modeled with distorted mesh (Fig. 5). This problem was intended to show the effect of combination of various NC modes, i.e., the basic and hierarchical NC modes, for the in-plane behavior of an element, in particular, of a distorted element. The material properties and boundary conditions are same to those of the example depicted in Fig. 3. The numerical results are tabulated in Table 3. When the basic NC modes ($\bar{N}_1$ and $\bar{N}_2$) are added, the behavior of elements NMD4-I to -IV is significantly

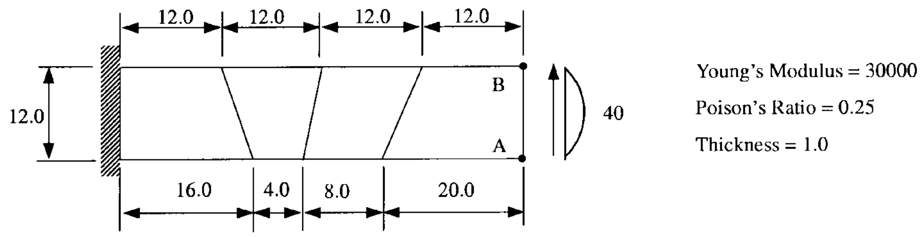

Fig. 5 Thick cantilever beam (distorted mesh)

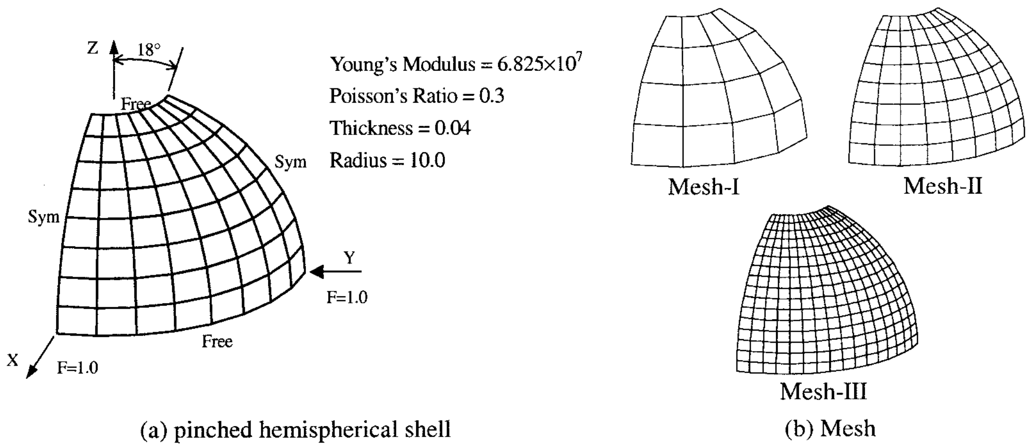

Fig. 6 Pinched hemispherical shell

improved. Furthermore, the elements NMD4-II and NMD4-III are highly accurate by virtue of combined effect of basic and hierarchical NC modes and selective integration scheme (Table 7).

The second test is the pinched hemispherical shell test suggested by NacNeal and Harder (1985). The material properties used in this test are given in Fig. 6. MacNeal and Harder suggested a value of 0.094 as a correct solution for the comparison of results for the displacement in the direction of applied load, and the more recent analyses suggest 0.093 (Simo *et al.* 1989). Thus, the test results are normalized to 0.093 in this paper and tabulated in Table 4. The elements NFS4-I to -IV in a series (Table 8) are free from membrane locking even though coarse meshes are used while elements NFS4-V to -VII suffer from severe membrane locking due to absence of coupling between membrane and bending actions in its formulation. In the formulation of membrane-locking-free elements from NFS4-I to -IV, hierarchical NC modes for membrane component and higher order NC modes for plate bending component are used. Detailed discussion of the role of hierarchical and higher order NC modes in eliminating the membrane locking can be found in the reference by Choi and Lee (2002c).

3.3 Higher order NC modes

The higher order NC modes ($\bar{N}_6$ and $\bar{N}_7$) have been successfully used in the formulations of a series of 8-node Mindlin plate bending elements previously (Choi and Kim 1989). In this case, the

Table 4 Results of pinched hemispherical shell

Elements	Component		Normalized displacements in direction of load			Behavior
	Plate	Membrane	Mesh-I	Mesh-II	Mesh-III	
NFS4-I	CPB4	NMD4-III	0.990	0.999	1.001	Very good (No locking)
NFS4-II	NPB4-I		0.998	1.002	1.002	Best (No locking)
NFS4-III	NPB4-II		1.000	1.002	1.002	
NFS4-IV	NPB4-III		1.000	1.002	1.002	
NFS4-V	NPB4-I	NMD4-I	0.139**	0.685**	0.974	Poor
NFS4-VI	NPB4-II		0.139**	0.685**	0.974	
NFS4-VII	NPB4-III		0.139**	0.685**	0.974	
Groenwold and Stander (1995)			0.939*	0.997	0.997	Good
Taylor (1987)			0.930*	1.012	1.005	
Choi *et al.* (1999)			0.945*	1.000	1.002	
Simo *et al.* (1989)			1.004	0.998	0.999	Best (No locking)

**membrane locking-severe
*membrane locking-moderate

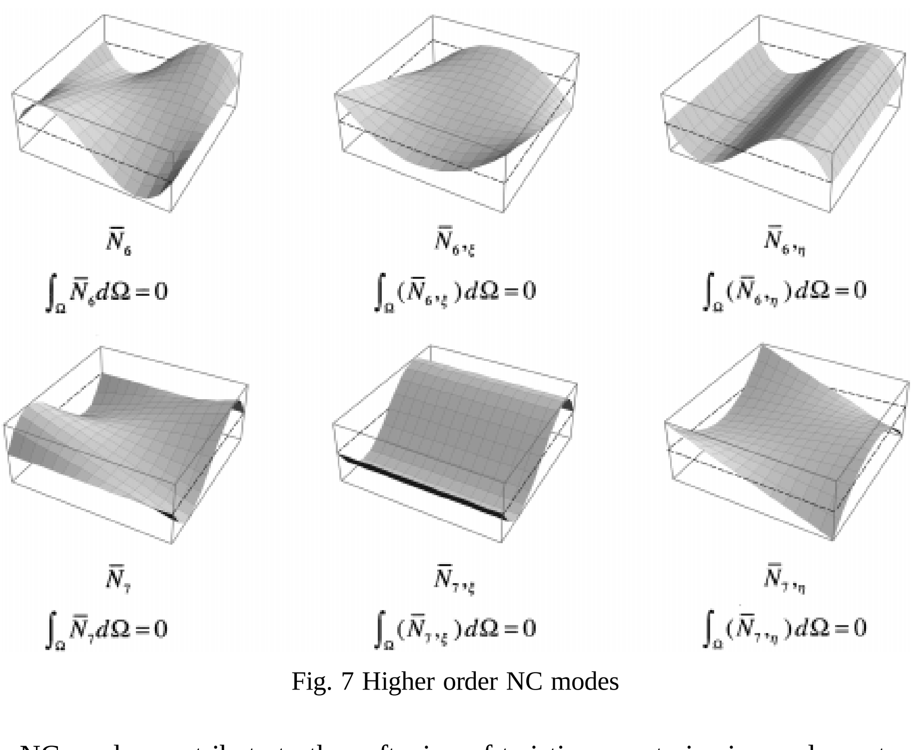

Fig. 7 Higher order NC modes

higher order NC modes contribute to the softening of twisting constrains in an element.

The advantage of use of the higher order NC modes is that the integration of both NC modes and

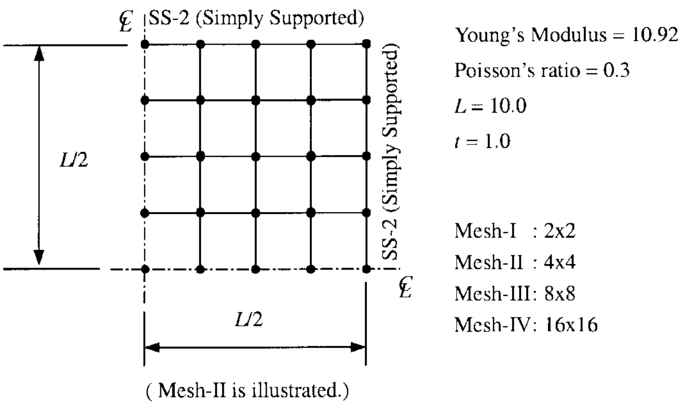

Fig. 8 Simply supported square plate (1/4 model)

Table 5 Results of simply supported thick square plate

Elements	Displacements at center				Normalized Value			
	Mesh-I	Mesh-II	Mesh-III	Mesh-IV	Mesh-I	Mesh-II	Mesh-III	Mesh-IV
NPB4-I†	42.29	42.64	42.71	42.72	0.990	0.998	0.999	1.000
NPB4-II†*	42.57	42.72	42.72	42.73	0.996	1.000	1.000	1.000
NPB4-III‡*	42.58	42.73	42.73	42.73	0.996	1.000	1.000	1.000
Bathe and Dvorkin (1985)	41.90	42.55	42.68	42.72	0.981	0.996	0.999	1.000
Ibrahimbegovic (1992)	44.71	43.23	42.85	42.76	1.046	1.012	1.003	1.001
Theory	42.728				1.000			

†Basic NC modes without bubble modes (1,2)
‡Basic NC modes with bubble modes (1,2,3)
*Higher order NC modes (6,7)

their derivatives over a domain of an element becomes zero (Fig. 7), and therefore, the correction constants (Eq. (6) and Eq. (8)) for the higher order NC modes become always zero. Thus, when these modes are applied to 4-node elements, the strain energy of an element does not change due to the NC modes. Choi and Lee (2002a) successfully used the higher order NC modes ($\bar{N}_6$ and $\bar{N}_7$) in the formulation of 4-node Mindlin plate bending elements for the first time. In their study, combination of basic and higher order NC modes are used in the rotational fields of 4-node Mindlin plate-bending elements and successfully incorporated into the formulation. The improvement achieved is due to the addition of higher order NC modes to basic NC modes that allows the element to have different amplitudes of rotations at the two opposite nodes of element (Choi and Lee 2002a).

The analysis for a thick square plate under uniform load $q = 1.0$ (Fig. 8) was carried out to examine the effects of the higher order NC modes. The length of the plate is 10.0, the thickness is 1.0, and the material properties are Young's modulus $E = 10.92$ and Poisson's ratio $\nu = 0.3$. The numerical results are tabulated in Table 5, together with those of Bathe and Dvorkin (1985) and Ibrahimbegovic (1992) for comparison. The theoretical solutions for thick plate are 42.728 (Timoshenko and Woinowsky-Krieger 1959). The elements 'NPB4-II' and 'NPB4-III' (Table 6), which have higher order non-conforming modes ($\bar{N}_6$ and $\bar{N}_7$), produced better results and faster convergence than those of other elements considered in this study. It is interesting to note that the elements 'NPB4-II' and 'NPB4-III', which has the bubble mode ($\bar{N}_3$) added to NC modes, behave similarly. The reason is that the bubble mode ($\bar{N}$) included in 'NPB4-III' is shifted by correction constant d_3 $(= -4/9)$, and thus it does not change the strain energy of the element to produce no consequence to the element behavior.

3.4 NC modes with other techniques

In general, the merits of NC modes are doubled if the other remedial schemes, e.g., selective/ modified integration technique and substitution of shear strain matrix, are used in a complementary manner (Choi and Kim 1989, Kim and Choi 1992, Choi *et al.* 1999a, 1999b).

In the first phase of development of NMD4-series elements (Choi *et al.* 2002d), two basic NC modes ($\bar{N}_1$ and $\bar{N}_2$) were added to displacement fields and the proper order of numerical integration was adopted. This element was designated as NMD4-I. The element NMD4-II was established by adding a set of hierarchical NC modes ($\bar{N}_4$ and $\bar{N}_5$) to NMD4-I based on the same integration scheme (Table 7). The element NMD4-II produces better solutions than NMD4-I, but the element may be rank deficient since the eigenvalue analysis of a single element matrix gives one extra zero energy mode besides three zero energy modes associated with rigid-body mode. A simple modification of integration scheme in forming NMD4-III, i.e., the use of 5-point integration scheme for the integration of submatrix $\bar{\boldsymbol{B}}^{*T}\boldsymbol{D}\bar{\boldsymbol{B}}^{*}$, removes spurious zero energy modes without deteriorating the high performance of NMD4-II (Choi *et al.* 2002d).

In the formulation of all the plate bending elements listed in Table 6, the shear strain matrix is substituted by the assumed shear strain matrix by mixed formulation (Choi and Lee 2002a). The shear-locking-free behaviors of these elements are mainly attributable to the contribution of substituted shear strain matrix.

4. Conclusions

The efficiency of various NC modes, i.e., basic NC modes, hierarchical NC modes, and higher order NC modes, which were successfully incorporated into the formulation of a series of new finite element, was discussed in this paper.

Fig. 9 summarizes the role of various NC modes studied in this paper. 1) when the basic NC modes are used in the formulation of in-plane elements (NMD4-I to -IV) and out-of-plane elements (NPB4-I to -III), the element behaviors can be significantly improved. 2) in addition to the basic NC modes, hierarchical NC modes, which were successfully used in the formulation of in-plane elements (NMD4-II to -IV), improve the behavior of an element in distorted mesh. 3) the higher order NC modes, used in the formulation of plate bending elements (NPB4-II and NPB4-III), produce better results than the element NPB4-I, which have basic NC modes only. 4) when the basic NC modes are combined with hierarchical and/or higher order NC modes, the elements become insensitive to mesh distortions in addition to the high accuracy and fast convergence of regular mesh. 5) the flat shell elements formulated by combining the membrane component, in which hierarchical NC modes are used, and the plate-bending component, in which higher order NC modes are used, do not show membrane locking. 6) the merits of various NC modes are doubled if

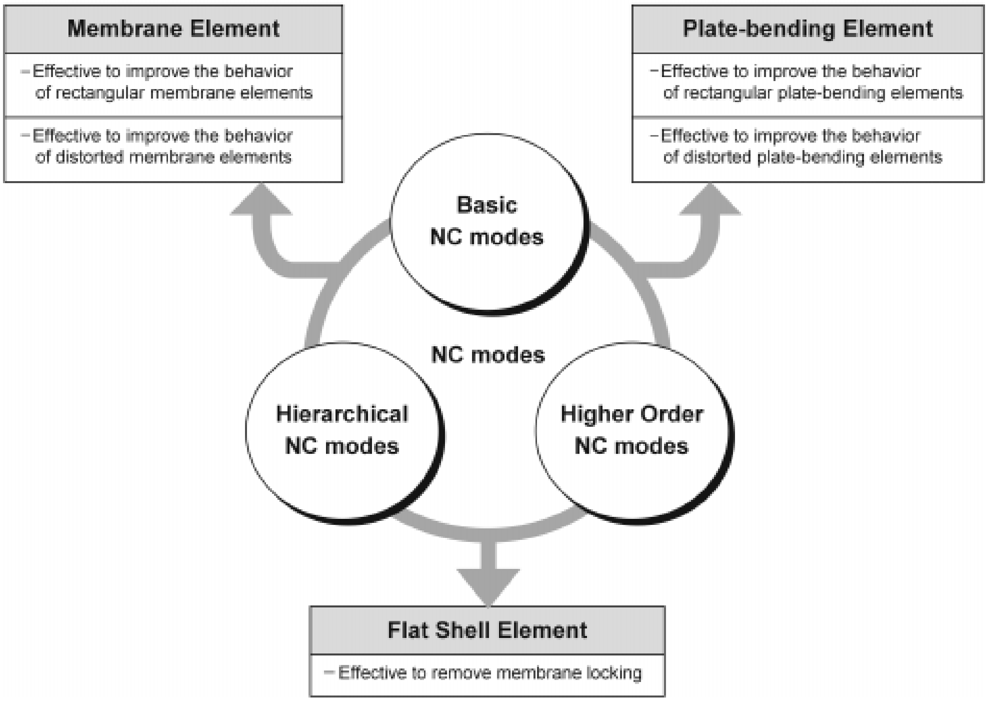

Fig. 9 Effectiveness of various NC modes

the other remedial schemes, e.g., selective/modified integration technique and substitution of shear strain matrix, are used in a complementary manner.

Appendix A: Non-conforming Mindlin plate elements

The establishment of a series of new Mindlin plate elements was proposed (Choi and Lee 2002a). Among the elements established, the elements designated as NPB4-II and NPB4-III are of special interests in the view of their excellent performance where NPB4 indicates '**4**-node **N**on-conforming **P**late **B**ending element' (Table 6).

Table 6 4-node non-conforming Mindlin plate bending elements

Elements	Non-conforming modes (θ_x, θ_y)	Integration scheme						Correction Methods
		$\boldsymbol{K}_{CC}$		$\boldsymbol{K}_{CN}$		$\boldsymbol{K}_{NN}$		
		Bending	Shear	Bending	Shear	Bending	Shear	
CPB4*	-	2 × 2	2 × 2	-	-	-	-	-
NPB4-I	$\overline{N}_1$, $\overline{N}_2$	2 × 2	2 × 2	2 × 2	2 × 2	2 × 2	2 × 2	DMM**
NPB4-II	$\overline{N}_1$, $\overline{N}_2$, $\overline{N}_6$, $\overline{N}_7$	2 × 2	2 × 2	3 × 3	2 × 2	3 × 3	2 × 2	DMM
NPB4-III	$\overline{N}_1 \sim \overline{N}_3$, $\overline{N}_6$, $\overline{N}_7$	2 × 2	2 × 2	3 × 3	2 × 2	3 × 3	2 × 2	DMM

*identical to MITC4 (Bathe and Dvorkin 1985)
**DMM : Direct Modification Method

Appendix B: Non-conforming membrane elements with drilling DOF

The stiffness matrices in Eq. (2) are written as

$$\boldsymbol{K}_{CC} = \int_V \begin{bmatrix} \boldsymbol{B}^T\boldsymbol{D}\boldsymbol{B} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{bmatrix} dV + \alpha\mu\int_V \begin{bmatrix} \boldsymbol{G}^T\boldsymbol{G} & \boldsymbol{G}^T\boldsymbol{N} \\ \boldsymbol{N}^T\boldsymbol{G} & \boldsymbol{N}^T\boldsymbol{N} \end{bmatrix} dV \tag{A.1a}$$

$$\boldsymbol{K}_{CN} = \int_V \begin{bmatrix} \boldsymbol{B}^T\boldsymbol{D}\overline{\boldsymbol{B}}^* \\ \boldsymbol{O} \end{bmatrix} dV + \alpha\mu\int_V \begin{bmatrix} \boldsymbol{G}^T\overline{\boldsymbol{G}}^* \\ \boldsymbol{N}^T\overline{\boldsymbol{G}}^* \end{bmatrix} dV \tag{A.1b}$$

$$\boldsymbol{K}_{NN} = \int_V \overline{\boldsymbol{B}}^{*T}\boldsymbol{D}\overline{\boldsymbol{B}}^* dV + \alpha\mu\int_V \overline{\boldsymbol{G}}^{*T}\overline{\boldsymbol{G}}^* dV \tag{A.1c}$$

where $\boldsymbol{B}$ and $\boldsymbol{G}$ are the strain-displacement matrices related to conforming displacement and $\overline{\boldsymbol{B}}^*$ and $\overline{\boldsymbol{G}}^*$ are the strain-displacement matrices modified by Direct Modification Method which are related to non-conforming displacement, respectively. And μ is the shear modulus and α is the problem dependent constant.

According to the non-conforming modes used in the formulation, a series of NMD4-I ~ IV (**4**-node **N**on-conforming **M**embrane element with **D**rilling DOF) were established (Choi *et al.* 2002d). Among the elements established, NMD4-III is of special interests in the view of its excellent performance (Table 7).

Table 7 4-node non-conforming membrane elements with drilling DOF

Elements	Non-conforming modes	Integration schemes						Correction Method
		$\boldsymbol{K}_{CC}$		$\boldsymbol{K}_{CN}$		$\boldsymbol{K}_{NN}$		
	u, v	$\boldsymbol{B}^T\boldsymbol{D}\boldsymbol{B}$	$\boldsymbol{G}^T\boldsymbol{G}$, $\boldsymbol{G}^T\boldsymbol{N}$, $\boldsymbol{N}^T\boldsymbol{N}$	$\boldsymbol{B}^T\boldsymbol{D}\overline{\boldsymbol{B}}^*$, $\boldsymbol{G}^T\overline{\boldsymbol{G}}^*$	$\boldsymbol{N}^T\overline{\boldsymbol{G}}^*$	$\overline{\boldsymbol{B}}^{*T}\boldsymbol{D}\overline{\boldsymbol{B}}^*$	$\overline{\boldsymbol{G}}^{*T}\overline{\boldsymbol{G}}^*$	
CMD4*	-	2×2		-				
NMD4-I	$\overline{N}_1$, $\overline{N}_2$	2×2		2×2	3×3	2×2		
NMD4-II	$\overline{N}_1$, $\overline{N}_2$, $\overline{N}_4$, $\overline{N}_5$	2×2		2×2	3×3	2×2		DMM
NMD4-III	$\overline{N}_1$, $\overline{N}_2$, $\overline{N}_4$, $\overline{N}_5$	2×2		2×2	3×3	5-point	2×2	
NMD4-IV	$\overline{N}_1 \sim \overline{N}_5$	2×2		2×2	3×3	3×3		

*CMD4 means '**4**-node **C**onforming **M**embrane element with **D**rilling DOF'

Appendix C: Non-conforming flat shell elements

A series of non-conforming flat shell elements (NFS4-I ~ VII) (Choi and Lee 2002b, Choi and Lee 2002c) can be established by the linear combination of plate bending elements (NPB4-series) (Choi and Lee 2002a) and membrane elements (NMD4-series) (Choi *et al.* 2002d), where 'NFS4' means '**4**-node **N**on-conforming **F**lat **S**hell element' (Table 8). At the final stage of the element

Table 8 Types of flat shell elements

Elements	Plate			Membrane		
	Elements	Non-conforming modes		Elements	Non-conforming modes	
		w	θ_x, θ_y		u, v	θ_z
NFS4-I	CPB4	-	-	NMD4-III	$\overline{N}_1$, $\overline{N}_2$, $\overline{N}_4$, $\overline{N}_5$	-
NFS4-II	NPB4-I	-	$\overline{N}_1$, $\overline{N}_2$			
NFS4-III	NPB4-II	-	$\overline{N}_1$, $\overline{N}_2$, $\overline{N}_6$, $\overline{N}_7$			
NFS4-IV	NPB4-III	-	$\overline{N}_1 \sim \overline{N}_3$, $\overline{N}_6$, $\overline{N}_7$			
NFS4-V	NPB4-I	-	$\overline{N}_1$, $\overline{N}_2$	NMD4-I	$\overline{N}_1$, $\overline{N}_2$	-
NFS4-VI	NPB4-II	-	$\overline{N}_1$, $\overline{N}_2$, $\overline{N}_6$, $\overline{N}_7$			
NFS4-VII	NPB4-III	-	$\overline{N}_1 \sim \overline{N}_3$, $\overline{N}_6$, $\overline{N}_7$			

formulation, the rigid link correction (Taylor 1987) is applied to consider the warped geometry. More detailed discussion can be found in the references by Choi and Lee (2002b, 2002c).

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