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A matrix displacement formulation for minimum weight design of frames

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Abstract. A static linear programming formulation for minimum weight design of frames that is based on a matrix displacement method is presented in this paper. According to elementary theory of plasticity, minimum weight design of frames can be carried out by using only the equilibrium equations, because the system is statically determinate when at an incipient collapse state. In the present formulation, a statically determinate released frame is defined by introducing hinges into the real frame and the bending moments in yield constraints are expressed in terms of unit hinge rotations and the external loads respectively, by utilizing the matrix displacement method. Conventional Simplex algorithm with some modifications is utilized for the solution of linear programming problem. As the formulation is based on matrix displacement method, it may be easily adopted to the weight optimization of frames with displacement and deformation limitations. Four illustrative examples are also given for comparing the results to those obtained in previous studies.

Key words: minimum weight design; optimal plastic design; linear programming.

1. Introduction

It is well known that in the first order elementary theory of plasticity where the plastic hinge concept is used, internal forces of the frame can be determined by using only on the equilibrium equations because the structure is statically determinate at an incipient collapse state. Thus, overall flexibility equations are ignored and deflection requirements are not included in the analysis. Furthermore, if the weight of structure and the yield constraints are expressed linearly in terms of the design variables, the minimum weight design process may be transformed into a linear programming problem and the structure can be designed as having minimum weight.

Heyman (1951, 1959, 1960) and Foulkes (1953, 1954) formulated the minimum weight design problem in the form of a linear objective function subject to linear constraints during the 1950s. In these studies, by using upper bound theorem (or kinematic method) and by considering all single degree of freedom mechanisms or just the elementary mechanisms, the cross-sectional properties of frame members that minimize the weight of structure have been determined. In the mentioned studies, the optimization process has been very difficult due to the large number of elementary mechanisms and the accuracy of the optimum solution, which assumes that all these mechanisms are correct.

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The first computerized optimum design method was developed by Livesley (1956) by using the lower bound theorem (or static method). In Livesley's method, the bending moments at all critical sections of frame are expressed in terms of external loads and redundant forces, then the yield constraints are constructed. The internal forces and the member plastic moments that satisfy the yield constraints and minimize the weight function are achieved as the solution of the optimization problem. In the following years, extensive theoretical and numerical studies have been carried out to extend the previous techniques (Prager and Shield 1967, Heyman 1971, Toakley 1968).

In recent years, various static and kinematic linear programming formulations have been developed for optimal design of frames. The mentioned kinematic linear programming formulations generally have aimed to reduce the number of elementary mechanisms and to generate these mechanisms automatically (Watwood 1979, Tam and Jennings 1992). In the static linear programming formulations however, condensing the tableau size by developing the revised linear programming techniques and the automatic construction of equilibrium equations from input data defining the frame geometry, have been the goal (Nguyen 1984, Jennings and Tam 1986).

In this paper, a static linear programming formulation that is based on the matrix displacement method is presented. In this method, a statically determinate released frame is defined by introducing hinges (called "introduced hinges") into the statically indeterminate real frame. The bending moments at the critical sections are obtained by analyzing the frame for unit introduced hinge rotations respectively and for the external loads. Thus, yield constraints are formulated in terms of introduced hinge rotations instead of redundant forces. Simplex Method with efficient modifications is utilized for the solution of linear programming problem. Thus, an upper bound variable is defined for positivity condition and initial feasible basic solution is obtained simply by applying a pivoting technique to conventional Simplex algorithm. Furthermore, it is explained how the displacement and plastic hinge rotation limitations are considered in order to adopt the formulation to weight optimization of frames that collapse before the mechanism load due to the excessive displacement and deformations.

2. Formulation of minimum weight design problem

2.1 Assumptions

The presented optimum design formulation can be applied to plane frames which comply with the following assumptions:

a) The frame behaves in a rigid-plastic manner in which only flexural action is significant.
b) The frame members are straight and prismatic.
c) The frame members are subjected to only concentrated loads.
d) The plastic hinges can only form at the ends of the members or at the concentrated load application points.

2.2 Yield constraints

Yield constraints are inequalities that specify that the bending moment at a potential plastic hinge point (called a "critical section") cannot exceed the member plastic bending moment. Yield constraints for a critical section on a frame member may be expressed as

$$|m| - m_p \leq 0 \tag{1}$$

where,

m is the bending moment of a critical section,

m_p is the plastic bending moment of a critical section.

The sign of bending moments are not known in the beginning of the optimization procedure, the inequality Eq. (1) gives two conditions for a critical section as

$$m - m_p \leq 0 \tag{2}$$

and

$$-m - m_p \leq 0 \tag{3}$$

It is well known that according to the elementary theory of plasticity, a structure is statically determinate at incipient collapse state and that bending moments may be determined depending only on the equilibrium equations.

The bending moment at a critical section of a frame which is transformed into statically determinate by introducing hinges, may be expressed in terms of those introduced hinge rotations and the external loads by superposition as

$$m_i = \sum_{j=1}^{n} (m)_{i(\phi_j = 1)}\phi_j + m_{0i} \qquad (i = 1 \text{ to } c) \tag{4}$$

In this expression,

n is the number of introduced hinges or degree of redundancy,

c is the number of critical sections,

ϕ_j is the rotation of jth introduced hinge,

$(m)_{i(\phi_j = 1)}$ is the bending moment at the ith critical section due to the unit value of ϕ_j introduced hinge rotation, and

m_{0i} is the bending moment at the ith critical section due to the external loads while the all introduced hinge rotations are equal to zero.

If the bending moment expression in (4) is written for all critical sections in matrix form, it yields

$$\boldsymbol{m} = \boldsymbol{m}_{\phi}\boldsymbol{\phi} + \boldsymbol{m}_{\mathbf{0}} \tag{5}$$

where,

$\boldsymbol{m}_\phi$ is a $c \times n$ matrix of bending moments due to the introduced hinge rotations. Any $m_{\phi_{ij}}$ element of the matrix is equal to bending moment at the ith critical section due to the $\phi_j = 1$ rotation, while the other introduced hinge rotations and external loads are equal to zero,

$\boldsymbol{\phi}$ is a $n \times 1$ vector of introduced hinge rotations,

$\boldsymbol{m}_{\mathbf{0}}$ is a $c \times 1$ vector of bending moments due to the external loads while all introduced hinge rotations are equal to zero.

Since the sign of the bending moments at critical sections are unknown in the beginning of the optimization procedure, according to expressions (2) and (3), the bending moment matrix $\boldsymbol{m}$ includes $2c$ rows.

If the bending moment matrix of critical sections in (5) is substituted in (1) and if the group number of members which consist of the same cross-section is represented as g, the yield constraints expressed in matrix form, are as follows,

$$\boldsymbol{m}_{\phi}\phi - \boldsymbol{t}\boldsymbol{m}_{p} + \boldsymbol{m}_{0} \leq \boldsymbol{0} \tag{6}$$

where,

$\boldsymbol{m}_p$ is a $g \times 1$ vector of plastic bending moments of the group members, and

$\boldsymbol{t}$ is a $2c \times g$ matrix which contains information about the plastic moment groups. For any critical section of the member, the elements of matrix $\boldsymbol{t}$ corresponding to the section plastic moment are equal to unity while the remaining elements are zero.

The elements of matrices $\boldsymbol{m}_{\phi}$ and $\boldsymbol{m}_{0}$ are easily obtained by analysing the statically indeterminate frame by the conventional matrix displacement method for $(n + 1)$ loading vectors that are due to the unit introduced hinge rotations and due to the external loads. Since the load vector due to the external loads is well known, only the fixed end forces due to the unit rotation of an introduced hinge located on any section of prismatic frame member is given in Fig. 1.

2.3 Weight function

In the present formulation, the weight of structure to be minimized is the objective function of the problem. If the group number of members which have the same cross-section is represented as g, assuming that a linear relationship exist between the weight per unit length and the plastic bending moment of element, the weight of structure can be written as

$$w = \sum_{k=1}^{g} l_k m_{pk} \tag{7}$$

where, m_{pk} is the plastic bending moment of group k members and l_k is the sum of the lengths of all group k members.

The weight of structure in terms of design variables can be expressed as follows,

$$w = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{l}^T \end{bmatrix} \begin{bmatrix} \phi \\ \boldsymbol{m}_p \end{bmatrix} \tag{8}$$

In this equation, $\boldsymbol{l}^T$ is transpose of $g \times 1$ vector of the group lengths.

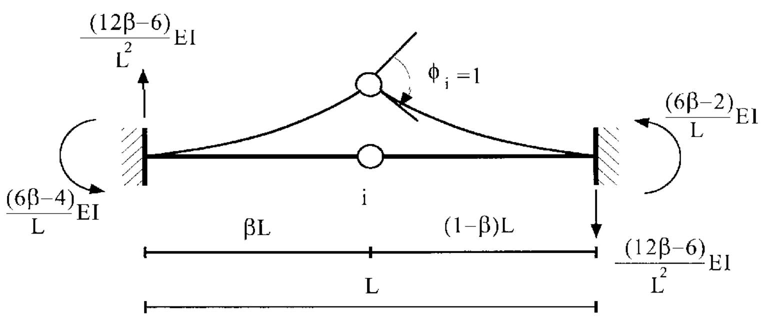

Fig. 1 Fixed end forces of a prismatic member due to the unit introduced hinge rotation

2.4 Additional constraints

For an optimum plastic design problem, it is generally sufficient to calculate the plastic bending moments that minimize the weight function while satisfying the yield constraints. In some cases however, the plastic bending moments should satisfy the required constructive conditions that plastic bending moments of the bottom story columns or beams are larger than those of upper story. These types of constraints should also be considered when formulating the optimization problem.

The constructive constraints related to the plastic bending moments may be expressed in different ways, such as

$$-m_{pi} + (m_{pi})_l \leq 0 \tag{9}$$

and

$$m_{pi} \leq m_{pk} \tag{10}$$

In these expressions, m_{pi} and m_{pk} are the plastic moments of members i and k, respectively, and $(m_{pi})_l$ is the limiting value for plastic moment m_{pi}.

In the present formulation, the limitations on nodal displacements or plastic hinge rotations may also be easily included in the constraints. These type of constraints are very important for the optimization of frames that collapse before the mechanism load due to the excessive displacements or plastic deformations of hinges. As the plastic hinge rotations are design variables of the problem, the limitations of those are directly constructed. However, the nodal displacement limitations are included in a simple manner without any important changes in the present formulation as follows:

1-The bending moments at critical sections are obtained by analysing the frame for the unit value of nodal displacement together with the unit hinge rotations and external loads.

2-The equilibrium equation in the direction of nodal displacement can not be satisfied as it is taken into account equal to zero in analysis of the frame for unit hinge rotations and external loads. Thus, an equilibrium equation must be added to the constraints stating that the sum of forces in direction of nodal displacement is equal to zero. These forces have been already obtained from the analysis of frame for unit nodal displacement, unit hinge rotations and external loads, together with the bending moments.

3-Since the nodal displacement is also taken into account as a new design variable, the limitation can be directly included in the constraints.

4-The new equation is eliminated by Gaussian algorithm as the constraints satisfy the equilibrium condition.

It is well known that weight optimization according to the collapse load is an iterative procedure and it is also needed non-linear push-over analysis tracking the formation of plastic hinges to collapse, to obtain a plastic hinge pattern at each iteration step. However, there is no need to consider the displacement and deformation limitations in optimal plastic design according to mechanism load, as the structure is statically determinate at an incipient collapse state.

3. Solution by simplex method

As derived in the previous sections, the weight function and yield constraints may be written in linear programming format, as

$$\text{Minimize } w = \begin{bmatrix} \mathbf{0} & \boldsymbol{l}^T \end{bmatrix} \begin{bmatrix} \phi \\ \boldsymbol{m_p} \end{bmatrix}$$

Subject to

$$\boldsymbol{m}_\phi \phi - \boldsymbol{t}\boldsymbol{m_p} + \boldsymbol{m_0} \leq \mathbf{0} \tag{11}$$

Thus, the minimum design problem is transformed into a linear programming problem which aims to determine the optimal design variables ϕ and $\boldsymbol{m_p}$. In this paper, the Simplex Method is utilized for the solution of linear programming problem.

The application of the Simplex Method to a minimum weight design problem requires that the design variables ϕ and $\boldsymbol{m_p}$ be positive. Although the elements of the plastic bending moment vector $\boldsymbol{m_p}$ are always positive, the elements of introduced hinge rotation vector ϕ may have either positive or negative values. Therefore, these design variables are expressed as the difference between two positive variables, as

$$\phi_i = \bar{\phi}_i - \varphi_u \tag{12}$$

where φ_u is an upper bound for all introduced hinge rotations.

Applying the above transformation to design variables ϕ, yield constraints become

$$\boldsymbol{m}_\phi \bar{\phi} - \boldsymbol{m}_{\varphi u} \varphi_u + \boldsymbol{t}\boldsymbol{m_p} + \boldsymbol{m_0} \leq \mathbf{0} \tag{13}$$

Finally, the minimum weight design problem may be formulated as

$$\text{Minimize } w = \begin{bmatrix} \mathbf{0} & 0 & \boldsymbol{l}^T \end{bmatrix} \begin{bmatrix} \bar{\phi} \\ \varphi_u \\ \boldsymbol{m_p} \end{bmatrix}$$

Subject to

$$\boldsymbol{m}_\phi \bar{\phi} - \boldsymbol{m}_{\varphi u} \varphi_u - \boldsymbol{t}\boldsymbol{m_p} + \boldsymbol{m_0} \leq \mathbf{0} \tag{14}$$

where, $\bar{\phi}$ is a $n \times 1$ vector of transformed introduced hinge rotations and $\boldsymbol{m}_{\varphi u}$ is a $2c \times 1$ vector of bending moments corresponding to the upper bound variable φ_u. The typical element of vector $\boldsymbol{m}_{\varphi u}$ is calculated by the sum of the row elements of matrix $\boldsymbol{m}_\phi$ as follows,

$$m_{\varphi u_i} = \sum_{j=1}^{n} m_{\phi j} \tag{15}$$

According to present formulation, the Simplex Tableau has $(n + g + 2)$ columns and $2c$ rows. It should be noted that, in contrast to Toakley (1968), Jennings and Tam (1986), a special technique was applied to minimize total number of the design variables by defining a unique upper bound variable for all introduced hinge rotations, instead of a modified technique that is based on incidence matrix storing. On the other hand, number of the rows is reduced to c by using the similar technique as in Toakley (1968), Jennings and Tam (1986).

Furthermore, if there are constructive constraints related to the cross sections or plastic hinge rotation capacities, those should be considered together with the yield constraints in the Simplex Tableau. However, the nodal displacement limitations may be added to constraints as explained in the previous chapter.

3.1 Determination of initial feasible basic solution

The Simplex Method, which is applied to the linear optimization problems, aims to obtain the optimal feasible solution starting from an initial feasible basic solution.

If the elements of the matrix $\boldsymbol{m_0}$ are all positive, then an initial feasible basic solution may be directly obtained. However, if matrix $\boldsymbol{m_0}$ has negative elements, mathematical techniques based on introducing new variables to the problem, are applied in general.

In this study, the initial feasible basic solution, or so-called the initial tableau, is obtained in a simple manner without increasing the number of variables. This technique, which is consistent with the structural optimization problem, is explained below.

The plastic bending moments, which satisfy the yield conditions are

$$m = m_p \tag{16}$$

at n different critical sections and

$$m < m_p \tag{17}$$

at the remaining critical sections while the elements of vector $\bar{\phi}$ and upper bound variable φ_u are equal to zero, form the initial feasible basic solution. Thus, the initial tableau is obtained simply by selecting the columns of matrix $\boldsymbol{t}$ as pivot columns and by applying the Simplex algorithm to these columns, respectively.

4. Automatic generation of introduced hinge locations

Especially, for the frames with high degree of redundancy, automatic generation of introduced hinge locations is necessary for the efficiency of the formulation. Hence, a method for automatic generation of introduced hinge locations is explained below for a specific problem.

Consider a fixed based frame with eighteen degrees of redundancy shown in Fig. 2. It is well known that for defining a statically determinate released frame, three hinges must be introduced over each closed loop. For this purpose, three hinges are introduced into the bottom section of the left hand side column, the beam and the bottom section of the right hand side column considering the positive sign convention of loops, respectively. Over the members of any loop, if an introduced hinge already exists on the bottom section of the left hand side column, the third introduced hinge must be located on the top section of the right hand side column. The frame element numbers over each closed loop should be given as input data for automatic generation of the introduced hinge locations. Furthermore, if a real hinge exists on the frame, location of it should be also given as

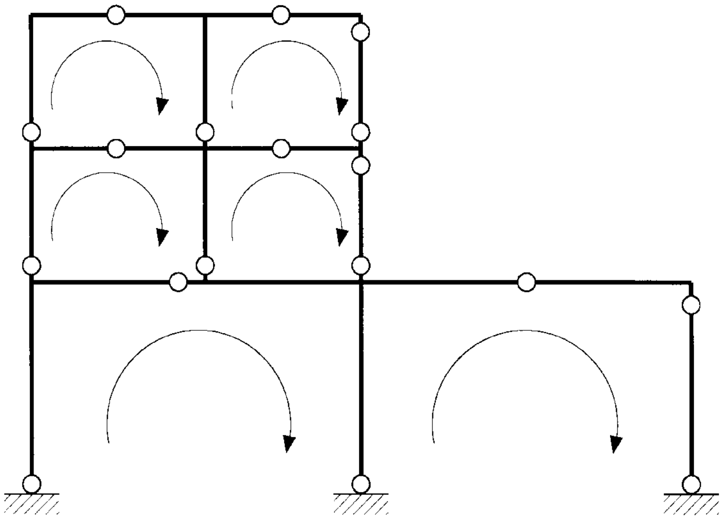

Fig. 2 Automatic generation of introduced hinge locations on a statically indeterminate frame

input data and the condition that the bending moment is equal to zero at the real hinge section should be considered in the analyses.

5. Computer implementation

For practical applications of the present matrix displacement formulation, a computer program has been developed. The program that is composed of tableau construction and optimization phases involves the following steps:

1) Read the input data consisting of system geometry, constraints, restraints, external loads, element numbers over the loops, member plastic moment group indicators and additional constraints related to the constructive conditions.
2) Generate the locations of introduced hinges.
3) Analyse the frame for unit introduced hinge rotations, respectively and for external loads, then construct the $\boldsymbol{m}_\phi$ and $\boldsymbol{m}_0$ matrices.
4) Construct the frame plastic moment group matrix $\boldsymbol{t}$.
5) Construct the group length vector $\boldsymbol{l}$ for the objective function.
6) Form the Simplex Tableau that includes the objective function and the yield constraints by establishing the constructed matrices.
7) Add an extra column corresponding to the upper bound variable φ_u to the Simplex Tableau due to the positivity condition. For this purpose, find the sum of each row elements of matrix $\boldsymbol{m}_\phi$ and change their signs.
8) If the optimization problem has any additional constraints related to the constructive conditions, add the mentioned constraints to the Simplex Tableau.
9) Apply the initial feasible basic solution procedure and form the initial tableau.
10) Apply the Simplex Method and find the optimal solution.
11) Transform the optimal introduced hinge rotations and give the optimal design variables.

6. Numerical examples

In this chapter, four numerical examples are given to illustrate the present formulation and to compare the results with those obtained in previous studies. The units are in *kN* and *m* for all examples. Since the structural frame is statically determinate at an incipient collapse, arbitrary values can be assumed for the inertia moments and the cross sectional areas of frame members in analyses. Thus, the bending stiffness of all frame elements are considered as $EI = 1000\ kNm^2$, shear and axial deformations are neglected in the analyses.

6.1 Example 1

For the first example, a single-bay, single-storey simple frame with three degrees of redundancy studied by Heyman (1971), Jennings and Tam (1986), is designed by the present formulation. The geometrical dimensions, external loads, plastic moment groups and critical section numbers are shown in Fig. 3. The frame is designed by selecting two different statically determinate released frames in order to show the optimum solution is independent of the introduced hinge locations.

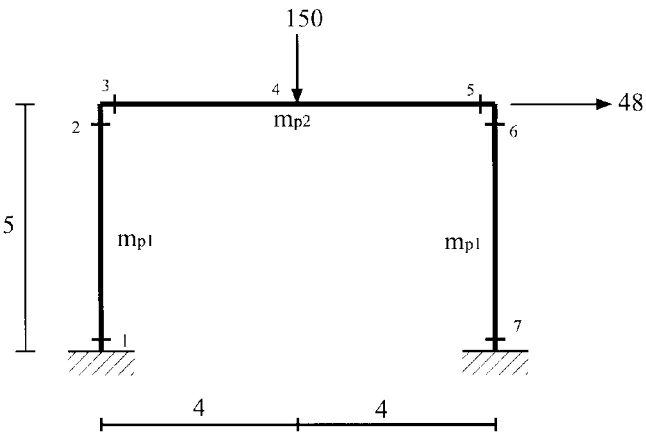

Fig. 3 Single-bay, single-storey frame

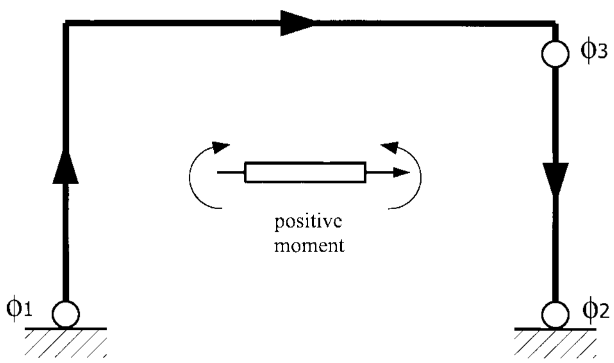

Fig. 4 Introduced hinge locations and sign convention for the first design

Since the frame has third degree of freedom redundancy, three hinges are introduced into different sections.

The present matrix displacement formulation is applied firstly on the frame shown in Fig. 4. As it is seen from Fig. 4, three hinges are introduced into sections 1, 7 and 6. Hence, the frame is analyzed for the unit introduced hinge rotations ($\phi_i = 1$), respectively, and for the external loads $\phi = 0$, and the bending moments of the critical sections are obtained in terms of design variables to construct the yield constraints. The mentioned constraints for each critical section and weight function are given in Table 1, in terms of the design variables.

After the variable transformation and initial feasible basic solution are obtained, the optimal solution is reached after 5 iterations by using the conventional Simplex algorithm. The values of design variables and the weight of frame for the optimal solution are given in Table 2. As can be seen in Table 2, the optimal plastic bending moments and the weight of structure are similar to those obtained by Heyman (1971), Jennings and Tam (1986) as

$$m_{p1} = 120$$

$$m_{p2} = 180$$

$$w = 2640$$

For the comparison, the yield constraints are constructed on the statically determinate released frame shown in Fig. 5. Hence, the third hinge is introduced into section 4, instead of section 6. The

Table 1 Yield constraints and the weight function for the first design

m	$\phi_1 = 1$	$\phi_2 = 1$	$\phi_3 = 1$	m_{p1}	m_{p2}	$\phi = 0$
m_1	402.28	244.46	–126.36	–1	0	–15.49
$-m_1$	–402.28	–244.46	126.36	–1	0	15.49
m_2	31.46	–126.36	16.26	–1	0	–66.92
$-m_2$	–31.46	126.36	–16.26	–1	0	66.92
m_3	31.46	–126.36	16.26	0	–1	–66.92
$-m_3$	–31.46	126.36	–16.26	0	–1	66.92
m_4	–47.45	–47.45	95.17	0	–1	185.72
$-m_4$	47.45	47.45	–95.17	0	–1	–185.72
m_5	–126.36	31.46	174.08	0	–1	–161.65
$-m_5$	126.36	–31.46	–174.08	0	–1	161.65
m_6	–126.36	31.46	174.08	–1	0	–161.65
$-m_6$	126.36	–31.46	–174.08	–1	0	161.65
m_7	244.46	402.28	31.46	–1	0	129.77
$-m_7$	–244.46	–402.28	–31.46	–1	0	–129.77
w	0	0	0	10	8	

Table 2 The design variables and weight of frame for optimal solution

ϕ_1	ϕ_2	ϕ_3	m_{p1}	m_{p2}	w
–0.48	0.28	–0.16	120	180	2640

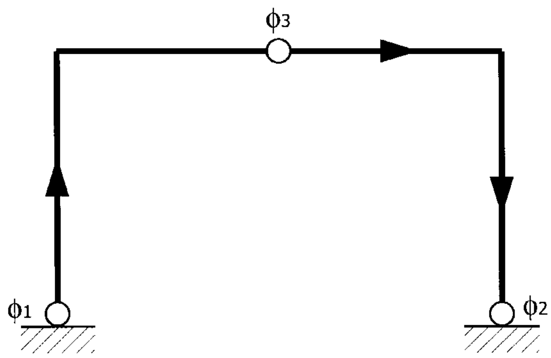

Fig. 5 Introduced hinge locations for the second design

Table 3 The yield constraints and the weight function for the second design

m	$\phi_1 = 1$	$\phi_2 = 1$	$\phi_3 = 1$	m_{p1}	m_{p2}	$\phi = 0$
m_1	402.28	244.46	–47.45	–1	0	–15.49
$-m_1$	–402.28	–244.46	47.45	–1	0	15.49
m_2	31.46	–126.36	95.17	–1	0	–66.92
$-m_2$	–31.46	126.36	–95.17	–1	0	66.92
m_3	31.46	–126.36	95.17	0	–1	–66.92
$-m_3$	–31.46	126.36	–95.17	0	–1	66.92
m_4	–47.45	–47.45	95.17	0	–1	185.72
$-m_4$	47.45	47.45	–95.17	0	–1	–185.72
m_5	–126.36	31.46	95.17	0	–1	–161.65
$-m_5$	126.36	–31.46	–95.17	0	–1	161.65
m_6	–126.36	31.46	95.17	–1	0	–161.65
$-m_6$	126.36	–31.46	–95.17	–1	0	161.65
m_7	244.46	402.28	–47.45	–1	0	129.77
$-m_7$	–244.46	–402.28	47.45	–1	0	–129.77
w	0	0	0	10	8	

Table 4 The design variables and the weight of structure for second design

ϕ_1	ϕ_2	ϕ_3	m_{p1}	m_{p2}	w
–0.4	0.2	–0.16	120	180	2640

mentioned constraints and the weight function are given in Table 3 for the second design.

After the application of variable transformation and initial feasible basic solution procedures, the design variables and the weight of structure are obtained by using the conventional Simplex algorithm and the optimal solution is given in Table 4. As can be seen in the Table 4, although the introduced hinge rotations are different from those obtained in the first design, the plastic bending

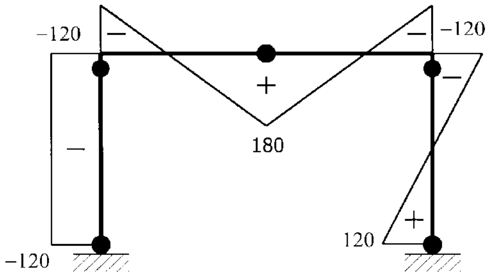

Fig. 6 Bending moment diagram for the optimal solution

moments and the weight of structure for the optimal solution are similar. The bending moment diagram for the optimal solution is shown in Fig. 6.

6.2 Example 2

As the second example, the two-bay, two-storey fixed frame having nine degrees of redundancy studied by Jennings and Tam (1986) is designed optimally by using the present formulation. The geometrical dimensions, external loads, plastic moment groups and the number of critical sections of the frame are shown in Fig. 7.

The introduced hinges are located by using the hinge generation technique that is previously explained. Hence, introduced hinges are located on sections 16, 13, 4, 20, 8, 11, 1, 7 and 12. The statically determinate released frame with 9 introduced hinges is shown in Fig. 8.

The frame is analyzed firstly for the unit introduced hinge rotations ($\phi_i = 1$), respectively and for the external loads ($\phi = 0$) to construct the yield constraints by using the present formulation. The bending moments for each critical section are given in Table 5, in terms of the introduced hinge rotations and the external loads. The weight function of frame in terms of the plastic bending moments is as follows

$$w = 12m_{p1} + 18m_{p2} + 36m_{p3} \tag{18}$$

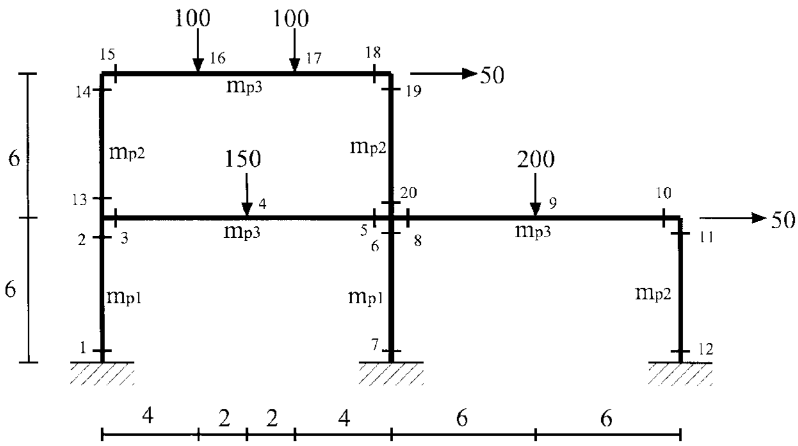

Fig. 7 Two-bay, two-storey frame

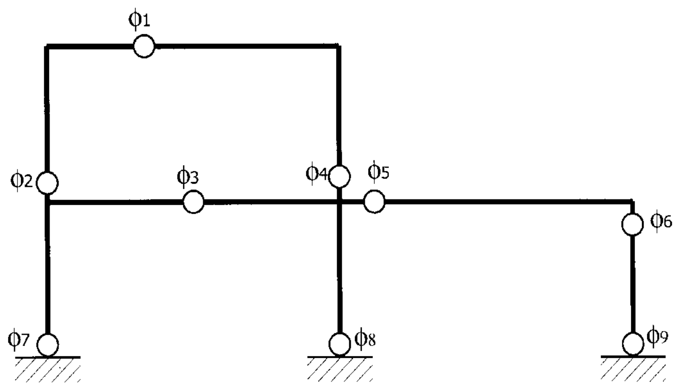

Fig. 8 Introduced hinge locations over the two-bay, two-storey frame

Table 5 Bending moments at critical sections

m	$\phi_1 = 1$	$\phi_2 = 1$	$\phi_3 = 1$	$\phi_4 = 1$	$\phi_5 = 1$	$\phi_6 = 1$	$\phi_7 = 1$	$\phi_8 = 1$	$\phi_9 = 1$	$\phi = 0$
m_1	6.55	–52.99	–16.51	–52.66	62.56	–76.26	436.98	163.63	136.22	–69.32
m_2	–8.09	144.19	38.29	100.86	–38.70	49.19	–70.18	–109.98	–88.98	–28.56
m_3	–4.38	–74.67	69.35	–14.81	–62.63	47.48	–17.18	–38.74	–69.03	–78.65
m_4	3.35	–31.06	73.99	–28.74	13.42	–3.44	–16.51	–20.87	–0.91	225.21
m_5	11.09	12.55	78.63	–42.67	89.47	–54.36	–15.84	–2.99	67.21	–370.92
m_6	–18.72	104.29	36.46	129.07	–68.66	–21.65	–131.05	–56.41	96.30	26.77
m_7	6.86	–71.24	–20.87	–62.30	-8.88	62.49	163.63	419.56	–139.88	70.25
m_8	–11.71	23.93	13.42	61.95	220.08	–56.46	62.56	–8.88	–14.98	–233.15
m_9	–3.04	12.82	4.99	19.10	81.81	58.86	–6.85	26.80	–25.20	339.37
m_{10}	5.62	1.71	–3.44	–23.75	–56.46	174.17	–76.26	62.49	–35.42	–288.11
m_{11}	5.62	1.71	–3.44	–23.75	–56.46	174.17	–76.26	62.49	–35.42	–288.11
m_{12}	–5.31	–19.95	–0.91	14.11	–14.98	–34.93	136.22	–139.88	425.97	227.65
m_{13}	–3.71	218.86	–31.06	115.67	23.93	1.71	–52.99	–71.24	–19.95	50.10
m_{14}	84.23	30.69	2.58	–80.85	–24.38	14.11	6.43	3.88	–16.67	–120.63
m_{15}	84.23	30.69	2.58	–80.85	–24.38	14.11	6.43	3.88	–16.67	–120.63
m_{16}	71.63	–3.71	3.35	–41.51	–11.71	5.62	6.55	6.86	–5.31	217.52
m_{17}	59.03	–38.11	4.12	–2.16	0.97	–2.87	6.66	9.84	6.04	155.67
m_{18}	46.43	–72.51	4.90	37.18	13.64	–11.35	6.77	12.81	17.39	–306.19
m_{19}	46.43	–72.51	4.90	37.18	13.64	–11.35	6.77	12.81	17.39	–306.19
m_{20}	–41.51	115.67	–28.74	233.70	61.953	–23.75	–52.66	–62.30	14.11	164.53

Table 6 Design variables and frame weight corresponding to optimal solution

ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8	ϕ_9	m_{p1}	m_{p2}	m_{p3}	w
0.8	0.167	0.9	–0.467	–0.3	0	0.3	–0.3	0	0	300	300	16200

Before the application of Simplex procedure, yield constraints are constructed and the initial feasible basic solution procedure is applied to form the initial tableau. Then, the optimal solution is reached after 18 iterations. The values of design variables and the weight of frame corresponding to the optimal solution are given in Table 6. As can be seen in Table 6, the optimal plastic bending moments and the weight of structure are similar as those obtained by Jennings and Tam (1986) as

$$m_{p1} = 0$$

$$m_{p2} = 300$$

$$m_{p3} = 300$$

$$w = 16200$$

6.3 Example 3

As a third example, a single-bay, two-storey fixed frame with six degrees of redundancy previously studied by Hodge (1969) is designed optimally by using the matrix displacement formulation. The geometrical dimensions, external loads, plastic moment groups and the number of critical sections of the frame are shown in Fig. 9. Six hinges are introduced into the bottom sections of columns and the middle sections of the beams in order to construct the yield constraints. Those introduced hinge locations are shown in Fig. 10.

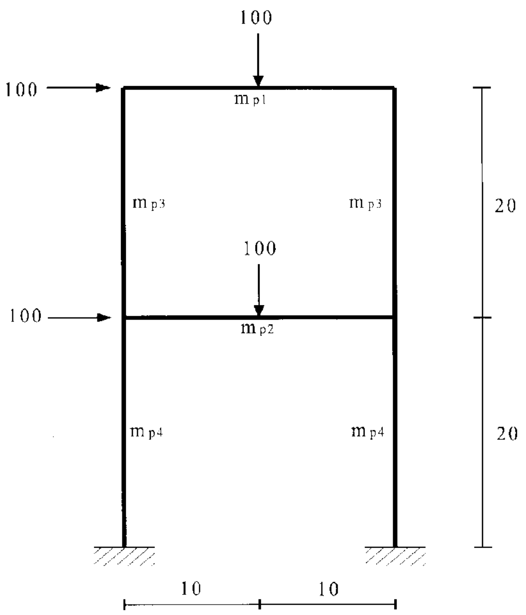

Fig. 9 Single-bay, two-storey frame

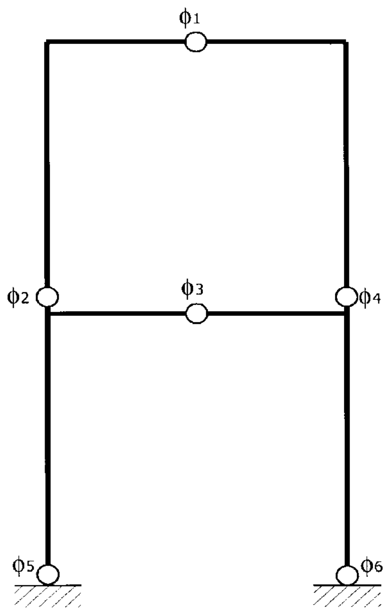

Fig. 10 Statically determinate released frame with introduced hinges

Table 7 Design variables and frame weight corresponding to optimal solution

ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	m_{p1}	m_{p2}	m_{p3}	m_{p4}	w
5	–3.33	5	–6.67	5	–5	500	1500	500	1000	100000

The frame is designed by using the present formulation and optimal solution is reached after 16 iterations. The values of design variables and the weight of frame corresponding to the optimal solution are given in Table 7. As can be seen in Table 7, the optimal plastic bending moments and the weight of structure are similar as those obtained by Hodge (1959) as

$$m_{p1} = 500$$

$$m_{p2} = 1500$$

$$m_{p3} = 500$$

$$m_{p4} = 1000$$

$$w = 100000$$

6.4 Example 4

As a last example, a single-bay, three-storey fixed frame with nine degrees of redundancy is designed optimally by using the present formulation. The geometrical dimensions, external loads, plastic moment groups and the number of critical sections of the frame are shown in Fig. 11. Introduced hinges are located on the bottom sections of columns and the middle sections of the beams, as shown in Fig. 12.

Firstly, the frame is designed by using the present formulation without additional constraints and the optimal solution is reached after 18 iterations. The values of the plastic bending moments and

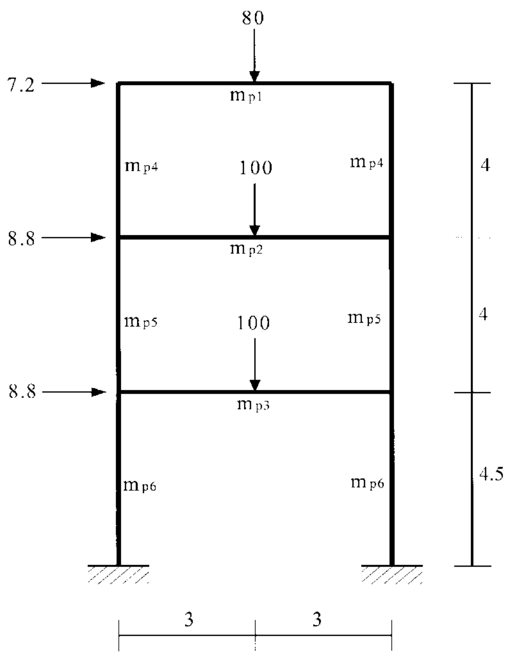

Fig. 11 Single-bay, three-storey frame

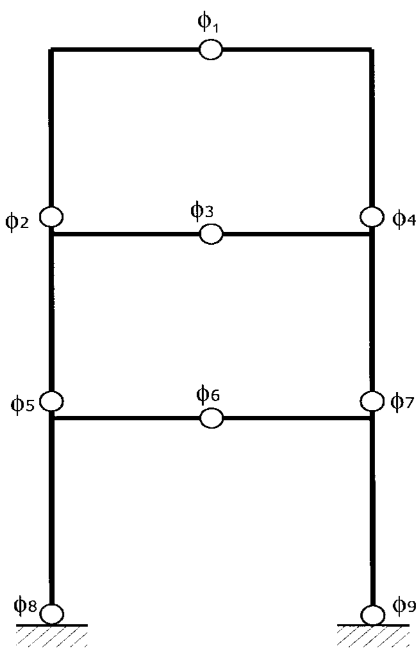

Fig. 12 Statically determinate released frame with introduced hinges

Table 8 Optimal design without additional constraints

m_{p1}	m_{p2}	m_{p3}	m_{p4}	m_{p5}	m_{p6}	w
60	97.33	81.62	60	37.33	44.29	2611

Table 9 Optimal introduced hinge rotations

ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8	ϕ_9
1.3E-2	–6.93E-2	4.16E-2	3.31E-2	–4.15E-2	–1.44E-2	1.55E-2	–7.2E-2	2.16E-2

Table 10 Optimal plastic bending moments and weight of structure

m_{p1}	m_{p2}	m_{p3}	m_{p4}	m_{p5}	m_{p6}	w
75.31	89.4	89.4	44.7	44.7	44.7	2642

the weight of frame corresponding to the optimal solution are given in Table 8.

After the first optimal design, eight additional constraits, which state that the plastic bending moments of bottom storey members should be larger than those of upper story members, are added to the problem and the optimal design procedure is repeated. The optimal solution is obtained after 21 iterations. The optimal values of the design variables and the weight of frame are given in Table 9 and Table 10, respectively.

7. Conclusions

A static linear programming formulation for optimum design of frames based on the matrix displacement approach is presented. In the formulation, yield constraints are expressed in terms of introduced hinge rotations on a statically determinate released frame instead of redundant forces in the matrix force method formulations. As the method is based on the matrix displacement method, yield constraints are constructed directly by using a conventional frame analysis program with efficient extensions. The optimization problem, in which introduced hinge rotations are chosen as design variables together with the member plastic bending moments, is solved by utilizing the conventional Simplex algorithm with special modifications. Thus, before the optimization phase, an upper bound variable is added to the problem due to the positivity condition and initial feasible basic solution, or so-called the initial tableau, is obtained in a simple manner that is consistent with the phenomenon of structural optimization. In contrast to the force method formulations, the present matrix displacement formulation may be adopted to the weight optimization of statically indeterminate frames that collapse before the mechanism load due to excessive displacements and plastic hinge deformations, since the nodal displacement and the plastic hinge rotation limitations may be easily considered.

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