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Applications of an improved estimator of the constitutive relation error to plasticity problems

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Abstract. This paper presents several applications of an improved estimator of the constitutive relation error (CRE) for plasticity problems. The cumulative aspect of the CRE estimator with respect to time is analyzed and we propose a first analysis of the local effectivity indexes of the CRE estimator in plasticity.

Key words: finite element method; plasticity; error estimation; local qualities.

1. Introduction

The monitoring of F.E. calculations in history-dependent nonlinear analysis is an important research topic. Three main approaches to the development of estimators can be found in the literature: the estimators introduced by Babuška and Rheinboldt (Babuška and Rheinboldt 1978), which use the equilibrium residuals to calculate the errors (Babuška and Rheinboldt 1982, Johnson and Hansbo 1992, Tie and Aubry 1992, Huerta *et al.* 1998, E. Stein and Schmidt 1998, Rannacher and Stuttmeier 1999, Cirak and Ramm 2000); the estimators introduced by Zienkiewicz (Zienkiewicz and Zhu 1987), which consist of comparing the finite element solution with a smoothed solution (Zienkiewicz and Zhu 1987, Coupez *et al.* 1998, Boroomand and Zienkiewicz 1998); the estimators introduced by Ladevèze (Ladevèze *et al.* 1986), which are based on the concept of error in the constitutive relation (Gallimard *et al.* 1996, Gallimard *et al*. 1997, Ladevèze and Moës 1998, Ladevèze *et al*. 1999, Gallimard *et al.* 2000). In such calculations, the quality of the finite element solution at t depends not only on the quality of the mesh, but also on the quality of the time discretization used since the beginning of the loading. Therefore, the error cannot be fully controlled simply by improving the mesh quality after a given time step. Error estimators which take into account all discretization errors over the whole time interval $[0, T]$ are indispensable in order to estimate the quality of the calculations. Estimators possessing such properties have been developed for small-strain problems in quasi-static plasticity and viscoplasticity from an *a posteriori* error estimator based on the error in the constitutive relation (Ladevèze *et al.* 1986). The CRE estimators thus obtained are the Drucker-type error estimator (Gallimard *et al.* 1996) and the dissipation error estimator (Ladevèze and Moës 1998). The state-of-the-art for time-dependent

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nonlinear F.E. analysis can be found in Ladevèze (2000).

Here, we will focus on the Drucker-type error estimator. The principle of the error in the constitutive relation (Ladevèze and Leguillon 1983, Ladevèze *et al.* 1986) is based on the separation of the equations of the problem into two groups. In what we call Drucker's error estimator (DCRE), the first group of equations combines the kinematic constraints with the equilibrium equation and the second group contains the constitutive relation. A displacement-stress pair $(\underline{U}_{KA}, \boldsymbol{\sigma}_{SA})$ which satisfies the first group of equations is constructed and its quality is estimated by measuring how well it verifies the second group of equations (i.e., the constitutive relation). The finite element displacement $\underline{U}_h$ verifies the kinematic constraints and the displacement $\underline{U}_{KA}$ can be obtained easily by $\underline{U}_{KA}(\underline{M}, t) = \underline{U}_h(\underline{M}, t)$. The real difficulty lies in the construction of a stress $\boldsymbol{\sigma}_{SA}(\underline{M}, t)$ which verifies the equilibrium.

The quality of this estimator depends on the quality of the stress fields recovered. In Gallimard *et al.* (2000), an improved recovery technique derived from (Ladevèze and Rougeot 1997), which takes into account both the constitutive relation and the error measure, was developed. This technique led us to the definition of an improved DCRE estimator in plasticity. The numerical tests developed in Gallimard *et al.* (2000) show that this improved error estimator yields a significant improvement in the global effectivity index thanks to the good quality of the recovered equilibrated stress field. The advantage of this global error estimator is that it helps control the plastic computation with little additional cost.

However, recent works have made the calculation of the local quality of the quantities considered possible. These works were developed in elasticity (Rannacher and Stuttmeier 1997, Cirak and Ramm 1998, Peraire and Patera 1998, Ladevèze *et al.* 1999, Prudhomme and Oden 1999, Strouboulis *et al*. 2000), but also for nonlinear problems in plasticity (Rannacher and Stuttmeier 1999, Cirak and Ramm 2000). They are based on the calculation of a finite element approximation of Green's functions defined by a dual problem. In addition, in plasticity, it necessary to linearize this dual problem at each time step. These approaches, although admittedly very interesting, have a relatively high cost and are especially difficult to introduce into an industrial code. A relatively inexpensive alternative was proposed for elasticity in Ladevèze *et al. (*1999). This method, based on the error estimator in the constitutive relation and on an improved technique for constructing equilibrated stress fields, makes it possible to measure the local quality of the stresses obtained by F.E. analysis directly.

In this paper, we study the behavior of the improved DCRE on several examples to see whether this direct method can be extended to plasticity. For this purpose, we analyze both the global effectivity of the error and the effectivity of the local contributions to the error. The advantage of the improved DCRE is that it is a global error which incorporates all discretization errors (due to the mesh, due to the discretization in time, due to the iterative algorithm used to solve the nonlinear problem...).

We show that the error at time t can be split into one part which depends only on the solution at t and another which takes into account the loading history. Only the first part can be controlled by modifying the mesh at time t. In this paper, we also study the behavior of the error estimator through time and, particularly, the cumulativeness of the local time errors in the case of complex loadings. For example, for a cyclic loading in plasticity, one can observe that the global effectivity of the error is independent of the number of cycles. The good behavior of the improved DCRE can be explained by its strong mechanical foundation.

2. Preliminaries

2.1 The reference problem to be solved

Let us consider that, under the assumption of small perturbations and small displacements, the structure lies in a domain Ω, bounded by $\partial\Omega$, which is independent of t. Over the time interval $[0, T]$, the structure is subjected to:

- a prescribed displacement $\underline{U}_d(\underline{M}, t)$ a portion $\partial_1\Omega$ of the boundary,
- a traction $\underline{F}_d(\underline{M}, t)$ the complementary portion $\partial_2\Omega$,
- a distribution of body forces $\underline{f}_d(\underline{M}, t)$ on the domain Ω.

In a time-dependent nonlinear calculation, the stress value at time t is a function of the history of the strain until time t, which can be defined at each point M of the structure Ω by the relation:

$$\boldsymbol{\sigma}(\underline{M}, t) = \boldsymbol{A}(\boldsymbol{\varepsilon}(\dot{\underline{U}}(\underline{M}, \tau)); \tau \leq t) \tag{1}$$

where $\boldsymbol{A}$ is an operator characterizing the material and $\boldsymbol{\varepsilon}$ is the strain field.

Let $\mathcal{U}_{ad}^{[0,T]}$ designate the space of the displacements which verify the kinematic constraints:

$$\mathcal{U}_{ad}^{[0,T]} = \{\underline{U}(\underline{M}, t) \in \mathcal{U}^{[0,T]} \text{ such that } \underline{U}|_{\partial_1\Omega}(\underline{M}, t) = \underline{U}_d(\underline{M}, t) \forall t \in [0, T]\} \tag{2}$$

where $\mathcal{U}^{[0,T]}$ is the space of the displacement fields $\underline{U}(\underline{M}, t)$ defined on $\Omega \times [0, T]$ and let $\mathcal{S}_{ad}^{[0,T]}$ designate the space of the stresses which are solutions to the equilibrium equations:

$$\mathcal{S}_{ad}^{[0,T]} = \{\boldsymbol{\sigma}(\underline{M}, t) \in \mathcal{S}^{[0,T]} \text{ such that } \forall \underline{U}^* \in \mathcal{U}_0 \forall t \in [0, T] \int_\Omega Tr[\boldsymbol{\sigma\varepsilon}(\underline{U}^*)]d\Omega = \int_\Omega \underline{f}_d^T \underline{U}^* d\Omega + \int_{\partial_2\Omega} \underline{F}_d^T \underline{U}^* dS\} \tag{3}$$

where $\mathcal{S}^{[0,T]}$ is the space of the stress fields $\boldsymbol{\sigma}(\underline{M}, t)$ and $\mathcal{U}_0 = \{\underline{U}(\underline{M}, t) \in \mathcal{U}^{[0,T]}$ such that $\underline{U}|_{\partial_1\Omega} = 0\}$.

Then, the nonlinear problem can be formulated in the following manner:

$$\text{Find } (\underline{U}(\underline{M}, t), \boldsymbol{\sigma}(\underline{M}, t)) \in \mathcal{U}_{ad}^{[0,T]} \times \mathcal{S}_{ad}^{[0,T]} \text{ which satisfies Eq. (1)} \tag{4}$$

We will denote $(\underline{U}_{ex}(\underline{M}, t), \boldsymbol{\sigma}_{ex}(\underline{M}, t))$ the solution of the nonlinear reference problem, Eq. (4).

2.2 Constitutive relation: the Prandtl-Reuss plastic model

The state of the material is characterized at each point by the total strain $\boldsymbol{\varepsilon}$, the inelastic strain $\boldsymbol{\varepsilon}^p$ and the cumulative plastic strain which is a scalar variable designated by p. The associated variables are the stress for $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}^p$ and the hardening parameter R for p. They verify the relation: $\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^p + \dot{\boldsymbol{\varepsilon}}^e$.

The free energy is of the form:

$$\rho\psi(\boldsymbol{\varepsilon}^e, p) = \frac{1}{2}Tr[\boldsymbol{\varepsilon}^e \boldsymbol{K} \boldsymbol{\varepsilon}^e] + G(p) \tag{5}$$

where G is a strictly convex function characterizing the hardening law.

The derivation of ψ leads to the state equations:

$$\boldsymbol{\sigma} = \boldsymbol{K}\boldsymbol{\varepsilon}^e \quad \text{and} \quad R = \frac{dG(p)}{dp} = g(p) \tag{6}$$

We use a nonlinear hardening law:

$$G(p) = \frac{H}{1+\alpha}p^{\alpha+1} \quad \text{and} \quad g(p) = Hp^{\alpha} \tag{7}$$

where H is a scalar quantity and $0 < \alpha < 1$.

The dissipation potential is the indicator function of the elasticity convex. Here, the elasticity convex is the set $(\boldsymbol{\sigma}, R)$ which satisfies:

$$z(\boldsymbol{\sigma}, R) = \|\boldsymbol{\sigma}^D\| - R - R_0 \leq 0 \tag{8}$$

where $\boldsymbol{\sigma}^D$ is the deviatoric part of $\boldsymbol{\sigma}$ and R_0 the initial yield stress.

Then, the evolution laws are given by:

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\frac{\partial z}{\partial \sigma} \quad \text{and} \quad -\dot{p} = \dot{\lambda}\frac{\partial z}{\partial R} \tag{9}$$

with:

$$\dot{\lambda} \geq 0 \quad \text{and} \quad [(z < 0, \dot{\lambda} = 0) \quad \text{or} \quad (z = 0, \dot{z} < 0, \dot{\lambda}\dot{z} = 0)] \tag{10}$$

2.3 The finite element solution

Within the framework of the F.E.M., an approximate solution to the problem of Eq. (4) is obtained by using an incremental method along with a finite element discretization $\boldsymbol{E}$ and a time discretization Δ.

Assuming that the histories of both the displacements and the stresses are known until t_{i-1}, the problem is then to calculate these histories on the increment $[t_{i-1}, t_i]$ (with $\Delta = \{t_1, \ldots, t_n\}$ and $t_1 = 0 < t_2 < \ldots < t_{n-1} < t_n = T$). A number of algorithms based on the displacement approach are available to solve this problem (Owen and Hinton 1986).

At the end of each time increment t_i, these algorithms provide:

- a finite element displacement field which satisfies the kinematic constraints

$$\underline{U}_h(\underline{M}, t_i) = N(\underline{M})^T \underline{q}(t_i) \tag{11}$$

where $N(\underline{M})$ designates the matrix of the shape functions and $\underline{q}(t_i)$ the vector of the nodal displacements at t_i;

- a stress field $\boldsymbol{\sigma}_h(\underline{M}, t_i)$ which satisfies the equilibrium equations of the finite element model at t_i;
- a stress field $\tilde{\boldsymbol{\sigma}}_h(\underline{M}, t_i)$ which is linked to $\underline{U}_h(\underline{M}, t_i)$ by the constitutive relation.

Assuming that the data are piecewise linear on $[0, T]$, it is easy to complete the F.E. solution on $[0, T]$ in order to obtain both a displacement $\underline{U}_h(\underline{M}, t)$ that satisfies the kinematic constraints and a stress field $\boldsymbol{\sigma}_h(\underline{M}, t)$ that satisfies the equilibrium equations of the finite element model on $[0, T]$.

3. Error in the constitutive relation

We consider models which satisfy Drucker's inequality (1964) strictly. For such models, the error in the constitutive relation was introduced in Ladevèze (1985) and associated error estimators appeared in Ladevèze *et al.* (1986) and in Gallimard *et al.* (1996, 1997).

3.1 Drucker's inequality

Let $(\boldsymbol{\varepsilon}, \boldsymbol{\sigma})$ and $(\underline{\boldsymbol{\varepsilon}}, \underline{\boldsymbol{\sigma}})$ be two arbitrary strain-stress pairs which satisfy the constitutive relation described in (1) on $[0, T]$, with $(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}) = (\underline{\boldsymbol{\varepsilon}}, \underline{\boldsymbol{\sigma}}) = (0, 0)$ at $t = 0$. The material is said to satisfy Drucker's inequality if it verifies (12). Moreover, if (13) is verified, the material is said to satisfy Drucker's stability

$$\forall t \in [0, T] \int_0^t Tr[(\boldsymbol{\sigma} - \underline{\boldsymbol{\sigma}})(\dot{\boldsymbol{\varepsilon}} - \dot{\underline{\boldsymbol{\varepsilon}}})]dt \geq 0 \tag{12}$$

$$\forall t \in [0, T] \int_0^t Tr[(\boldsymbol{\sigma} - \underline{\boldsymbol{\sigma}})(\dot{\boldsymbol{\varepsilon}} - \dot{\underline{\boldsymbol{\varepsilon}}})]dt = 0 \Leftrightarrow \forall t \in [0, T] (\boldsymbol{\varepsilon}, \boldsymbol{\sigma}) = (\underline{\boldsymbol{\varepsilon}}, \underline{\boldsymbol{\sigma}}) \tag{13}$$

Let us introduce the cumulated plastic strain p associated with $(\boldsymbol{\varepsilon}, \boldsymbol{\sigma})$ and $R = g(p)$ and the cumulated plastic strain $\underline{p}$ associated with $(\boldsymbol{\varepsilon}, \boldsymbol{\sigma})$ and $\underline{R} = g(\underline{p})$. It was shown in Ladevèze and Pelle (2001) that:

$$\begin{aligned}\int_0^t Tr[(\boldsymbol{\sigma} - \underline{\boldsymbol{\sigma}})(\dot{\boldsymbol{\varepsilon}} - \dot{\underline{\boldsymbol{\varepsilon}}})]dt = {} & \frac{1}{2}\Big(Tr[(\boldsymbol{\sigma}|_t - \underline{\boldsymbol{\sigma}}|_t) K^{-1} (\boldsymbol{\sigma}|_t - \underline{\boldsymbol{\sigma}}|_t)] \\ & + (R|_t - \underline{R}|_t)(p|_t - \underline{p}|_t)\Big) + \int_0^t (i_z^2 + i_{\varepsilon^p}^2 + i_{pz}^2) dt \end{aligned} \tag{14}$$

with

$$i_z^2 \geq 0 \quad i_{\varepsilon^p}^2 \geq 0 \quad i_{pz}^2 \geq 0$$

$$i_z^2 = \frac{1}{2}(\dot{p}(R - \underline{R} - (p - \underline{p}) g'(\underline{p})) + \dot{\underline{p}}(\underline{R} - R - (\underline{p} - p) g'(p))) \tag{15}$$

$$i_{pz}^2 = (\dot{p} - \dot{\underline{p}})(z(\boldsymbol{\sigma}, R) - z(\underline{\boldsymbol{\sigma}}, \underline{R})) \tag{16}$$

$$i_{\varepsilon^p}^2 = (\dot{p}\|\boldsymbol{\sigma}^D\| + \dot{\underline{p}}\|\underline{\boldsymbol{\sigma}}^D\|)\left(1 - \frac{Tr[\boldsymbol{\sigma}^D \underline{\boldsymbol{\sigma}}^D]}{\|\boldsymbol{\sigma}^D\| \cdot \|\underline{\boldsymbol{\sigma}}^D\|}\right) \tag{17}$$

where $\boldsymbol{\sigma}^D$ and $\underline{\boldsymbol{\sigma}}^D$ are the deviatoric parts of $\boldsymbol{\sigma}$ and $\underline{\boldsymbol{\sigma}}$ respectively.

Remark: For the sake of completeness, the splitting of $\int_0^t Tr[(\boldsymbol{\sigma} - \underline{\boldsymbol{\sigma}})(\dot{\boldsymbol{\varepsilon}} - \dot{\underline{\boldsymbol{\varepsilon}}})]dt$ is detailed in the appendix.

3.2 Error measure based on Drucker's inequality

Let $s_{ad} = (\underline{U}_{KA}, \boldsymbol{\sigma}_{SA})$ be an admissible pair belonging to $\mathcal{U}_{ad}^{[0,T]} \times \mathcal{S}_{ad}^{[0,T]}$ and let us define the scalar quantity:

$$\eta(\underline{M}, t, s_{ad}) = \int_0^t Tr[(\boldsymbol{\sigma}_{SA}(\underline{M}, t) - \boldsymbol{\sigma}_{KA}(\underline{M}, t))(\dot{\boldsymbol{\varepsilon}}_{SA}(\underline{M}, t) - \dot{\boldsymbol{\varepsilon}}_{KA}(\underline{M}, t)]dt \quad (18)$$

where $\boldsymbol{\sigma}_{KA}$ is the stress field associated with the displacement $\underline{U}_{KA}$ through the constitutive relation Eq. (1) on $\Omega \times [0, T]$ and $\boldsymbol{\varepsilon}_{SA}$ is the strain field associated with $\boldsymbol{\sigma}_{SA}$.

For a material which satisfies Drucker's inequality strictly:

- $\eta(\underline{M}, t, s_{ad})$ is positive or zero on $\Omega \times [0, T]$,
- $\eta(\underline{M}, t, s_{ad}) = 0$ on $\Omega \times [0, T]$ if and only if $(\underline{U}_{KA}, \boldsymbol{\sigma}_{SA})$ is the exact solution to the reference problem (2.1).

In order to evaluate the quality of $s_{ad} = (\underline{U}_{KA}, \boldsymbol{\sigma}_{SA})$ as an approximate solution to the model problem, the previous relations lead us to the definition of the following *time-global* error measure at T:

$$e_T = \sup_{t \in [0, T]} e[0, t] \quad (19)$$

where the contribution to the error over [0, t] is:

$$e^2[0, t] = \int_\Omega \eta(\underline{M}, t, s_{ad})d\Omega \quad (20)$$

This error measure takes into account all error sources.

3.3 Recovery of the improved admissible solution on Ω × [0,T]

We will assume that the data is piecewise linear over $[0, T]$ (an assumption which, in practice, is not restrictive). Therefore, the displacement $\underline{U}_{KA}$ can be easily recovered from the finite element solution $\underline{U}_h(\underline{M}, t_i)$ by linear interpolation:

$$\underline{U}_{KA}(\underline{M}, t) = \frac{t - t_{i-1}}{t_i - t_{i-1}}\underline{U}_h(\underline{M}, t_i) + \frac{t_i - t}{t_i - t_{i-1}}\underline{U}_h(\underline{M}, t_{i-1}) \quad (21)$$

We focus on the construction of $\boldsymbol{\sigma}_{SA}^{IMP}(\underline{M}, t_i)$, where $\boldsymbol{\sigma}_{SA}^{IMP}(\underline{M}, t)$ is obtained by linear interpolation. Following (Ladevèze and Rougeot 1997), we introduce a weak prolongation condition in order to relate the statically admissible stress to the finite element stress.

$$\forall E \in \boldsymbol{E} \forall j \in \bar{\boldsymbol{I}} \subset \boldsymbol{I} \quad \int_E Tr[(\boldsymbol{\sigma}_{SA}^{IMP}(\underline{M}, t_i) - \boldsymbol{\sigma}_h(\underline{M}, t_i))\boldsymbol{\varepsilon}(\underline{\omega}_j)]d\Omega = 0 \quad (22)$$

where $\bar{\boldsymbol{I}}$ is associated with the non-vertex nodes.

Let us designate by $\mathcal{S}_{ad}^{t_i}$ the space of the stresses which satisfy the equilibrium equations at t_i:

$$\mathcal{S}_{ad}^{t_i} = \{\boldsymbol{\sigma} \in \mathcal{S} \text{ such that } \forall \underline{U}^* \in \mathcal{U}_0$$
$$\int_\Omega Tr[\boldsymbol{\sigma}\boldsymbol{\varepsilon}(\underline{U}^*)]d\Omega =$$
$$\int_\Omega \underline{f}_d(\underline{M}, t_i)^T \underline{U}^* d\Omega + \int_{\partial_2\Omega} \underline{F}_d(\underline{M}, t_i)^T \underline{U}^* dS\} \quad (23)$$

where $\mathcal{S}$ is the space of the stress fields and let us define $\mathcal{S}^{\Delta}_{ad,h}$ as the set of stresses:

$$\mathcal{S}^{\Delta}_{ad,h} = \{\{\boldsymbol{\sigma}(\underline{M}, t_1), \ldots, \boldsymbol{\sigma}(\underline{M}, t_n)\} \in \mathcal{S}^n \text{ such that}$$

$$\forall i \in \{1, \ldots, n\} \;\; \boldsymbol{\sigma}(\underline{M}, t_i) \in \mathcal{S}^{t_i}_{ad} \text{ and } \boldsymbol{\sigma}(\underline{M}, t_i) \text{ satisfies Eq. (22)}\} \tag{24}$$

$\boldsymbol{\sigma}^{IMP}_{SA}(\underline{M}, t_i)$ $(\forall i \in \{1, \ldots, n\})$ is obtained by solving the following minimization problem:

$$\text{Find } \{\boldsymbol{\sigma}^{IMP}_{SA}(\underline{M}, t_1), \ldots, \boldsymbol{\sigma}^{IMP}_{SA}(\underline{M}, t_n)\} \in \mathcal{S}^{\Delta}_{ad,h} \text{ which minimizes}$$
$$\text{the error in the constitutive relation } e_T \text{ defined in Eq. (19)} \tag{25}$$

Remark : The practical construction of $\boldsymbol{\sigma}^{IMP}_{SA}$ is detailed in Gallimard *et al.* (2000).

3.4 The improved DCRE error estimator

Let $s^{IMP}_{ad} = (\underline{U}_{KA}, \boldsymbol{\sigma}^{IMP}_{SA})$ be the improved admissible pair. The definition of the improved error estimator (improved DCRE) is given by:

$$e^{IMP}_T = \sup_{t \in [0,T]} e^{IMP}[0, t] \tag{26}$$

where the contribution to the error on [0, t] is given by:

$$e^{IMP}[0, T] = \int_{\Omega} \eta(\underline{M}, t, s^{IMP}_{ad}) d\Omega \tag{27}$$

The stress field $\boldsymbol{\sigma}^{IMP}_{SA}$ is obtained by minimizing the DCRE in a subspace of the space of admissible stress fields. Within that subspace, $\boldsymbol{\sigma}^{IMP}_{SA}$ is the best stress field possible. This is the reason why this error estimator is very effective.

Let us define a contribution to the error on element E over [0, t] as:

$$e^2_E[0, t] = \int_E \eta(\underline{M}, t, s^{IMP}_{ad}) dE \tag{28}$$

the relative error is given by:

$$\varepsilon_T = \sup_{t \in [0,T]} e^{IMP}[0, t]/D$$
$$D^2 = 2\int_0^T \int_{\Omega} [Tr[\boldsymbol{\sigma}_{KA} \dot{\boldsymbol{\varepsilon}}_{KA}] + Tr[\boldsymbol{\sigma}^{IMP}_{SA} \dot{\boldsymbol{\varepsilon}}^{IMP}_{SA}]] d\Omega \, dt \tag{29}$$

4. Analysis of the error measure

An exact error measure based on Drucker's inequality can be defined between the exact solution and the prolongation over $[0, T]$ of the finite element solution $\underline{U}_h(\underline{M}, t_i){:}\underline{U}_{KA}(\underline{M}, t)$.

$$e^2_{ex}[0, t] = \int_{\Omega} \eta(\underline{M}, t, (\underline{U}_{KA}, \boldsymbol{\sigma}_{ex})) d\Omega \tag{30}$$

4.1 Effectivity index

The performance of the global error estimator can be evaluated by calculating a global effectivity index defined on $[0, t]$:

$$\zeta(t) = \frac{e^{IMP}[0, t]}{e_{ex}[0, t]} \tag{31}$$

The error estimator is accurate if $\zeta(t)$ is close to 1. It is on the conservative side if $\zeta(t)$ is greater than 1.

In order to measure the local performance of the error estimator, one compares the local contributions to the error with the exact local error on an element E of the mesh calculated between 0 and t (Eq. 28). A local effectivity index can be defined by:

$$\zeta_E[0, t] = \frac{e_E[0, t]}{e_{ex, E}[0, t]} \tag{32}$$

Note: In practice, the exact solution is not known analytically. To represent this exact solution, we use a very refined finite element mesh and a very refined time approximation.

4.2 Definition of the error indicators

By integrating the constitutive relation, we can derive from $\underline{U}_{KA}$ the cumulated plastic strain p_{KA}, the yield stress $R_{KA} = g(p_{KA})$ and the stress σ_{KA}. Relation (14) allows us to partition the error measure defined by Eq. (30):

$$\begin{aligned}\eta(\underline{M}, t, \underline{U}_{KA}, \boldsymbol{\sigma}_{ex}) = {} & \frac{1}{2}(Tr[(\boldsymbol{\sigma}_{ex}|_t - \boldsymbol{\sigma}_{KA}|_t)K^{-1}(\boldsymbol{\sigma}_{ex}|_t - \boldsymbol{\sigma}_{KA}|_t)] \\ & + (R_{ex}|_t - R_{KA}|_t)(p_{ex}|_t - p_{KA}|_t)) \\ & + \int_0^t (i_z^2(\underline{M}, t, \underline{U}_{KA}, \boldsymbol{\sigma}_{ex}) + i_{\varepsilon^p}^2(\underline{M}, t, \underline{U}_{KA}, \boldsymbol{\sigma}_{ex}) \\ & + i_{pz}^2(\underline{M}, t, \underline{U}_{KA}, \boldsymbol{\sigma}_{ex}))dt \end{aligned} \tag{33}$$

with

$$i_z^2 \geq 0 \quad i_{\varepsilon^p}^2 \geq 0 \quad i_{pz}^2 \geq 0$$

$$\begin{aligned} i_z^2(\underline{M}, t, \underline{U}_{KA}, \boldsymbol{\sigma}_{ex}) = {} & \frac{1}{2}(\dot{p}_{ex}(R_{ex} - R_{KA} - (p_{ex} - p_{KA})g'(p_{KA})) \\ & + \dot{p}_{KA}(R_{KA} - R_{ex} - (p_{KA} - p_{ex})g'(p_{ex}))) \end{aligned} \tag{34}$$

$$i_{pz}^2(\underline{M}, t, \underline{U}_{KA}, \boldsymbol{\sigma}_{ex}) = (\dot{p}_{ex} - \dot{p}_{KA})(z(\boldsymbol{\sigma}_{ex}, R_{ex}) - z(\boldsymbol{\sigma}_{KA}, R_{KA})) \tag{35}$$

$$i_{\varepsilon^p}^2(\underline{M}, t, \underline{U}_{KA}, \boldsymbol{\sigma}_{ex}) = (\dot{p}_{ex}\|\boldsymbol{\sigma}_{KA}^D\| + \dot{p}_{KA}\|\boldsymbol{\sigma}_{ex}^D\|)\left(1 - \frac{Tr[\boldsymbol{\sigma}_{ex}^D\ \boldsymbol{\sigma}_{KA}^D]}{\|\boldsymbol{\sigma}_{ex}^D\| \cdot \|\boldsymbol{\sigma}_{KA}^D\|}\right) \tag{36}$$

where $\sigma_{\bullet}^{D}$ is the deviatoric part of $\sigma_{\bullet}$ and $g'(p) = \dfrac{dg}{dp}$.

Thus, the DCRE estimator includes "instant" parts, which depend only on the variables at t, and cumulative parts with respect to time, which depend on the history of the loading. Here, we are proposing an interpretation of these cumulative parts:

$I_{ex,z}[0, t]$, $I_{ex,\varepsilon^p}[0, t]$ and $I_{ex,pz}[0, t]$ defined in Eqs. (37), (38) and (39) are positive, monotonically increasing quantities.

$$I_{ex,z}[0,t] = \int_{\Omega}\int_{0}^{t} i_z(\underline{M}, t, \underline{U}_{KA}, \boldsymbol{\sigma}_{ex})dt\, d\Omega \tag{37}$$

$$I_{ex,\varepsilon^p}[0,t] = \int_{\Omega}\int_{0}^{t} i_{pz}(\underline{M}, t, \underline{U}_{KA}, \boldsymbol{\sigma}_{ex})dt\, d\Omega \tag{38}$$

$$I_{ex,pz}[0,t] = \int_{\Omega}\int_{0}^{t} i_{pz}(\underline{M}, t, \underline{U}_{KA}, \boldsymbol{\sigma}_{ex})dt\, d\Omega \tag{39}$$

$I_{ex,z}[0, t]$ is equal to zero if $\forall(\underline{M}, t) \in [0, t]$, $p_{ex} = p_{KA}$. This quantity can be used as an error indicator for the history of the cumulated plastic strain.

$I_{ex,\varepsilon^p}[0, t]$ is equal to zero if $\forall(\underline{M}, t) \in [0, t]$, $s_{ex} = \alpha s_{KA}$. This quantity can be used as an error indicator in the direction of the deviatoric part of the stress or in the direction of the plastic strain rate.

$I_{ex,pz}[0, t]$ is equal to zero if $\forall(\underline{M}, t) \in [0, t]$ the exact solution and the KA solution have the same plastic zone. This quantity can be used as an error indicator for the history of the plastic zone.

Since the exact solution is unknown, in order to estimate $I_{ex,z}[0, t]$, $I_{ex,\varepsilon^p}[0, t]$ and $I_{ex,pz}[0, t]$, we use the improved SA solution and calculate the following quantities:

$$I_z[0,t] = \int_{\Omega}\int_{0}^{t} i_z(\underline{M}, t, \underline{U}_{KA}, \boldsymbol{\sigma}_{SA}^{IMP})dt\, d\Omega \tag{40}$$

$$I_{\varepsilon^p}[0,t] = \int_{\Omega}\int_{0}^{t} i_{\varepsilon^p}(\underline{M}, t, \underline{U}_{KA}, \boldsymbol{\sigma}_{SA}^{IMP})dt\, d\Omega \tag{41}$$

$$I_{pz}[0,t] = \int_{\Omega}\int_{0}^{t} i_{pz}(\underline{M}, t, \underline{U}_{KA}, \boldsymbol{\sigma}_{SA}^{IMP})dt\, d\Omega \tag{42}$$

The cumulative part of the error I_h is defined by the sum of these indicators:

$$I_h[0,t] =_{def} (I_z^2[0,t] + I_{\varepsilon^p}^2[0,t] + I_{pz}^2[0,t])^{1/2} \tag{43}$$

In the same way, we also define a local error indicator on each element E. These local error indicators will be designated by $I_{z,E}[0, t]$, $I_{\varepsilon^p,E}[0, t]$ and $I_{pz,E}[0, t]$. Finally, it is possible to calculate local and global effectivity indexes for these indicators as in Section 3.2.

The "instant" part of the error I_s is defined by:

$$I_s(t) =_{def} \left(\int_{\Omega}\frac{1}{2}((\boldsymbol{\sigma}_{ex}|_t - \boldsymbol{\sigma}_{KA}|_t)^T K^{-1}(\boldsymbol{\sigma}_{ex}|_t - \boldsymbol{\sigma}_{KA}|_t) + (R_{ex}|_t - R_{KA}|_t)(p_{ex}|_t - p_{KA}|_t))d\Omega\right)^{1/2} \tag{44}$$

The following relation is derived from (33):

$$e[0,t] = (I_s^2(t) + I_h^2[0,t])^{1/2} \tag{45}$$

5. Analysis of the effectiveness of the improved DCRE estimator

In Gallimard *et al*. (2000), we showed that the improved DCRE estimator leads to better global effectivity indexes than the standard version of the DCRE (Gallimard *et al.* 1996). This is particularly true for anisotropic meshes or nearly perfect plasticity. Besides, on all the examples considered, these global effectivity indexes were greater than 1.

Here, we consider the local effectivity indexes calculated directly with the improved DCRE estimator. The example studied is described on Fig. 1. The loading is monotonous from 0 to 15,000 N/mm. The material parameters are: Young's modulus $E = 200{,}000$ Mpa, Poisson ratio $\nu = 0.3$, initial yield stress $R_0 = 240$ Mpa, hardening modulus $H = 1000$ Mpa, hardening exponent $\alpha = 0.5$.

Two different meshes (one coarse, one refined) of 6-node triangular elements were used to study the behavior of the improved DCRE estimator. The plastic deformation at the end of the loading is shown on Fig. 2 the coarse mesh and on Fig. 3 for the refined mesh.

The evolutions of the global errors and of the global effectivity indexes with respect to kinematic time are represented on Fig. 4 and Fig. 5. One can observe that the behavior of the improved DCRE estimator and that of the exact error are similar and that the effectivity index varies between 1.8 and 3. The plasticity effect, however, is more pronounced for the coarse calculation.

In order to study the local effectivity indexes, we eliminated the elements in which the error density was less than 2% of the average error density in the whole structure. The local effectivity indexes were calculated for the coarse finite element analysis at the first (elastic) time step (Fig. 6) and at the end of the loading (Fig. 7). The minimum local effectivity index was 1.02 for the first time step and 1.42 for the last time step. We plotted the effectivity indexes for the refined finite element analysis at the same instants (Fig. 8 and Fig. 9). In this case, the minimum effectivity indexes were 1.20 (elasticity) and 1.38 (plasticity). Let us observe that both calculations led to local

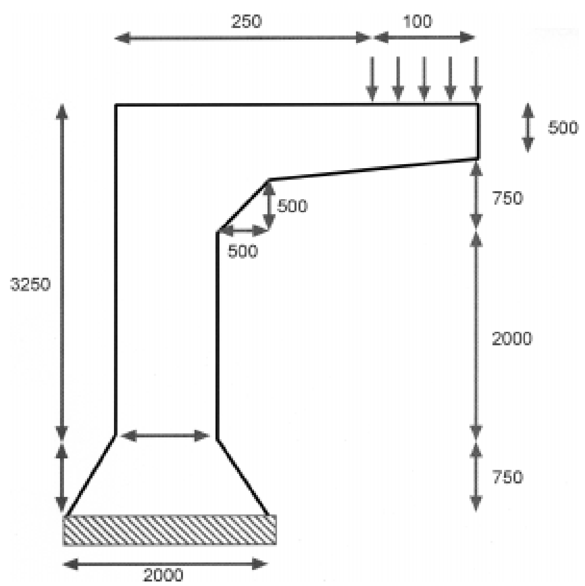

Fig. 1 Applied loading

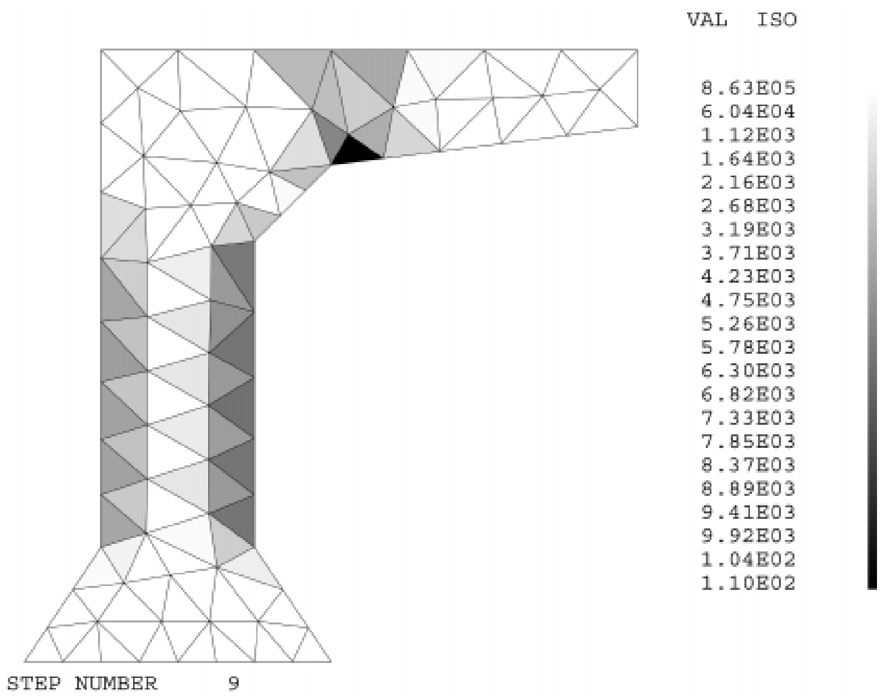

Fig. 2 Cumulated plastic deformation

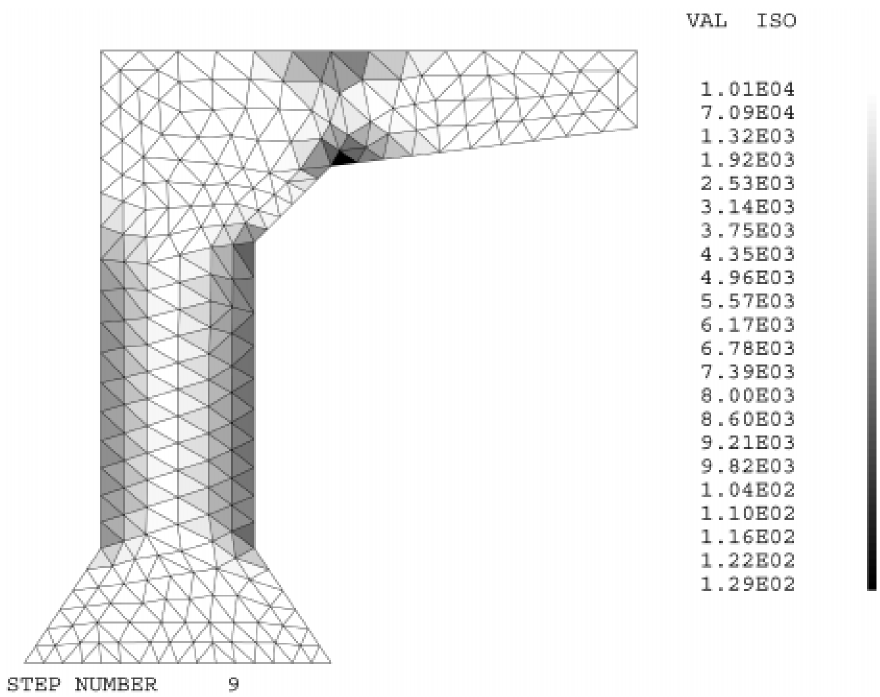

Fig. 3 Cumulated plastic deformation

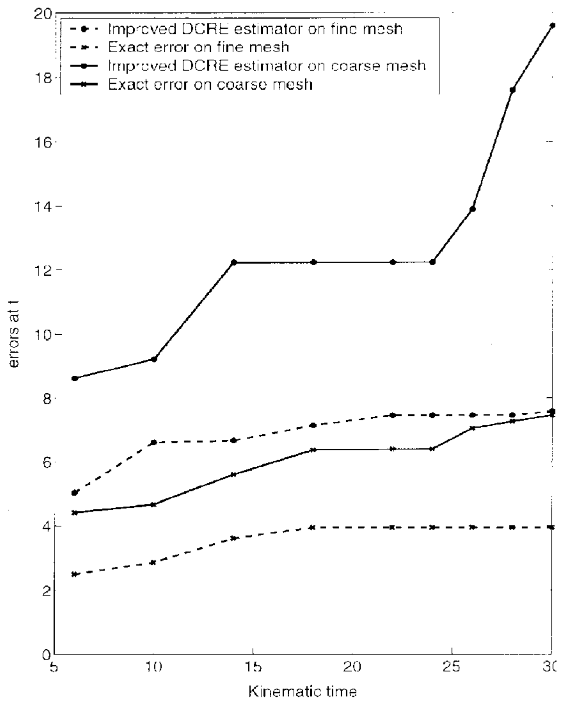

Fig. 4 Improved DRCE estimator

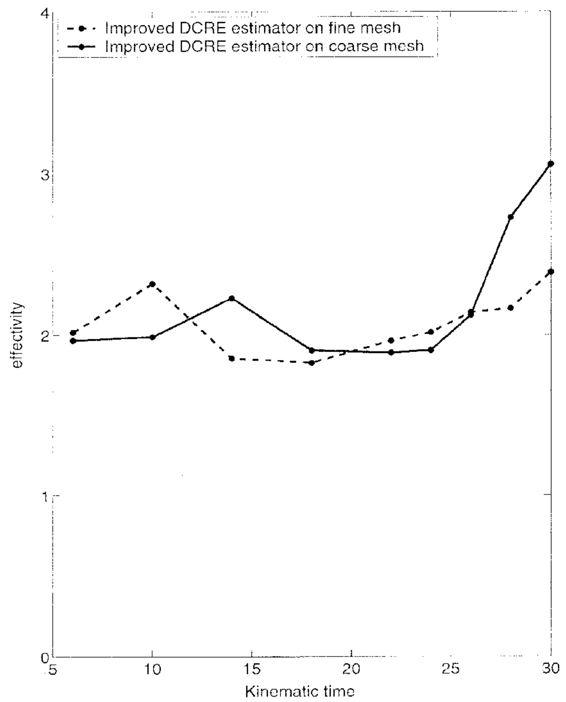

Fig. 5 Global effectivity index

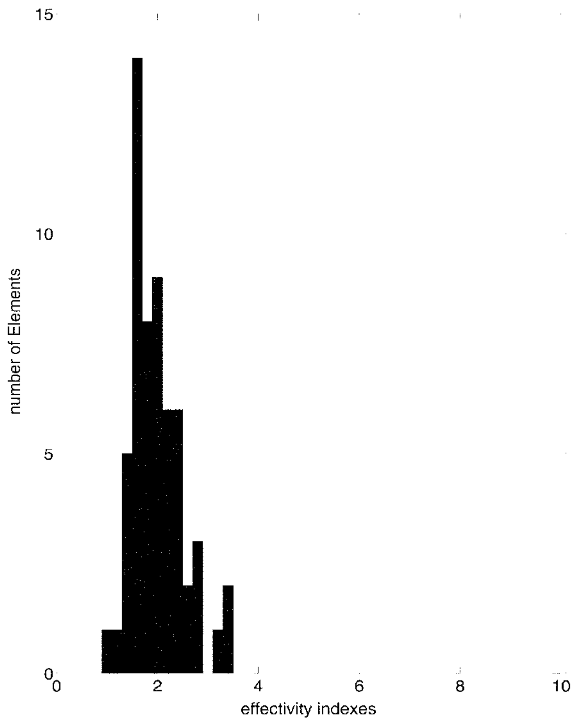

Fig. 6 Local effectivity indexes for the coarse calculation, first time step

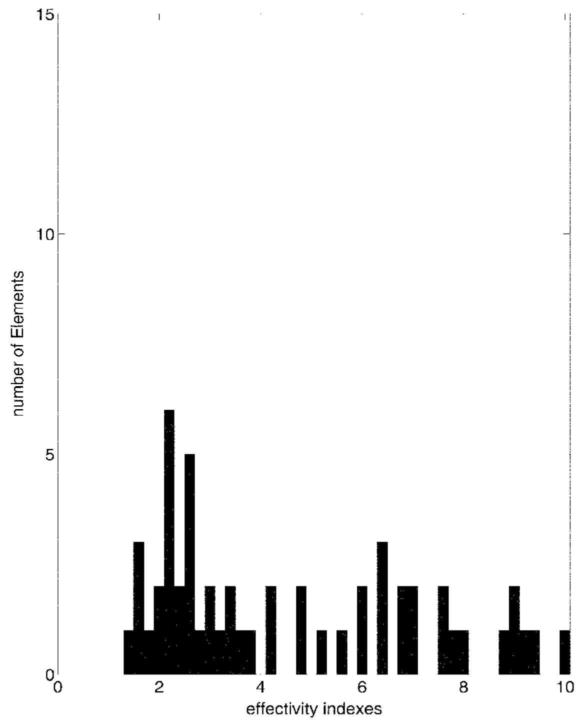

Fig. 7 Local effectivity indexes for the coarse calculation, last time step

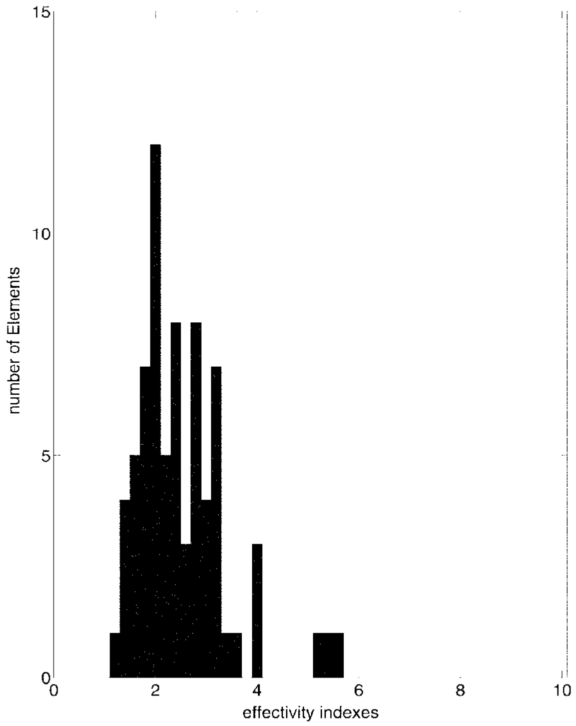

Fig. 8 Local effectivity indexes for the refined calculation, first time step

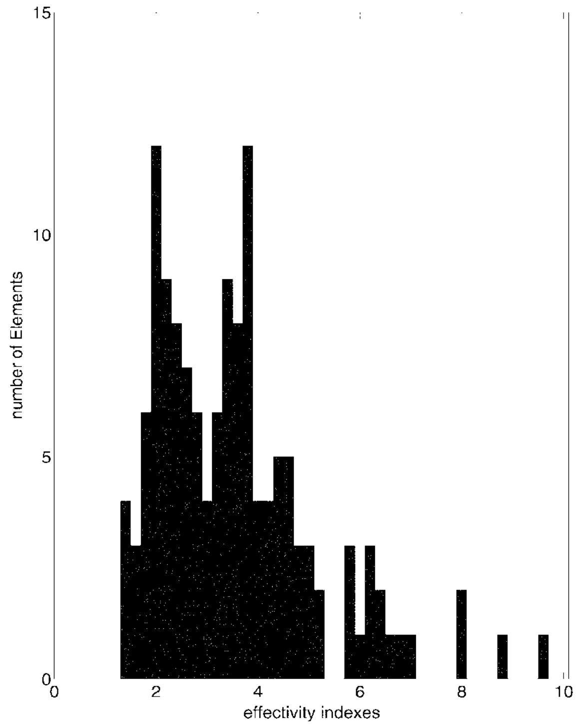

Fig. 9 Local effectivity indexes for the refined calculation, last time step

effectivity indexes larger than one and that, although the effectivity indexes were more scattered when the structure was plastified, their values remained mostly under 4.

6. Analysis of the effectivity indexes of the error indicators

In this section, we compare the error indicators introduced above with the reference quantities $I_{ex,z}[0, T]$, $I_{ex,\varepsilon^p}[0, T]$ and $I_{ex,pz}[0, T]$. We use the example described on Fig. 1 again. The results are shown on Table 1 for the coarse calculation and on Table 2 for the refined calculation. We can observe that the error indicators we introduced are greater than the reference quantities.

In the following example, we show that one can use these indicators to obtain information about local quantities. We are considering the calculation of $I_{pz,E}$ in the refined analysis. The local values are represented on Fig. 10 (calculated values) and on Fig. 11 (exact values). The qualitative

Table 1 Global effectivity indexes: coarse calculation

	I_z	I_{ε^p}	I_{pz}
Calculated quantities	0.639	0.699	0.508
Reference quantities	0.165	0.270	0.202
Effectivity indexes	3.0	2.6	2.7

Table 2 Global effectivity indexes: refined calculation

	I_z	I_{ε^p}	I_{pz}
Calculated quantities	0.240	0.246	0.180
Reference quantities	0.081	0.144	0.065
Effectivity indexes	3.0	1.7	2.8

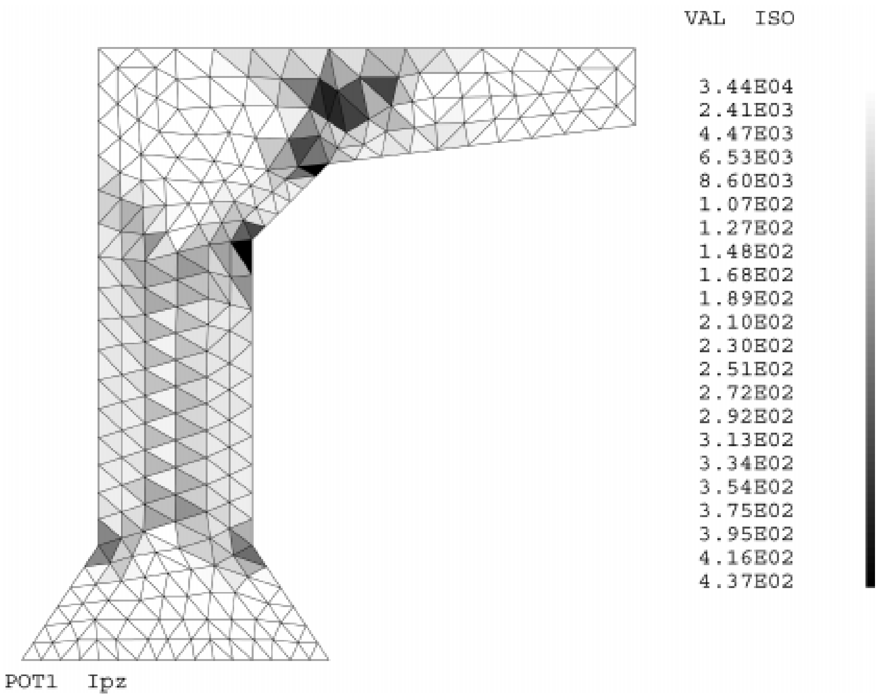

Fig. 10 Calculated I_{pz} on the refined mesh

comparison of Fig. 10 and Fig. 11 shows that $I_{pz}[0, T]$ describes $I_{ex,pz}[0, T]$ correctly. In order to achieve a numerical comparison, we calculated the local effectivity indexes on $I_{pz,E}$ for the elements with a significant error indicator. The results are shown on Fig. 12. We can observe that the effectivity indexes vary between 1.4 and 6.

7. Cumulative aspects of the improved DCRE estimator

For a first study of the behavior of the error in the constitutive relation in the case of structures subjected to cyclic loading, we considered the very simple case of the Prandtl-Reuss plasticity theory. For such a model, the limit cycle of plasticity is reached after only a few cycles and it leads to the stabilization of the cumulated plastic strain p and of the elasticity convex. The example studied is described on Fig. 13. The material parameters are: Young's modulus $E = 200000$ Mpa,

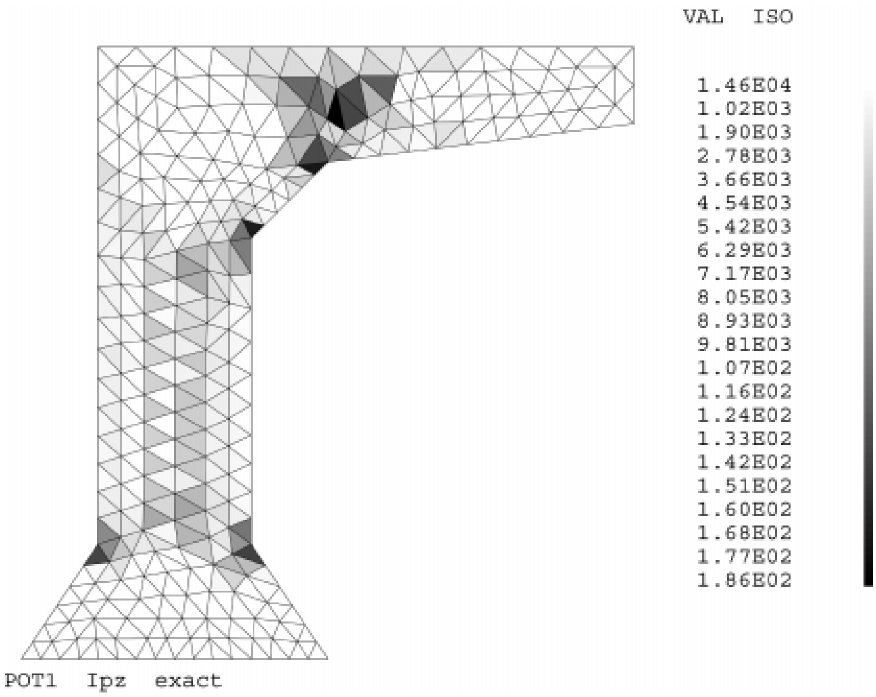

Fig. 11 Exact I_{pz} on the refined mesh

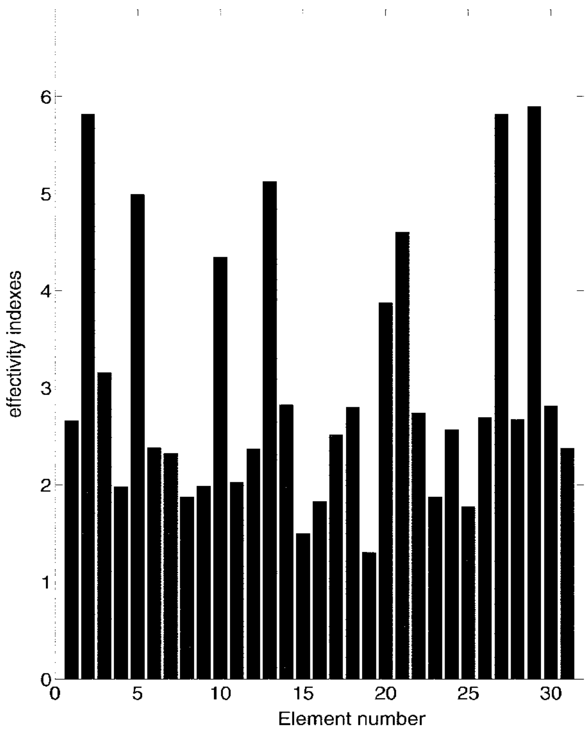

Fig. 12 Values of I_{pz} on the refined mesh

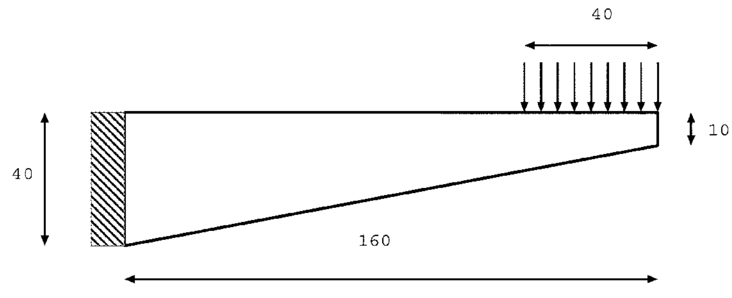

Fig. 13 Loading

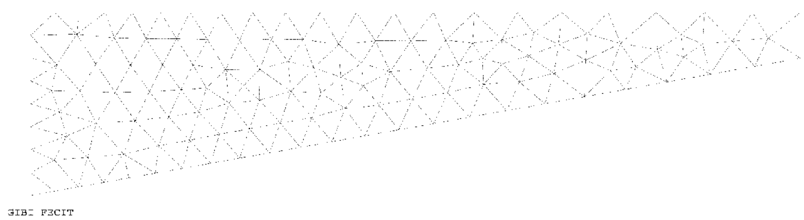

Fig. 14 Mesh with 6-node triangular elements

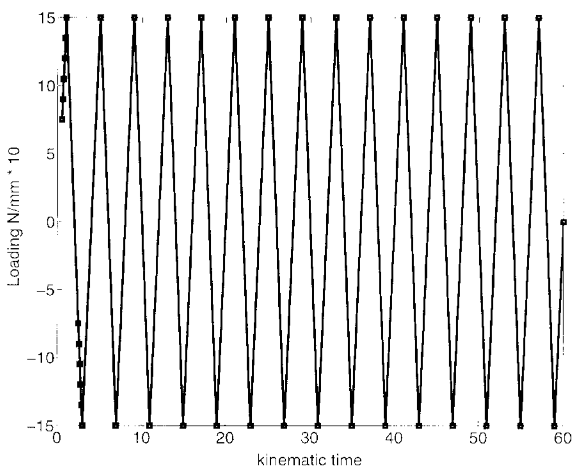

Fig. 15 15-cycle loading

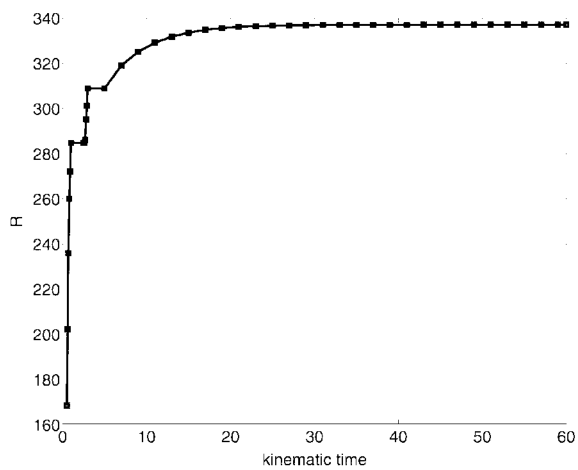

Fig. 16 Yield stress

Poisson's ratio $\nu = 0.3$, initial yield stress $R_0 = 240$ Mpa, hardening modulus $H = 1000$ Mpa, hardening exponent $\alpha = 0.5$. The mesh is shown on Fig. 14.

The cyclic loading is represented on Fig. 15. The limit cycle was reached after 10 cycles with an accuracy of 10.e-5 on the value of the yield stress (Fig. 16).

The evolution of the error estimator is represented on Fig. 17. The error kept increasing as the yield stress increased. When the limit cycle was reached after the 10th cycle, the error remained constant. In order to study the cumulative part of the error as shown in Section (4.2), we split the

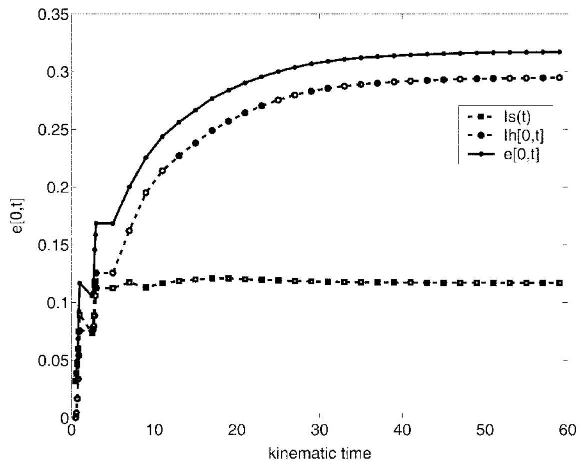

Fig. 17 Evolution of the improved DCRE estimator

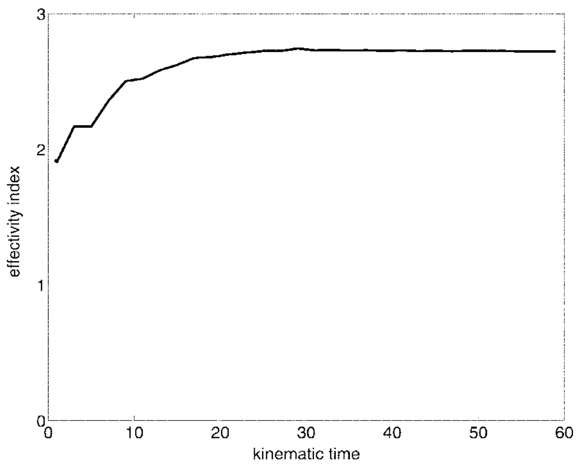

Fig. 18 Evolution of the effectivity index

global error into two parts: an "instant" part $I_s(t)$ and a cumulative part $I_h[0, t]$. The evolutions of these error indicators are represented on Fig. 17. We can observe that the evolution of the global error contribution $e[0, t]$ was due to the indicator on the cumulative part and that $I_s(t)$ remained constant after the first cycle. To show the stability of the global error estimator, the effectivity index was calculated using a very fine finite element analysis. The error estimator did not increase with the iterations.

8. Conclusions

In this paper, we showed not only that the improved DCRE leads to a good estimate of the global error, but also that the local effectivity indexes calculated from the contributions to the error of an element *E* are good-quality indicators. Since the error measure proposed is a global error measure which includes all the sources of error, we proposed to split this error measure into two parts, one of which depends only on the solution at *t* while the other integrates the history of the loading. We introduced error indicators associated with these parts. We showed on examples that the global effectivity and the local effectivity of these indicators are good. Finally, we showed on a simple cyclic loading example that the global quality of the improved DCRE remains constant during cycles. In the case of cyclic loading, some quantities depend on the history and others do not. We are currently working on the definition of specific error estimators for quantities which do not depend on the history.

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Appendix

For the Prandtl-Reuss plasticity model

$$\int_0^t Tr[(\boldsymbol{\sigma}-\underline{\boldsymbol{\sigma}})(\dot{\boldsymbol{\varepsilon}}-\dot{\underline{\boldsymbol{\varepsilon}}})]dt = \int_0^t \underbrace{Tr[(\boldsymbol{\sigma}-\underline{\boldsymbol{\sigma}})(\dot{\boldsymbol{\varepsilon}}^e-\dot{\underline{\boldsymbol{\varepsilon}}}^e)]}_{(A)}\, dt \tag{46}$$

$$+\int_0^t \underbrace{Tr[(\boldsymbol{\sigma}-\underline{\boldsymbol{\sigma}})(\dot{\boldsymbol{\varepsilon}}^p-\dot{\underline{\boldsymbol{\varepsilon}}}^p)]}_{(B)}\, dt = \tag{47}$$

we have

$$\int_0^t (A)dt = Tr[(\boldsymbol{\sigma}-\underline{\boldsymbol{\sigma}})K^{-1}(\boldsymbol{\sigma}-\underline{\boldsymbol{\sigma}})]_{|t} \tag{48}$$

and the evolution law is classically written:

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{p}\frac{\boldsymbol{\sigma}^D}{\|\boldsymbol{\sigma}^D\|} \tag{49}$$

and if $\dot{\boldsymbol{\varepsilon}}^p \neq 0$ then

$$z(\boldsymbol{\sigma}, R) = \|\boldsymbol{\sigma}^D\| - R - R_0 \tag{50}$$

Thus,

$$(B) = Tr\left[(\boldsymbol{\sigma}-\underline{\boldsymbol{\sigma}})\left(\dot{p}\frac{\boldsymbol{\sigma}^D}{\|\boldsymbol{\sigma}^D\|} - \dot{\underline{p}}\frac{\underline{\boldsymbol{\sigma}}^D}{\|\underline{\boldsymbol{\sigma}}^D\|}\right)\right] \tag{51}$$

Introducing Eq. (50):

$$(B) = \dot{p}(z(\boldsymbol{\sigma}, R) + R + R_0) + \dot{\underline{p}}((z(\underline{\boldsymbol{\sigma}}, \underline{R}) + \underline{R} + R_0)$$

$$-(\dot{p}\|\underline{\boldsymbol{\sigma}}^D\| + \dot{\underline{p}}\|\boldsymbol{\sigma}^D\|)\frac{Tr[\boldsymbol{\sigma}^D\underline{\boldsymbol{\sigma}}^D]}{\|\underline{\boldsymbol{\sigma}}^D\|\|\boldsymbol{\sigma}^D\|} \tag{52}$$

After reorganizing the terms:

$$(B) = (\dot{p} - \dot{\underline{p}})(z(\boldsymbol{\sigma}, R) - z(\underline{\boldsymbol{\sigma}}, \underline{R})) + (\dot{p} - \dot{\underline{p}})(R - \underline{R})$$
$$+(\dot{p}\|\underline{\boldsymbol{\sigma}}^D\| + \dot{\underline{p}}\|\boldsymbol{\sigma}^D\|)\left(1 - \frac{Tr[\boldsymbol{\sigma}^D\underline{\boldsymbol{\sigma}}^D]}{\|\underline{\boldsymbol{\sigma}}^D\|\|\boldsymbol{\sigma}^D\|}\right) \quad (53)$$

If we note that

$$\dot{R} = \dot{p}\frac{dg}{dp} = \dot{p}g'(p) \quad (54)$$

we obtain:

$$(B) = (\dot{p} - \dot{\underline{p}})(z(\boldsymbol{\sigma}, R) - z(\underline{\boldsymbol{\sigma}}, \underline{R}))$$
$$+\frac{1}{2}((R - \underline{R})(p - \underline{p}))$$
$$+\frac{1}{2}(\dot{p}(R - \underline{R} - g'(p)(p - \underline{p})) + \dot{\underline{p}}(\underline{R} - R - g'(\underline{p})(\underline{p} - p)))$$
$$+(\dot{p}\|\underline{\boldsymbol{\sigma}}^D\| + \dot{\underline{p}}\|\boldsymbol{\sigma}^D\|)\left(1 - \frac{Tr[\boldsymbol{\sigma}^D\underline{\boldsymbol{\sigma}}^D]}{\|\underline{\boldsymbol{\sigma}}^D\|\|\boldsymbol{\sigma}^D\|}\right) \quad (55)$$

This expression leads directly to (14).