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# Dynamic elastic local buckling of piles under impact loads

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**Abstract.** A dynamic elastic local buckling analysis is presented for a pile subjected to an axial impact load. The pile is assumed to be geometrically perfect. The interactions between the pile and the surrounding soil are taken into account. The interactions include the normal pressure and skin friction on the surface of the pile due to the resistance of the soil. The analysis also includes the influence of the propagation of stress waves through the length of the pile to the distance at which buckling is initiated and the mass of the pile. A perturbation technique is used to determine the critical buckling length and the associated critical time. As a special case, the explicit expression for the buckling length of a pile is obtained without considering soil resistance and compared with the one obtained for a column by means of an alternative method. Numerical results obtained show good agreement with the experimental results. The effects of the normal pressure and the skin friction due to the surrounding soil, self-weight, stiffness and geometric dimension of the cross section on the critical buckling length are discussed. The sudden change of buckling modes is further considered to show the 'snap-through' phenomenon occurring as a result of stress wave propagation.

**Key words:** dynamic buckling; pulse buckling; piles; high velocity impact; stress wave; perturbation; critical buckling length.

## 1. Introduction

Pile foundation is one of the most important kinds of foundations used in construction engineering (Das and Sargand 1999, Heelis *et al.* 1999). Traditionally, the design of a pile was based on the satisfaction of the strength criteria. However, due to the fact that slender piles made of high strength materials, e.g., piles made of fiber-reinforced composites (Iskander and Hassan 1998, Han and Frost 1999), are now increasingly used in engineering practices, the necessity of investigating the instability of these piles have become evident. Under impulsive axial compressive loads, slender piles may encounter the so-called 'dynamic instability' or 'dynamic buckling', which was, as a matter of fact, initially found in the early 1930's.

It is considerably more difficult to solve a dynamic buckling problem than to solve static one. Literature reviews have shown that many studies have been carried out for dynamic buckling of columns or beams under impact loads (Lindberg and Florence 1987, Simitses 1987 & 1990,

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Karagiozova and Jones 1992 & 1996). In general, dynamic buckling models for columns or beams fall into the following three categories: (1) nonconservative systems subjected to follower forces i.e., flutter, where two natural frequencies coincide with each other resulting in the amplitude of vibrations growing without bound (see, e.g., Lee and Kuo 1991, Lee 1996, Kim and Choo 1998, Langthjem and Sugiyama 2000); (2) parametric resonance associated with the growing transverse vibration induced by pulsating periodic axial or tangential load (Svensson 1996, Turhan 1998, Yeh and Chen 1998); and (3) suddenly-loaded conservative systems that are characterized by the applied dynamic load (Hao *et al*. 2000, Cui *et al.* 1999 & 2000). The investigation presented in this paper falls into the third category and concerns the high velocity impact buckling or “pulse buckling” (Liderberg and Florence 1987). Most of the previous studies on dynamic buckling of columns or beams considered almost exclusively overall buckling response at the late stage of the impact (Ari-Gur and Elishakoff 1997, Kenny *et al*. 2000, Lepik 2000). In these studies, the duration of the applied load was assumed to be sufficiently long compared with the time for a stress wave to travel the whole length of the column. As a result, the effect of the axial stress wave propagation was not considered. To the authors’ best knowledge, the investigations on local buckling, which is induced by axial stress waves at an early stage of the impact, are rather limited.

Among the others, Lee and Ettestad (1983) studied dynamic buckling of a column under given impact velocities. Two different stages were considered: (a) before the primary axial compressive wave front reached the fixed end, and (b) after the primary axial wave has reached the end and reflected back several times. Wei *et al*. (1988) and Tang and Zhu (1998) also studied either experimentally or theoretically the local buckling phenomenon of columns. In their studies, the columns were assumed to be geometrically perfect since it was believed that imperfections had negligible effects on the general behavior of local buckling modes. For piles (columns surrounded with soils), directional instability under follower loads was studied by Burguss (1975, 1976) using the Rayleigh-Ritz method, and by Omar (1980) using a finite difference method. Recently, Shen and Gao (1992) studied parametric resonance of piles. Literature reviews showed that little work had been reported in the area of dynamic local buckling of piles.

It is known that for impact loading of intermediate velocity, which is measured in mini-second order, the dynamic buckling of an impacted bar normally occurs well after the stress wave has reached the end of the bar and may have reflected back many times. In such a situation, the effect of stress propagation can be ignored. The duration of the applied impact load, however, is an important parameter that affects the buckling behavior of the bar (Hao *et al*. 2000). In contrast to this, instability often occurs at an early stage of the impact before stress waves reach the end of the bar if it is subjected to an impact load of high velocity that is measured in a micro-second order. In consequence, the influence of stress wave propagation must be considered while the effect of impact load duration can be neglected.

In this paper, the dynamic elastic local buckling of a geometrically perfect pile is studied. The interactions between the pile and its surrounding soil, i.e., the normal pressure of the soil and the skin friction acting on the pile, are fully taken into account. The self-weight of the pile is also included. A perturbation technique is used to determine the critical buckling length and the associated critical time.

## 2. Dynamical elastic local buckling of a pile

Consider a geometrically perfect pile of length $l_0$ pumped into the earth. The interactions between

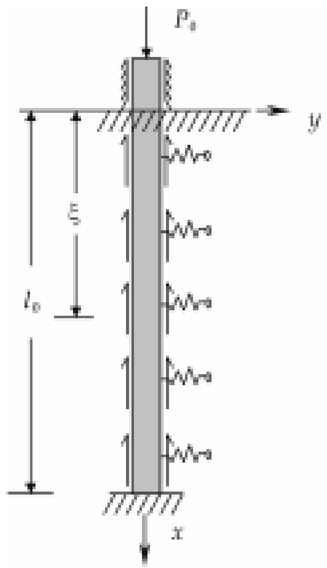


Fig. 1 A fully embedded pile subjected to impact applied load

the pile and the surrounding soil are represented by the normal pressure and the skin friction acting on the surface of the pile (Fig. 1). It is assumed that the soil has a linearly increasing stiffness of sub-grade reaction and the friction is uniformly distributed over the length of pile. The linear stiffness is denoted by $k_\sigma(= mx)$, where $m$ is the coefficient of sub-grade reaction and $x$ is the longitudinal coordinate that originates from the ground surface. The shear stress due to the skin friction is assumed to be constant and denoted by $k_\tau(= \tau)$ (see, for example. Han and Frost 1999, Heelis *et al.* 1999). The pile is subjected to an impact load of high velocity. Hence, it is assumed that local buckling occurs well before the stress wave reaches the end of the pile and thus the effect of stress wave propagation is considered. Because of the nature of the impact, the load duration is assumed to be longer than the critical buckling time at which the initial buckling occurs and the length of the pile is long enough so that the reflection of stress wave can be ignored.

During the course of the installation or in service, a pile may be unavoidably subjected to an axial impact load. When the impulsive magnitude is high enough, the aforementioned local buckling may occur. In such a situation, wave propagation and local buckling can be considered as described below.

Consider a pile subject to the following step load

$$N(0, t) = \begin{cases} P_0 & 0 \le t \le T \\ 0 & (t > T) \end{cases} \tag{1}$$

where $T$ is the impact duration and $P_0$ is the magnitude of the impact load. To consider the local buckling occurring within the load duration and the stress wave reaches the end of the pile, it is assumed that $t_{cr} \le T$ and $t_{cr} \le l_0 / c_0$, where $t_{cr}$ is the critical time when the stress wave propagates to the section at $x = \xi(= c_0 t)$ and the first buckling mode of the pile is observed. $c_0(= \sqrt{E / \rho_0})$ is the elastic wave velocity. $E$ and $\rho_0$ are, respectively, the elastic modulus and the mass density of the pile. At an arbitrary time instance $t$, the internal axial force of the pile is solved from the axial equation of motion of the pile and can be expressed as follows (Stronge 2000):

$$N(x,t) = \begin{cases} P_0 & 0 \le x \le \xi \\ 0 & x > \xi \end{cases} \tag{2}$$

It is worthwhile to mention that the axial inertia has been included in the solution of the equation of motion in the axial direction, while the stress wave reflections from the end of the pile and the transmission across the interface between the pile and its surrounding soil are ignored.

The equation of motion of the pile in the transverse direction can be written as

$$EI\frac{\partial^4 y}{\partial x^4} + (P_0 + \rho_0 gAx - u\tau x)\frac{\partial^2 y}{\partial x^2} + mxby + \rho_0 A\frac{\partial^2 y}{\partial t^2} = 0 \tag{3}$$

where $y$ is the transverse deflection of the pile; $b$ is the pile width or diameter; $u$ is the perimeter of the pile; $A$ is the area of cross-section and $EI$ is the flexural stiffness.

Various boundary conditions may be imposed on the impacted end ($x = 0$). One of the conditions that are frequently used to approximate the impacted end is a clamped end (Burguss 1975 & 1976, Shen and Gao 1992). The boundary conditions for the wave front at $x = \xi$, which is called "motion edge", is also assumed to be clamped so that the wave front compatibility conditions can be satisfied (Lee and Ettestad 1983). As a result, the boundary conditions of the pile are

$$y(0,t) = \frac{\partial y}{\partial x}(0,t) = y(\xi,t) = \frac{\partial y}{\partial x}(\xi,t) = 0 \tag{4}$$

Letting $r^2 = I/A$, where $r$ is the radius of gyration of the cross-section of the pile and introducing the following dimensionless parameters

$$\bar{P} = \frac{P_0}{EA},\ \bar{y} = \frac{y}{r},\ \bar{x} = \frac{x}{r},\ \bar{\xi} = \frac{\xi}{r},$$
$$\varepsilon_1 = \frac{(u\tau - \rho_0 gA)r^3}{EI},\ \varepsilon_2 = -\frac{mbr^5}{EI},\ \bar{t} = \frac{c_0 t}{r} \tag{5}$$

Eq. (3) can be written in the following dimensionless form

$$\frac{\partial^4 \bar{y}}{\partial \bar{x}^4} + (\bar{P} - \varepsilon_1 \bar{x})\frac{\partial^2 \bar{y}}{\partial \bar{x}^2} - \varepsilon_2 \overline{xy} + \frac{\partial^2 \bar{y}}{\partial \bar{t}^2} = 0 \tag{6}$$

The boundary conditions (4), therefore, become

$$\bar{y}(0,\bar{t}) = \frac{\partial \bar{y}}{\partial \bar{x}}(0,\bar{t}) = \bar{y}(\bar{\xi},\bar{t}) = \frac{\partial \bar{y}}{\partial \bar{x}}(\bar{\xi},\bar{t}) = 0 \tag{7}$$

Eq. (6) can be solved by the method of separation of variables, as described below:

$$\bar{y}(\bar{x},\bar{t}) = \bar{X}(\bar{x})\bar{T}(\bar{t}) \tag{8}$$

where $\bar{T}(\bar{t})$ is an amplifying factor and $\bar{X}(\bar{x})$ is a buckling mode. They are functions of $\bar{t}$ and $\bar{x}$ alone, respectively.

Substituting Eq. (8) into Eq. (6) yields

$$\frac{\overline{X}''''(\overline{x}) + \overline{P}\,\overline{X}''(\overline{x}) - \varepsilon_1 \overline{x}\,\overline{X}''(\overline{x}) - \varepsilon_2 \overline{x}\,\overline{X}(\overline{x})}{\overline{X}(\overline{x})} = -\frac{\ddot{\overline{T}}(\overline{t})}{\overline{T}(\overline{t})} \tag{9}$$

where the primes and dots denote, respectively, differentials with respect to $\overline{x}$ and $\overline{t}$ . Both sides of Eq. (9) must be equal to a non-positive constant, $-\beta^2$, so that the transverse motion of the pile is divergent rather than oscillatory. Thus, the following two ordinary differential equations are obtained.

$$\ddot{\overline{T}}(\overline{t}) - \beta^2 \overline{T}(\overline{t}) = 0 \tag{10}$$

$$\overline{X}''''(\overline{x}) + \overline{P}\,\overline{X}''(\overline{x}) - \varepsilon_1 \overline{x}\,\overline{X}''(\overline{x}) - \varepsilon_2 \overline{x}\,\overline{X}(\overline{x}) + \beta^2 \overline{X}(\overline{x}) = 0 \tag{11}$$

Accordingly, the boundary conditions for Eq. (11) are

$$\overline{X}(0) = \overline{X}'(0) = \overline{X}(\overline{\xi}) = \overline{X}'(\xi) = 0 \tag{12}$$

The general solution of Eq. (10) is

$$\overline{T}(\overline{t}) = A_1 ch\beta\overline{t} + A_2 sh\beta\overline{t} \tag{13}$$

Eq. (11) is a differential equation with variable coefficients and, therefore, it is difficult in general to seek a closed form solution. In the following part of this section, the perturbation method is used to obtain an asymptotic solution.

It can be seen from Eq. (5) that the values of $\varepsilon_1$ and $\varepsilon_2$ are much smaller than unit. Hence, to form an asymptotic solution of Eq. (11), the buckling mode can be expressed in the form of following perturbation expansion.

$$\overline{X}(\overline{x}, \varepsilon_1, \varepsilon_2) = \sum_{m=0}^{q} \sum_{n=0}^{q} \varepsilon_1^m \varepsilon_2^n \overline{X}_{mn}(\overline{x}) \tag{14}$$

that represents a solution of the $q$-th order, where $q = 0, 1, 2, ..., \infty$.

Substituting Eq. (14) into Eq. (11) and comparing terms of equal powers of both $\varepsilon_1$ and $\varepsilon_2$ yield a set of perturbation equations. They are

$$\overline{X}''''_{mn}(\overline{x}) + \overline{P}\,\overline{X}''_{mn}(\overline{x}) + \beta^2 \overline{X}_{mn}(\overline{x}) = \overline{x}\,\overline{X}''_{m-1n}(\overline{x}) + \overline{x}\,\overline{X}_{mn-1}(\overline{x}) \ (m, n = 0, 1, 2) \tag{15}$$

where the mode component $X_{mn}(\overline{x})$ is identical to zero when either $m$ or $n$ is negative. For each combination of $m$ and $n$, the differential equation is solved as described below.

When $m = 0$ and $n = 0$, Eq. (15) becomes

$$\overline{X}''''_{00}(\overline{x}) + \overline{P}\,\overline{X}''_{00}(\overline{x}) + \beta^2 \overline{X}_{00}(\overline{x}) = 0 \tag{16}$$

and the general solution can be easily obtained as

$$\overline{X}_{00}(\overline{x}) = B_1 \sin k_1 \overline{x} + B_2 \cos k_1 \overline{x} + B_3 \sin k_2 \overline{x} + B_4 \cos k_2 \overline{x} \tag{17}$$

where $k_1$ and $k_2$ are real constants and defined as

$$\left.\begin{array}{c} k_1 \\ k_2 \end{array}\right\} = \sqrt{\frac{\overline{P} \mp \sqrt{\overline{P}^2 - 4\beta^2}}{2}} \tag{18}$$

In Eq. (18), $\beta \leq \overline{P}/2$ must be satisfied so that any exponential mode is discarded. This is in consistence with the concept of a conservative system of the type under consideration. If $\beta = 0$, the solution is reduced to the special case for a static buckling problem. From Eq. (18), it can be seen that $\beta$ is an important parameter that determines the wavelength and mode shape of the pile.

When $m = 1$, $n = 0$, Eq. (15) becomes

$$\overline{X}''''_{10}(\overline{x}) + \overline{P}\,\overline{X}''_{10}(\overline{x}) + \beta^2 \overline{X}_{10}(\overline{x}) = \overline{x}\,\overline{X}''_{00}(\overline{x}) \tag{19}$$

Eq. (19) is an inhomogeneous differential equation that can be solved after substituting Eq. (17) into Eq. (19). The general solution is

$$\begin{aligned} \overline{X}_{10}(\overline{x}) = {} & B_1[\alpha_{101}\sin k_1\overline{x} + \beta_{101}\cos k_1\overline{x}] + B_2[\alpha_{101}\cos k_1\overline{x} - \beta_{101}\sin k_1\overline{x}] \\ & + B_3[\alpha_{102}\sin k_2\overline{x} + \beta_{102}\cos k_2\overline{x}] + B_4[\alpha_{102}\cos k_2\overline{x} - \beta_{102}\sin k_2\overline{x}] \end{aligned} \tag{20}$$

where

$$\begin{aligned} s_i &= \overline{P} - 2k_i^2 \qquad (i = 1, 2) \\ \alpha_{10i} &= \left(\frac{k_i^2}{s_i^2} - \frac{1}{4s_i^2}\right)x \quad (i = 1, 2) \\ \beta_{10i} &= \frac{k_i}{4s_i}x^2 \quad (i = 1, 2) \end{aligned} \tag{21}$$

By following exactly the same procedure, for arbitrarily selected $m$ and $n$,

$$\begin{aligned} \overline{X}_{mn}(\overline{x}) = {} & B_1[\alpha_{mn1}\sin k_1\overline{x} + \beta_{mn1}\cos k_1\overline{x}] + B_2[\alpha_{mn1}\cos k_1\overline{x} - \beta_{mn1}\sin k_1\overline{x}] \\ & + B_3[\alpha_{mn2}\sin k_2\overline{x} + \beta_{mn2}\cos k_2\overline{x}] + B_4[\alpha_{mn2}\cos k_2\overline{x} - \beta_{mn2}\sin k_2\overline{x}] \end{aligned} \tag{22}$$

where

$$\alpha_{01i} = \left(-\frac{1}{s_i^2} + \frac{1}{4k_i^2 s_i}\right)x \quad (i = 1, 2)$$

$$\beta_{01i} = -\frac{1}{4k_i s_i}x^2 \quad (i = 1, 2)$$

$$\alpha_{11i} = \frac{1}{16s_i^2}x^4 + \left(-\frac{5k_i^2}{s_i^4} - \frac{1}{2s_i^3} - \frac{1}{16k_i^2 s_i^2}\right)x^2 \quad (i = 1, 2)$$

$$\beta_{11i} = \left(-\frac{5k_i}{6s_i^3} + \frac{1}{24k_i s_i^2}\right)x^3 + \left(\frac{20k_i^3}{s_i^5} + \frac{7k_i}{s_i^4} - \frac{1}{16k_i^3 s_i^2}\right)x \quad (i = 1, 2)$$

$$
\begin{aligned}
\alpha_{20i} &= -\frac{k_i^2}{32s_i^2}x^4 + \left(\frac{5k_i^4}{2s_i^4} + \frac{k_i^2}{s_i^3} - \frac{3}{32s_i^2}\right)x^2 \quad (i=1,2) \\
\beta_{20i} &= \left(\frac{5k_i^3}{12s_i^3} + \frac{k_i}{16s_i^2}\right)x^3 + \left(-\frac{10k_i^5}{s_i^5} - \frac{13k_i^3}{2s_i^4} + \frac{k_i}{4s_i^3} - \frac{3}{32k_is_i^2}\right)x \quad (i=1,2) \\
\alpha_{02i} &= -\frac{1}{32k_i^2s_i^2}x^4 + \left(\frac{5}{2s_i^4} - \frac{1}{2k_i^2s_i^3} + \frac{5}{32k_i^4s_i^2}\right)x^2 \quad (i=1,2) \\
\beta_{02i} &= \left(\frac{5}{12k_is_i^3} - \frac{5}{48k_i^3s_i^2}\right)x^3 + \left(-\frac{10k_i}{s_i^5} - \frac{1}{2k_is_i^4} - \frac{1}{4k_i^3s_i^3} + \frac{5}{32k_i^5s_i^2}\right)x \quad (i=1,2) \\
\alpha_{21i} &= \left(\frac{7k_i^2}{32s_i^4} + \frac{11}{384s_i^3}\right)x^5 - \left(\frac{30k_i^4}{s_i^6} + \frac{65k_i^2}{4s_i^5} + \frac{19}{32s_i^4} + \frac{1}{32k_i^2s_i^3}\right)x^3 \\
&\quad + \left(\frac{720k_i^6}{s_i^8} + \frac{730k_i^4}{s_i^7} + \frac{160k_i^2}{s_i^6} + \frac{7}{2s_i^5} - \frac{1}{16k_i^2s_i^4} + \frac{3}{64k_i^4s_i^3}\right)x \quad (i=1,2) \\
\beta_{21i} &= \frac{k_i}{128s_i^3}x^6 - \left(\frac{25k_i^3}{8s_i^5} + \frac{49k_i}{48s_i^4} - \frac{5}{384k_is_i^3}\right)x^4 \\
&\quad + \left(\frac{180k_i^5}{s_i^7} + \frac{275k_i^3}{2s_i^6} + \frac{65k_i}{4s_i^5} + \frac{3}{32k_is_i^4} - \frac{3}{64k_i^3s_i^3}\right)x^2 \quad (i=1,2) \qquad (23) \\
\alpha_{12i} &= \left(-\frac{7}{32s_i^4} + \frac{5}{384k_i^2s_i^3}\right)x^5 + \left(\frac{30k_i^2}{s_i^6} + \frac{25}{4s_i^5} + \frac{3}{32k_i^2s_i^4} - \frac{3}{32k_i^4s_i^3}\right)x^3 \\
&\quad - \left(\frac{720k_i^4}{s_i^8} + \frac{470k_i^2}{s_i^7} + \frac{47}{s_i^6} - \frac{3}{2k_i^2s_i^5} + \frac{5}{16k_i^4s_i^4} - \frac{9}{64k_i^6s_i^3}\right)x \quad (i=1,2) \\
\beta_{12i} &= -\frac{1}{128k_is_i^3}x^6 + \left(\frac{25k_i}{8s_i^5} + \frac{5}{48k_is_i^4} + \frac{11}{384k_i^3s_i^3}\right)x^4 \\
&\quad - \left(\frac{180k_i^3}{s_i^7} + \frac{145k_i}{2s_i^6} + \frac{7}{4k_is_i^5} - \frac{17}{32k_i^3s_i^4} + \frac{9}{64k_i^5s_i^3}\right)x^2 \quad (i=1,2) \\
\alpha_{22i} &= \frac{1}{1024s_i^4}x^8 - \left(\frac{113k_i^2}{96s_i^6} + \frac{53}{192s_i^5} - \frac{17}{4608k_i^2s_i^4}\right)x^6 \\
&\quad + \left(\frac{985k_i^4}{4s_i^8} + \frac{615k_i^2}{4s_i^7} + \frac{449}{32s_i^6} - \frac{7}{48k_i^2s_i^5} - \frac{61}{3072k_i^4s_i^4}\right)x^4 \\
&\quad - \left(\frac{13260k_i^6}{s_i^{10}} + \frac{14130k_i^4}{s_i^9} + \frac{15147k_i^2}{4s_i^8} + \frac{1447}{8s_i^7} - \frac{119}{32k_i^2s_i^6} - \frac{1}{32k_i^4s_i^5} - \frac{37}{1024k_i^6s_i^4}\right)x^2 \quad (i=1,2)
\end{aligned}
$$

$$\beta_{22i} = -\left(\frac{3k_i}{64s_i^5} + \frac{1}{256k_i s_i^4}\right)x^7 + \left(\frac{161k_i^3}{8s_i^7} + \frac{271k_i}{32s_i^5} + \frac{67}{320k_i s_i^5} + \frac{13}{7680k_i^3 s_i^4}\right)x^5$$
$$-\left(\frac{2210k_i^5}{s_i^9} + \frac{3725k_i^3}{2s_i^8} + \frac{7847k_i}{24s_i^7} + \frac{83}{24k_i s_i^6} - \frac{23}{96k_i^3 s_i^5} + \frac{37}{1536k_i^5 s_i^4}\right)x^3$$
$$+\left(\frac{53040k_i^7}{s_i^{11}} + \frac{69780k_i^5}{s_i^{10}} + \frac{26322k_i^3}{s_i^9} + \frac{10821k_i}{4s_i^8} + \frac{113}{8k_i s_i^7} - \frac{33}{32k_i^3 s_i^6} - \frac{1}{16k_i^5 s_i^5} + \frac{37}{1024k_i^7 s_i^4}\right)x \quad (i = 1, 2)$$

By using Eqs. (14) and (22), solutions up to an arbitrary order of approximation can be constructed. Without loss of generality, the solutions of up to the 2nd order approximation are given below.

$$(\overline{X})^{(q)}(\overline{x}) = B_1[f_1^{(q)}(\overline{x})\sin k_1\overline{x} + g_1^{(q)}(\overline{x})\cos k_1\overline{x}] + B_2[f_1^{(q)}(\overline{x})\cos k_1\overline{x} - g_1^{(q)}(\overline{x})\sin k_1\overline{x}]$$
$$+ B_3[f_2^{(q)}(\overline{x})\sin k_2\overline{x} + g_2^{(q)}(\overline{x})\cos k_2\overline{x}] + B_4[f_2^{(q)}(\overline{x})\cos k_2\overline{x} - g_2^{(q)}(\overline{x})\sin k_2\overline{x}] \quad (q = 0, 1, 2) \tag{24}$$

where

$$f_i^{(0)}(\overline{x}) = 1; \quad (i = 1, 2)$$
$$g_i^{(0)}(\overline{x}) = 0; \quad (i = 1, 2)$$
$$f_i^{(1)}(\overline{x}) = f_i^{(0)}(\overline{x}) + \varepsilon_1\alpha_{10i} + \varepsilon_2\alpha_{01i} + \varepsilon_1\varepsilon_2\alpha_{11i}; \quad (i = 1, 2)$$
$$g_i^{(1)}(\overline{x}) = g_i^{(0)}(\overline{x}) + \varepsilon_1\beta_{10i} + \varepsilon_2\beta_{01i} + \varepsilon_1\varepsilon_2\beta_{11i}; \quad (i = 1, 2)$$
$$f_i^{(2)}(\overline{x}) = f_i^{(1)}(\overline{x}) + \varepsilon_1^2\alpha_{20i} + \varepsilon_2^2\alpha_{02i} + \varepsilon_1^2\varepsilon_2\alpha_{21i} + \varepsilon_1\varepsilon_2^2\alpha_{12i} + \varepsilon_1^2\varepsilon_1^2\alpha_{22i}; \quad (i = 1, 2)$$
$$g_i^{(2)}(\overline{x}) = g_i^{(1)}(\overline{x}) + \varepsilon_1^2\beta_{20i} + \varepsilon_2^2\beta_{02i} + \varepsilon_1^2\varepsilon_2\beta_{21i} + \varepsilon_1\varepsilon_2^2\beta_{12i} + \varepsilon_1^2\varepsilon_1^2\beta_{22i}; \quad (i = 1, 2) \tag{25}$$

and $\overline{X}^q(\overline{x})$ is the $q$-th order solution. To find the final solution of the problem, the buckling mode must satisfy the boundary conditions described in Eq. (12), i.e.,

$$\overline{X}(0) = B_2 + B_4 = 0$$
$$\overline{X}'(0) = B_1(F_1^{(q)})'(0) + B_2(G_1^{(q)})'(0) + B_3(F_2^{(q)})'(0) + B_4(G_2^{(q)})'(0) = 0$$
$$\overline{X}(\overline{\xi}) = B_1F_1^{(q)}(\overline{\xi}) + B_2G_1^{(q)}(\overline{\xi}) + B_3F_2^{(q)}(\overline{\xi}) + B_4G_2^{(q)}(\overline{\xi}) = 0$$
$$\overline{X}'(\overline{\xi}) = B_1(F_1^{(q)})'(\overline{\xi}) + B_2(G_1^{(q)})'(\overline{\xi}) + B_3(F_2^{(q)})'(\overline{\xi}) + B_4(F_2^{(q)})'(\overline{\xi}) = 0 \tag{26}$$

where

$$F_1^{(q)}(\overline{x}) = f_i^{(q)}(\overline{x})\sin k_i\overline{x} + g_i^{(q)}(\overline{x})\cos k_i\overline{x} \quad (i = 1, 2; \quad q = 0, 1, 2)$$

$$G_1^{(q)}(\overline{x}) = f_i^{(q)}(\overline{x})\cos k_i\overline{x} - g_i^{(q)}(\overline{x})\sin k_i\overline{x} \quad (i = 1, 2; \quad q = 0, 1, 2)$$

Here $q$ is the order of approximation. Eq. (26) forms a set of homogeneous linear equation system in terms of $B_i(i = 1, \ldots, 4)$, the characteristic determinant of which is

$$\Delta = \det\begin{vmatrix} 0 & 1 & 0 & 1 \\ (F_1^{(q)})'(0) & (G_1^{(q)})'(0) & (F_2^{(q)})'(0) & (G_2^{(q)})'(0) \\ F_1^{(q)}(\overline{\xi}) & G_1^{(q)}(\overline{\xi}) & F_2^{(q)}(\overline{\xi}) & G_2^{(q)}(\overline{\xi}) \\ (F_1^{(q)})'(\overline{\xi}) & (G_1^{(q)})'(\overline{\xi}) & (F_2^{(q)})'(\overline{\xi}) & (G_2^{(q)})'(\overline{\xi}) \end{vmatrix} \tag{27}$$

Eq. (27) is the characteristic determinant in terms of two independent parameters, i.e., the dimensionless buckling length, $\overline{\xi}$, and the separation constant, $\beta^2$. Hence, the dynamic buckling of a pile can be uniquely defined by the associated buckling wavelength and the instant location of the stress front. The moving stress front continuously changes the effective length over which the pile may buckle in a local mode. For a given impact load, the pile only buckles when the stress front arrives at a critical section, from which the distance to the impact end is measured as the critical buckling length.

Mathematically, buckling means that there is bifurcation in the solution of perturbation equation. Hence, the existence of a nontrivial solution of Eq. (28) requires

$$\Delta = 0 \tag{28a}$$

According to Jiang *et al*. (1992), the dynamic modes must also satisfy

$$\frac{\partial \Delta}{\partial \overline{\xi}} = 0 \tag{28b}$$

so that the obtained buckling modes do not violate the restriction of linear analysis. Eqs. (27), (28) and (18) provide the solution for the dynamic buckling of the impacted pile, from which the critical buckling length $l_{cr} = r \times \overline{\xi}$ and the associated critical buckling time $t_{cr} = l_{cr}/c_0$ can be calculated.

## 3. Numerical examples

As part of the validation of the present method, the buckling of a pile is studied first without considering the interactions between the pile and its surrounding soil. The solution can be quickly found by substituting $\overline{X}^{(0)}(\overline{x})$ into Eqs. (28) and (18), i.e.,

$$k_1 k_2 = \beta = \frac{n_1 n_2 \pi^2}{\overline{\xi}^2} = \overline{P}\frac{n_1 n_2}{n_1^2 + n_2^2}$$

Table 1 Buckling length and buckling time for uniform section bars

| | | $P_0$ (KN ) | | | | |
|---|---|---|---|---|---|---|
| | | 19.17 | 21.45 | 27.99 | 33.23 | 35.24 |
| $l_{cr}$ (mm) | Tang (1998) | 604.80 | 539.48 | 457.29 | 427.79 | 413.04 |
| | Present | 578.20 | 551.88 | 481.30 | 440.67 | 428.05 |
| $t_{cr}$ (μs) | Tang (1998) | 120.96 | 107.90 | 91.46 | 85.56 | 82.61 |
| | Present | 115.64 | 110.38 | 96.26 | 88.13 | 85.61 |

where $n_1$ and $n_2$ are two positive integers that are either even or odd. This solution is identical to the one obtained by Wei *et al.* (1988). The dynamic buckling length and critical buckling time of the pile are compared in Table 1 with the experimental results due to Tang and Zhu (1998). In the calculation, the force time duration is assumed to be larger than the calculated critical buckling time. The pile has a rectangular cross-section of 10 mm × 7.3 mm. The material properties of the pile are $E$ = 200 GPa and $c_0 = 5.0 \times 10^3$ m/s.

After the successful comparisons, the dynamic buckling of piles having different cross sections or materials is further considered. The results are shown in Figs. 2-5 and Table 2, where $P_0/EA$ and $l_{cr}/r$ are, respectively, the dimensionless impact load and critical buckling length.

First, the convergence study of the approximation solution introduced in the preceding section is illustrated in Fig. 2, where the buckling load of a steel pile with square section is shown against the critical buckling length. In the calculations, the following geometrical and material properties are used.

$$E = 200 \text{ GPa}, \quad \rho_0 = 7800 \text{ kg/m}^3, \quad \tau = 20 \text{ KN/m}^2, \quad m = 7500 \text{ KN/m}^4, \quad A = 0.04 \text{ m}^2.$$

The results show that the convergence rate is fast and a second order approximation is normally sufficient to obtain a satisfactory solution.

The parametrical study shown in Table 2 is carried out for the same pile for five cases where different combinations of the parameters, i.e., mass, normal soil pressure and skin friction, are considered. The five cases are listed in Table 3.

The critical buckling lengths of the five cases for six different impact loads are obtained and

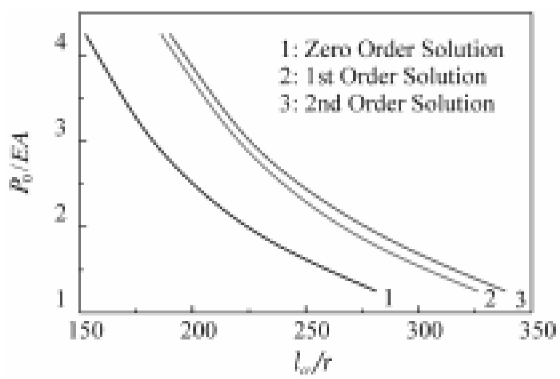


Fig. 2 Convergence analysis of the approximation solutions

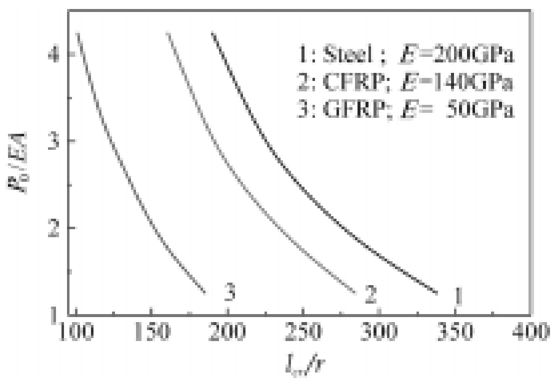


Fig. 3 Effect of elastic modulus on the critical buckling length of a pile

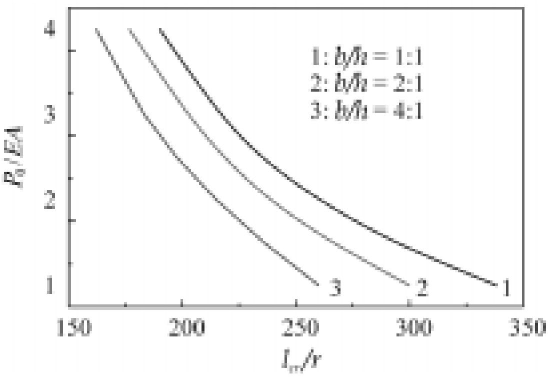


Fig. 4 Effect of geometric dimension on the critical buckling length of a pile

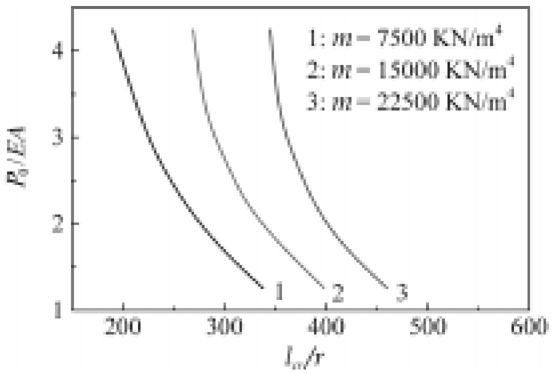


Fig. 5 Effect of soil reactions on the critical buckling length of a pile

Table 2 Critical buckling length and the associated buckling loads

| $P_0/EA(\times 10^{-3})$ | | 1.25 | 1.75 | 2.25 | 2.75 | 3.25 | 4.25 |
|---|---|---|---|---|---|---|---|
| $l_{cr}/r$ | Case1 | 281.25 | 237.68 | 210.08 | 190.13 | 174.48 | 152.58 |
| | Case2 | 279.55 | 237.27 | 210.05 | 189.90 | 174.20 | 152.40 |
| | Case3 | 337.65 | 290.40 | 255.80 | 233.20 | 215.55 | 189.60 |
| | Case4 | 342.60 | 293.20 | 257.00 | 234.20 | 216.30 | 190.00 |
| | Case5 | 338.46 | 292.76 | 256.60 | 234.13 | 216.18 | 189.94 |

Table 3 Cases considered in the parametric study

| | Mass | Normal soil pressure | Skin friction |
|---|---|---|---|
| Case 1 | No | No | No |
| Case 2 | Yes | No | Yes |
| Case 3 | Yes | Yes | No |
| Case 4 | No | Yes | Yes |
| Case 5 | Yes | Yes | Yes |

presented in Table 2. It can be observed that the results for the first two cases are very close but substantially lower than the results of the other three cases. From this observation, it can be concluded that the normal pressure on the pile has a predominant effect on the critical buckling length of the pile, while the effects of other factors are negligible.

Fig. 3 shows the dimensionless impact load against the critical buckling length for a steel pile, a Carbon Fiber Reinforced Polymer (CFRP) pile and a Glass Fiber Reinforced Polymer (GFRP) pile, respectively. The piles all have a square cross section of $A = 0.04\,\text{m}^2$. The normal pressure and the shear resistance of the soil are defined, respectively, by $m = 7500\,\text{KN/m}^4$ and $\tau = 20\,\text{KN/m}^2$. The materials of the piles have the following properties:

Steel : $E$ = 200 GPa, $\rho_0 = 7800\ \text{kg/m}^3$;
CFRP : $E$ = 140 GPa, $\rho_0 = 1600\ \text{kg/m}^3$;
GFRP : $E$ = 50 GPa, $\rho_0 = 1800\ \text{kg/m}^3$.

Table 4 Snap-through of the buckling modes under $P_0 = 1.0\times10^4$ KN

| Modes | I | II | III | IV |
|---|---|---|---|---|
| $l_{cr}$ (m) | 19.541 | 26.466 | 29.075 | 33.636 |
| $t_{cr}$ (ms) | 3.859 | 5.227 | 5.742 | 6.643 |
| $\beta$ ($\times 10^{-4}$) | 4.039 | 5.031 | 2.878 | 5.501 |

For all three cases, the critical buckling length increases as the applied impact load decreases. The steel pile has the longest critical buckling length while the GFRP pile has the shortest one. This is obviously because the steel pile has the highest elastic modulus and hence the highest stiffness. The results show that the stiffness of a material has a significant effect on the critical buckling length.

Fig. 4 shows the effect of the geometrical dimensions of a pile's cross section on the critical buckling length for steel piles having various rectangular cross sections. The piles have the same cross section area but different values of side ratios, i.e., $b/h$ = 1:1, 2:1 and 4:1, where $h$ is the width of pile's cross section. It can be seen from Fig. 4 that a decrease of the side ratio increases the critical buckling length. Hence, it can be concluded that subjected to the same impact load, piles with square sections have the longest critical buckling lengths.

Fig. 5 shows the effect of the coefficient of sub-grade reaction ($m = 7500$ KN/m$^4$, 15000 KN/m$^4$ and 22500 KN/m$^4$) on the critical buckling length for a steel pile with a square cross section. As expected, these results show that the critical buckling length is deceased by decreasing the coefficient of sub-grade reaction of the surrounding soil.

In dynamic buckling analysis, another important phenomenon that needs to be further investigated in the future is as the elastic stress wave transmits along a pile surrounded by soil of uniform stiffness, a sudden change of mode shape may occur. (Hayashi and Sano 1972) if the restrains at both the impact end and the stress front are assumed to be rigid. This phenomenon was called 'snap-through' in Wei *et al.* (1988) and is caused by the propagation of the stress wave and the increase of the distance between the impact end and the stress front. Table 4 shows the 'snap-through' of the first four buckling modes for the steel pile with square cross section and subjected to $P_0 = 1.0 \times 10^4$ KN. From Table 4, it can be seen that the value of $\beta$ for the third buckling mode drops abruptly, which indicates that a sharp decrease of buckling wavelength, and hence, a 'snap through' has occurred between the second and the third modes.

## 4. Conclusions

A new approach for the dynamic elastic local buckling of piles subjected to high velocity impact loads has been presented. Numerical illustrations of the method have been carried out. In the analysis the interactions between the piles and the surrounding soil, including the influence of the normal resistance of the soil and the skin friction along the piles, were considered. The self-weight of piles and the stress wave propagation were also taken into account. The method is based on the use of the perturbation technique that results in the solutions of a series of ordinary differential equations with constant coefficients.

The numerical results obtained are the critical buckling lengths and critical buckling times for the piles subjected to various loads and soil conditions. The results showed that dynamical elastic local buckling occurred before the stress wave reached the ends of the piles. The parametric investigation also showed that the effects of skin friction and pile's weight on the instability were small and could be ignored, while the elastic modulus, geometrical dimension and the coefficient of sub-grade reaction had significant effects.

It was observed that as the stress front, where a movable rigid restrain was imposed, traveled towards the end of the pile, a snap through occurred. This was also observed by (Hayashi and Sano 1972) where a bar with movable pinned end was considered. To have a better understanding of this phenomenon, further research is needed. This includes a study of the effect of the surrounding soil on the occurrence of the snap through.

## References

Ari-Gur, J. and Elishakoff, I. (1997), "Dynamic instability of a transversely isotropic column subjected to a compression pulse", *Comput. & Struct*., **62**(5), 811-815.

Burguss, I.W. (1975), "A note on the directional stability of driven piles", *Geotechnique*, **25**(2), 413-416.

Burguss, I.W. (1976), "The stability of slender piles during driving", *Geotechnique*, **26**(2), 281-292.

Cui, S., Cheong, H.K. and Hao, H. (1999), "Experimental study of dynamic buckling of simply-supported columns under axial slamming", *J. Eng. Mech.*, ASCE, **125**(5), 513-520.

Cui, S., Cheong, H.K. and Hao, H. (2000), "Experimental study of dynamic post-buckling characteristics of columns under fluid-soil slamming", *J. Eng. Struct.*, **22**, 647-656.

Das, Y.C. and Sargand, S.M. (1999), "Forced vibrations of laterally loaded piles", *Int. J. Solids Struct.*, **36**, 4975-4989.

Han, J. and Frost, J.D. (1999), "Buckling of vertically loaded fiber-reinforced polymer piles", *J. Reinf. Plast. Comp.*, **18**(4), 290-318.

Hao, H., Cheong, H.K. and Cui, S. (2000), "Analysis of imperfect column buckling under intermediate velocity impact", *Int. J. Solids Struct.*, **37**, 5297-5313.

Hayashi, T. and Sano, Y. (1972), "Dynamic buckling of elastic bars (the case of high velocity impact)", *Bull.* JSME, **15**(88), 1176-1184.

Heelis, M.E., Pavlovic, M.N. and West, R.P. (1999) "The stability of uniform-friction piles in homogeneous and non-homogeneous elastic foundations", *Int. J. Solids Struct.*, **36**, 3277-3292.

Iskander, M.G. and Hassan, M. (1998) "State of the practice review in FRP composite piling", *J. Compos. Construction*, **2**(3), 116-120.

Jiang, Z.B., Zhu, Z.X. and Li, Y.C. (1992), "An approximate theory for stress wave induced axisymmetric buckling of an elastic thin cylindrical shell", In: *Proc. Int. Sym. Intense Dynamic Loading and Its Effects*, Sichuan Univ. Press, Chengdu, China, 538-542.

Karagiozova, D. and Jones, N. (1992), "Dynamic buckling of a simply elastic-plastic model under pulse loading", *Int. J. Non-linear Mech.*, **27**(6), 981-1005.

Karagiozova, D. and Jones, N. (1996), "Multi-degrees of freedom model for dynamic buckling of a elastic-plastic structure", *Int. J. Solids Struct.*, **33**(23), 3377-3398.

Kenny, S., Pegg, N. and Taheri, F. (2000), "Dynamic elastic buckling of a slender beam with geometric imperfections subject to an axial impulse", *Finite Elements in Analysis and Design*, **35**, 227-246.

Kim, J.H. and Choo, Y.S. (1998), "Dynamic stability of a free-free Timoshenko beam subjected to a pulsating follower force", *J. Sound Vib*., **216**(4), 623-636.

Langthjem, M.A. and Sugiyama, Y. (2000), "Dynamic stability of columns subjected to follower loads: A survey", *J. Sound Vib*., **238**(5), 809-851.

Lee, H.P. (1996), "Damping effects on the dynamic stability of a rod subject to intermediate follower loads", *Comput. Methods. Appl. Mech. Eng*., **131**(1-2), 147-157.

Lee, L.H.N. and Ettestad, K.L. (1983), "Dynamic buckling of an ice strip by axial impact", *Int. J. Impact Eng.*, **1**(4), 343-356.

Lee, S.Y. and Kuo, Y.H. (1991), "Non-conservative stability of intermediate spring supported columns with an elastically restrained and support", *Int. J. Solids Struct.*, **28**(9), 1129-1137.

Lepik, U. (2000), "On dynamic buckling of elastic-plastic beams", *Int. J. Non-linear Mech.*, **35**, 721-734.

Lindberg, H.E. and Florence, A.L. (1987), *Dynamic Pulse Buckling-Theory and Experiment*, Martinus Nijoff, Dordrecht.

Omar, R.M. and Poskitt, T.T. (1980), "Directional stability of piles during driving", *Numerical Methods in Offshore Piling*, ICE, London, 19-28.

Shen, X.Y. and Gao, D.Z. (1992), "Effect of skin friction on the directional stability of driven piles during driving", *Appl. Mathematics Mech*., **13**(12), 1143-1148.

Simitses, G.J. (1987), "Instability of dynamically-loaded structures", *Appl. Mech. Rev.*, **40**(10), 1403-1408.

Simitses, G.J. (1990), *Dynamic Stability of Suddenly Loaded Structure*, Springer, New York.

Stronge, W.J. (2000), *Impact Mechanics*, Cambridge University Press, Cambridge, UK

Svensson, I. (1996), "Dynamic buckling of a beam with transverse constraints", *Nonlinear Dynamics*, **11**(4), 315-328.

Tang, L.Q. and Zhu, Z.X. (1998), "Experimental study on the impact buckling and post-buckling of elastic bars", *Explosion and Shock Waves*, **18**(2), 97-102(in Chinese).

Turhan, O. (1998), "A generalized Bolotin's method for stability limit determination of parametrically excited systems", *J. Sound Vib*., **216**(5), 851-863.

Wei, Y., Zhu, Z.X. and Li, Y.C. (1988), "Buckling of rods under impact load", *J. Experimental Mech*., **3**(3), 258-263.

Weller, T., Abramovich, H. and Yaffe, R. (1989), "Dynamic buckling of beams and plates subjected to axial impact", *Comput. & Struct.*, **32**(3/4), 835-851.

Yeh, M.K. and Chen, C.C. (1998), "Dynamic instability of a general column under a periodic load in the direction of the tangency coefficient at any position", *J. Sound Vib*., **217**(4), 665-689.