Structural Engineering and Mechanics, Vol. 13, No. 3 (2002) 261-276
DOI: http://dx.doi.org/10.12989/sem.2002.13.3.261

Mixed finite element model for laminated composite beams

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Abstract. A novel, 6-node, two-dimensional mixed finite element (FE) model has been developed to analyze laminated composite beams by using the minimum potential energy principle. The model has been formulated by considering four degrees of freedom (two displacement components *u*, *w* and two transverse stress components σ_z, τ_{xz}) per node. The transverse stress components have been invoked as nodal degrees of freedom by using the fundamental elasticity equations. Thus, the present mixed finite element model not only ensures the continuity of transverse stress and displacement fields through the thickness of the laminated beams but also maintains the fundamental elasticity relationship between the components of stress, strain and displacement fields throughout the elastic continuum. This is an important feature of the present formulation, which has not been observed in various mixed formulations available in the literature. Results obtained from the model have been shown to be in excellent agreement with the elasticity solutions for thin as well as thick laminated composite beams. A few results for a cross-ply beam under fixed support conditions are also presented.

Key words: mixed finite element; minimum potential energy principle; laminated composite beam.

1. Introduction

The ever-increasing use of composite materials in advanced technology areas, where precision and reliability play a paramount role, demands a clear understanding of their behaviour and performance under severe operating conditions. An understanding of failure due to delamination is of considerable importance. It has been concluded from many investigations that the high interlaminar shear stress on the free edge of angle-ply laminates can cause edge delamination (Pipes and Pagano 1970), as can the interlaminar normal stress in cross-ply laminates (Rybicki 1971). Thus, delamination has become a problem of significant concern in any analysis and design of advanced fibre-reinforced composite structures. A theory that can predict all of these stresses accurately is necessary for a clear understanding of the failure mechanism involved in the delamination of laminated composites.

The behaviour of composite laminates can be characterized by a complex three dimensional state of stress, evidencing high interlaminar stresses caused by the inherent anisotropy and mismatches of material properties of such structural members (Jones 1975). Elastic solutions for layered plates (Srinivas and Rao 1970, Pagano 1970, Pagano and Hatfield 1972) indicate that interlaminar continuity of transverse normal and shear stresses as well as displacement fields through the thickness of the plates are essential requirements for their analysis. Thus, a layer-wise analysis is

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often required for laminated composite structures. Various displacement based layer-wise theories and finite elements (FEs) have been proposed by Reddy (1987), Soldatos (1992), Wu and Kuo (1993), Wu and Hsu (1993) and others. These theories and corresponding finite element models have been reported to provide satisfactory results for both global (e.g., deflections and flexural stresses) and local values (e.g., transverse stresses) of thin and thick laminates. However, in all these models only the continuity of displacement fields through the thickness was satisfied. The continuity of transverse stress components at the interface could not be enforced.

A layer-wise mixed finite element model with displacement and transverse stress components as primary variables can easily satisfy the requirements of transverse stress continuity in addition to the continuity of displacement fields through the thickness of the laminated composite structure. Transverse stress components can be evaluated directly through such a mixed FE model. Thus, integration of the equilibrium equations can be avoided. This is helpful as they involve differentiation of in-plane stresses and displacement fields, thereby introducing further approximations in the calculation of transverse stresses. Wu and Lin (1993), for example, presented a two-dimensional mixed finite element scheme based on a local high-order displacement model for the analysis of sandwich structure, where displacement continuity conditions at the interface between layers were regarded as the constraints and the interlaminar stresses were introduced as the Lagrange multiplier. Shi and Chen (1992) developed a three-dimensional mixed FE model based on a global-local laminate variational model. The model proposed a mixed use of a hybrid element within a high precision stress solution region in the thickness direction of the laminate and a conventional displacement finite element in the remaining. Carrera (1996, 1998, 1999), has done extensive work on the development of a mixed finite element model. Firstly, a mixed plate element was developed (Carrera 1996) as an extension to a C^o Reissner-Mindlin plate element by using Reissner's mixed variational principle, in which the zig-zag variation of in-plane displacement fields through the thickness was ensured by including additional terms in the standard Reissner-Mindlin displacement model, whereas the transverse displacement field was kept unchanged. Stress degrees of freedom were introduced by assuming transverse shear stress fields. As a further development, Carrera (1998, 1999) also introduced the transverse normal stress field into the FE model. Because of the fact that in any mixed finite element model developed by using Reissner's variational principle, the stress fields are assumed independently of the displacement fields, the fundamental elastic relations cannot be satisfied exactly.

A 6-node plane stress mixed finite element model has been developed in the present formulation, by using the minimum potential energy principle. The transverse stress quantities (σ_z and τ_{xz}) are invoked from the assumed displacement fields by using the fundamental equations of elasticity. Thus, the model presented here differs from the previous FE models in the following ways:

(1) Fundamental elastic relations between stress and displacement fields at all the points of an elastic continuum have been maintained in the present FE model.

(2) The model satisfies exactly the requirements of through-thickness continuity of transverse stress components and continuous displacement fields. Thus, it can appropriately model a composite laminated structural member of any number of lay-ups of different materials (i.e., *N*-layered FE model).

(3) The formulation is simple and has a greater physical appeal as the model is based on the minimum energy principle.

(4) Because the transverse stress components are the nodal degrees of freedom in the present FE model, their computation does not require the integration of equilibrium equations, which normally

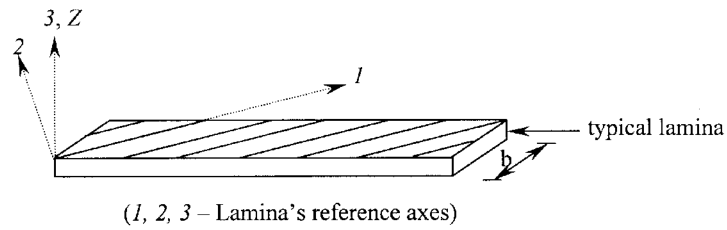
reduces the accuracy of these stress quantities.

2. Theoretical formulation

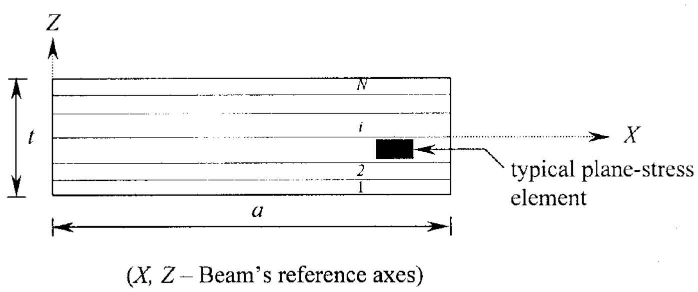
An anisotropic composite laminated beam consisting of N-layers of orthotropic laminae shown in Figs. 1(a, b) has been considered for finite element analysis. The beam has been discretized into a number of plane stress elements. Each element lies completely within a lamina; no element crosses the interface between any two successive laminae.

2.1 Kinematics

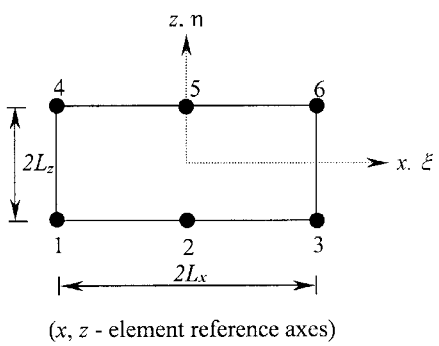
A six-node plane stress mixed finite element model shown in Fig. 1c has been developed by



a. Geometry of i^{th} lamina with its positive set of reference axes



b. Geometry of laminated beam with positive set of reference axes



c. Geometry of six-node plane stress mixed finite element

Fig. 1 (a-c) Geometry of laminated beam and mixed finite element considering the displacement fields $u(x, z)$ and $w(x, z)$ having quadratic variation along longitudinal axis x and cubic variation along transverse axis z. The displacement fields can be expressed as

$$u(x,z)=\sum_{i=1}^{3} g_i a_{0i}+z\sum_{i=1}^{3} g_i a_{1i}+z^2\sum_{i=1}^{3} g_i a_{2i}+z^3\sum_{i=1}^{3} g_i a_{3i} \quad (1)$$

$$w(x,z)=\sum_{i=1}^{3} g_i b_{0i}+z\sum_{i=1}^{3} g_i b_{1i}+z^2\sum_{i=1}^{3} g_i b_{2i}+z^3\sum_{i=1}^{3} g_i b_{3i} \quad (2)$$

where
$$g_1=\frac{\xi}{2}(\xi-1),\ g_2=1-\xi^2,\ g_3=\frac{\xi}{2}(1+\xi),\ \xi=x/L_x \quad (3)$$

Further, the generalized co-ordinates a_{mi} and b_{mi} (m = 0, 1, 2, 3; i = 1, 2, 3) are functions of element coordinate axis 'z'. The element's coordinate axes x, z are parallel to the laminate coordinates X, Z.

It may be noted that the variation of displacement fields has been assumed to be cubic along the thickness of element although there are only two nodes along 'z' axis of an element (Fig. 1c). Such a variation is required for invoking transverse stress components σ_z and τ_{xz} as the nodal degrees-of-freedom in the present formulation. Further, it also ensures parabolic variation of transverse stresses through the thickness of an element.

2.2 Constitutive equations

Each lamina in the laminate has been considered to be in the state of plane stress in X-Z plane so that the constitutive relation for a typical i^{th} lamina with reference to the coordinate system can be shown to be

$$\{\sigma\}=[D]\{\in\} \quad (4a)$$

Here
$$\{\sigma\}=[\sigma_x\ \sigma_z\ \sigma_{xz}]^T \quad (4b)$$

$$[D]=\begin{bmatrix} D_{11} & D_{13} & 0 \\ D_{13} & D_{33} & 0 \\ 0 & 0 & D_{55} \end{bmatrix} \quad (4c)$$

$$\{\in\}=[\in_x\ \ \in_z\ \ \gamma_{xz}]^T \quad (4d)$$

The coefficients D_{mn} are the reduced elastic constants.

2.3 Finite element formulation

The transverse stresses can be obtained from the constitutive equations, Eq. (4a) and the strain displacement relations as

$$\tau_{xz}=D_{55}\gamma_{xz}=D_{55}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \quad (5)$$

$$\sigma_z=D_{13}\in_x+D_{33}\in_z=D_{13}\frac{\partial u}{\partial x}+D_{33}\frac{\partial w}{\partial z} \quad (6)$$

Eqs. (5) and (6) respectively, can be rearranged in the following form

$$\left(\frac{\tau_{xz}}{D_{55}}-\frac{\partial w}{\partial x}\right)=\frac{\partial u}{\partial z} \tag{7}$$

$$\frac{1}{D_{33}}\left(\sigma_z-D_{13}\frac{\partial u}{\partial x}\right)=\frac{\partial w}{\partial z} \tag{8}$$

Substituting Eqs. (1), (2), (7) and (8) at each node of the element, the following expressions for the displacement fields $u(x, z)$ and $w(x, z)$ can be obtained

$$u(x,y)=\sum_{n=1}^{6} g_i\left(f_q u_n+f_p\frac{\partial u_n}{\partial z}\right) \tag{9}$$

$$w(x,y)=\sum_{n=1}^{6} g_i\left(f_q w_n+f_p\frac{\partial w_n}{\partial z}\right) \tag{10}$$

where 'n' is the node number (1, 2,…, 6) of the element, shown in Fig. 1c. u_n and w_n are the nodal displacement degrees-of-freedom and $\frac{\partial u_n}{\partial z}$ and $\frac{\partial w_n}{\partial z}$ contain nodal transverse stress degrees-of-freedom. That is how nodal transverse stress terms (i.e., stress degrees-of-freedom) are brought into the expression of displacement fields. Further,

$$\left.\begin{array}{c} i = 1, 2, 3 \text{ for the nodes with } \xi = -1,\ \xi = 0 \text{ and } \xi = 1, \text{ respectively.} \\ \eta = z/L_z;\ q = 1, 2;\ p = 3, 4; \text{ for the nodes with } \eta = -1 \text{ and } \eta = 1, \text{ respectively.} \\ f_1 = \frac{1}{4}(2-3\eta+\eta^3);\quad f_2 = \frac{1}{4}(2+3\eta-\eta^3); \\ f_3 = \frac{L_z}{4}(1-\eta-\eta^2+\eta^3);\quad f_4 = \frac{L_z}{4}(-1-\eta+\eta^2+\eta^3); \end{array}\right\} \tag{11}$$

Eqs. (9) and (10) can be rewritten in the standard finite element form as

$$\{u\}=[u\quad w]^T=[N]\{q\} \tag{12a}$$

where

$$[N]=[\underline{N}_1\ \underline{N}_2\ \underline{N}_3\ \underline{N}_4\ \underline{N}_5\ \underline{N}_6] \tag{12b}$$

$$\{q\}=[\underset{\sim}{q}_1^T\ \underset{\sim}{q}_2^T\ \underset{\sim}{q}_3^T\ \underset{\sim}{q}_4^T\ \underset{\sim}{q}_5^T\ \underset{\sim}{q}_6^T]^T \tag{12c}$$

Further,

$$\{q_n\} = [u_n\quad w_n\quad \tau_n\quad \sigma_n]^T \tag{13}$$

and

$$[N_n]=\left[\begin{array}{c:c:c:c} g_i f_q & -g_i' f_p & g_i f_q\dfrac{1}{D_{55}} & 0 \\ \hdashline -g_i' f_p\dfrac{D_{13}}{D_{33}} & g_i f_q & 0 & g_i f_p\dfrac{1}{D_{33}} \end{array}\right] \tag{14}$$

n, i, q and p are same as expressed in Eq. (11). Furthermore,

$$g_i' = \frac{\partial g_i}{\partial x} \tag{15}$$

The total potential energy Π of the laminated beam can be obtained from

$$\Pi = b\left[\frac{1}{2}\int_A \{\in\}^T\{\sigma\}dxdz - \int_A \{q\}^T\{p_b\}dxdz - \int \{q\}^T\{p_t\}dx\right] \tag{16}$$

where $\{p_b\}$ is the body force vector per unit volume, and $\{p_t\}$ is traction load vector acting on an edge of the laminated beam.

The strain vector $\{\in\}$ and the stress vector $\{\sigma\}$ can be expressed as

$$\{\in\} = [B]\{q\} \tag{17}$$

and

$$\{\sigma\} = [D][B]\{q\} \tag{18}$$

Here

$$[B] = [\underline{B}_1\ \underline{B}_2\ \underline{B}_3\ \underline{B}_4\ \underline{B}_5\ \underline{B}_6] \tag{19}$$

with

$$[B_n] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \end{bmatrix} [N_n] = \left[\begin{array}{c:c:c:c} g_i' f_q & -g_i'' f_p & g_i' f_p \frac{1}{D_{55}} & 0 \\ \hdashline -g_i' \bar{f}_p \frac{D_{13}}{D_{33}} & g_i \bar{f}_q & 0 & g_i \bar{f}_p \\ \hdashline g_i \bar{f}_q - g_i'' f_p \frac{D_{13}}{D_{33}} & g_i'(f_q - \bar{f}_p) & g_i \bar{f}_p \frac{1}{D_{55}} & g_i' \bar{f}_p \frac{1}{D_{33}} \end{array}\right] \tag{20}$$

n, i, q and p are same as expressed in Eq. (11). Further,

$$g_i'' = \frac{\partial^2 g_i}{\partial x^2}; \quad \text{and} \quad \bar{f}_j = \frac{\partial f_j}{\partial z}; \ j = q \ \text{ or } \ p; \tag{21}$$

Minimization of the total potential energy functional, Eq. (16), yields the element property matrix $[K]^e$ and the element influence vector $\{f\}^e$ as

$$[K]^e = b\int_{-L_x}^{L_x}\int_{-L_z}^{L_z} [B]^T[D][B]dxdz \tag{22}$$

$$\{f\}^e = b\int_{-L_x}^{L_x}\int_{-L_z}^{L_z} [N]^T\{p_b\}dxdz + \int [N]^T\{p_t\}dx \tag{23}$$

The global equation can be obtained in the following form, after assembly

$$[K]\{Q\} = \{F\} \tag{24}$$

where $[K]$, $\{Q\}$ and $\{F\}$ are, respectively, the global property matrix, the global degrees of freedom vector and the global influence vector.

3. Numerical results and discussion

A computer program incorporating the present two-dimensional mixed theory has been developed in FORTRAN-90 for the analysis of symmetric/unsymmetric composite and sandwich laminated beams. Numerical computations have been performed for various examples. Results have been compared with the available closed form elastic and finite element solutions wherever these are available in the literature.

Numerical investigations have been undertaken for a homogeneous orthotropic beam and two symmetric cross-ply laminated beams, simply supported at the ends and a fixed supported symmetric cross-ply laminated beam. All the beams are subjected to sinusoidal transverse load on their top surface. The boundary conditions and the material properties are tabulated in Tables 1 and 2, respectively. Only half of the beam has been considered in the analysis, exploiting symmetry. The intensity of sinusoidal loading can be expressed as

$$\hat{q}(X) = q_0 \sin\frac{\pi X}{a} \tag{25}$$

where q_0 represents the peak intensity of distributed load.

The elastic solution for composite laminates under cylindrical bending by Pagano (1969), and various other analytical and finite element solutions available in the literature have been used for proper comparison of the results obtained through the present analysis. The present formulation was found to yield converging results for simply supported beams, by considering 6 to 8 elements along X-direction of the beam, along with the discretization of the thickness (i.e., along Z-direction) in

Table 1 Boundary conditions

Boundary conditions for cross-ply laminated beams (simple support condition)		
Edge	B.C. on displacement field	B.C. on stress field
i) At $X = 0$	$w = 0$	–
ii) At $X = a/2$	$u = 0$	$\tau_{xz} = 0$
iii) At $Z = t/2$	–	$\sigma_z = \hat{q}(X)$, $\tau_{xz} = 0$
iv) At $Z = -t/2$	–	$\sigma_z = \tau_{xz} = 0$
Boundary conditions for cross-ply laminated beams (fixed support condition)		
Edge	B.C. on displacement field	B.C. on stress field
i) At $X = 0$	$u = w = 0$	–
ii) At $X = a/2$	$u = 0$	$\tau_{xz} = 0$
ii) At $Z = t/2$	–	$\sigma_z = \hat{q}(X)$, $\tau_{xz} = 0$
iii) At $Z = -t/2$	–	$\sigma_z = \tau_{xz} = 0$

Table 2 Material properties of Graphite/Epoxy composite (Pagano 1969)

E_1 = 172.4 GPa (25×10^6 psi)	E_2 = 6.89 GPa (10^6 psi)
$G_{12} = G_{13}$ = 3.45 GPa (0.5×10^6 psi)	G_{23} = 1.378 GPa (0.2×10^6 psi)
$\nu_{12} = \nu_{13} = \nu_{23} = 0.25$	

such a way that the ratio L_x/L_z in an element was between 10 to 15. On the other hand, 6 to 8 elements along *X*-direction of the beam, along with the L_x/L_z ratio in an element to be between 40 to 50 was found suitable for convergence of the results while dealing with the fixed support condition.

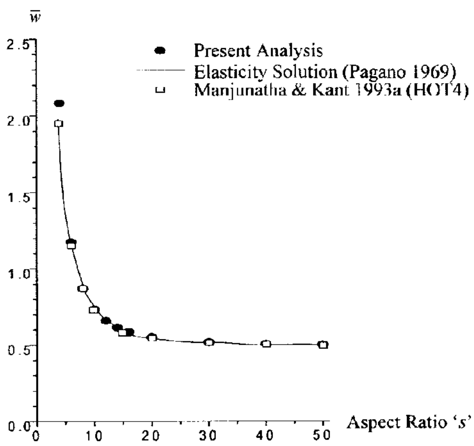
The following normalization factors have been used in all the numerical examples considered here for a comparison of the results.

$$\left.\begin{array}{c} \bar{Z}=\dfrac{Z}{t};\ \ \bar{u}=\dfrac{E_2 u(0,Z)}{tq_0};\ \ \bar{w}=\dfrac{100E_2 t^3 w(a/2,0)}{q_0 a^4}; \\ \bar{\sigma}_x=\dfrac{\sigma_x(a/2,0)}{q_0};\ \ \bar{\sigma}_z=\dfrac{\sigma_z(a/2,Z)}{q_0};\ \ \bar{\tau}_{xz}=\dfrac{\tau_{xz}(0,\,Z)}{q_0}; \\ \text{Aspect Ratio } s=\dfrac{a}{t}; \end{array}\right\} \qquad (26)$$

Illustrative numerical examples considered in the present work have been discussed next.

Example 1. A homogeneous orthotropic beam with fibers oriented along the *X*-direction, simply supported at the ends and subjected to sinusoidal loading on the top edge has been considered in numerical investigations. The variation of normalized transverse displacement ($\bar{w}$) with different aspect ratios, *s*, has been presented in Fig. 2a, whereas the variation of normalized in-plane normal stress ($\bar{\sigma}_x$) and transverse shear stress ($\bar{\tau}_{xz}$) through the thickness for aspect ratio, s = 4 have been presented in Figs. 2b and 2c, respectively. Results from elasticity analysis (Pagano 1969) and higher order theory (Manjunatha and Kant 1993a) have also been presented in all the figures. Results obtained through the present formulation show excellent matching with the elasticity solutions presented by (Pagano 1969), thereby validating the methodology adopted for the formulation of the present mixed finite element model.

Example 2. A two-layered cross-ply (0°/90°) laminated beam, simply supported at the ends and



(a)

Fig. 2 (a) Variation of normalized transverse displacement ($\bar{w}$) with aspect ratio *s* ($s = a/t$) at (a/2,0) for a simply supported homogeneous orthotropic beam under sinusoidal loading.

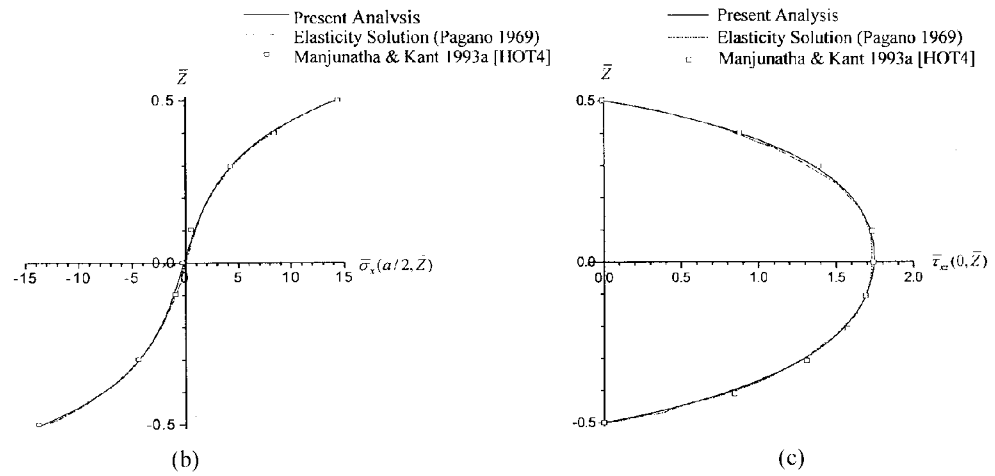


Fig. 2 (b, c). Through-thickness variation of normalized (b) in-plane normal stress ($\bar{\sigma}_x$) and (c) transverse shear stress ($\bar{\tau}_{xz}$) for a simply supported homogeneous orthotropic beam with s=4 under sinusoidal loading

Table 3 Comparison of maximum transverse displacement, in-plane stress and transverse shear stress for simply supported laminated beam under sinusoial loading (lamination scheme: 0°/90°)

Aspect Ratio 's'	Stress/ Displace-ment	Present Analysis	Elasticity Solution [Pagano 1969]	Engblom and Ochoa 1985	Wen-Jinn and Sun 1987	Lu and Liu 1992	Manjunatha & Kant 1993b Direct Integration		Shimpi and Ghugal 1999
							HOSTB7	HOSTB8	
4	$\bar{\sigma}_x$ (a/2,t/2)	3.8247	3.8359	2.9821	–	3.5714	3.7500	3.7680	3.9650
	$\bar{\sigma}_x$ (a/2,-t/2)	–29.9383	–29.9745	–27.7932	–	–30.0000	–26.9700	–26.9200	–30.2980
	$\bar{\tau}_{xz}$ (*max.*)	2.7500	2.7300	–	–	–	2.823	2.822	–
	$\bar{w}$ (a/2,0)	4.7636	4.7675	–	4.5950	4.7773	4.2839	4.2903	4.7431
10	$\bar{\sigma}_x$ (a/2,t/2)	19.7709	–	–	–	20.0000	19.6700	19.7300	19.8999
	$\bar{\sigma}_x$ (a/2,-t/2)	–175.995	–	–	–	175.000	–173.100	–173.000	–176.870
	$\bar{\tau}_{xz}$ (*max.*)	7.5670	–	–	–	–	7.2850	7.2840	–
	$\bar{w}$ (a/2,0)	2.9540	2.9568	–	2.9520	3.0000	2.8947	2.8965	2.9743

Note: – represents result not available

subjected to sinusoidal loading (Eq. 25) on the top edge has been considered in this example. Both layers have identical thickness. Table 3 presents a comparison of normalized maximum transverse displacement, in-plane stress and transverse shear stress for the simply supported beam with $s = 4$, 10. Elastic solution by (Pagano 1969), and analytical/FE results obtained by Engblom and Ochoa (1985), Wen-Jinn and Sun (1987), Lu and Liu (1992), Manjunatha and Kant (1993b), and Simpi and Ghugal (1999) have been employed for comparison. The results from the present analysis have been found to match extreamly well with the elastic solutions as compared to any other solutions presented by various researchers. Variation of the normalized transverse displacement ($\bar{w}$) against the aspect ratio 's' has been presented in Fig. 3a. The variation of normalized in-plane normal stress ($\bar{\sigma}_x$), transverse normal and shear stresses ($\bar{\sigma}_z$ and $\bar{\tau}_{xz}$) and in-plane displacement ($\bar{u}$) through the

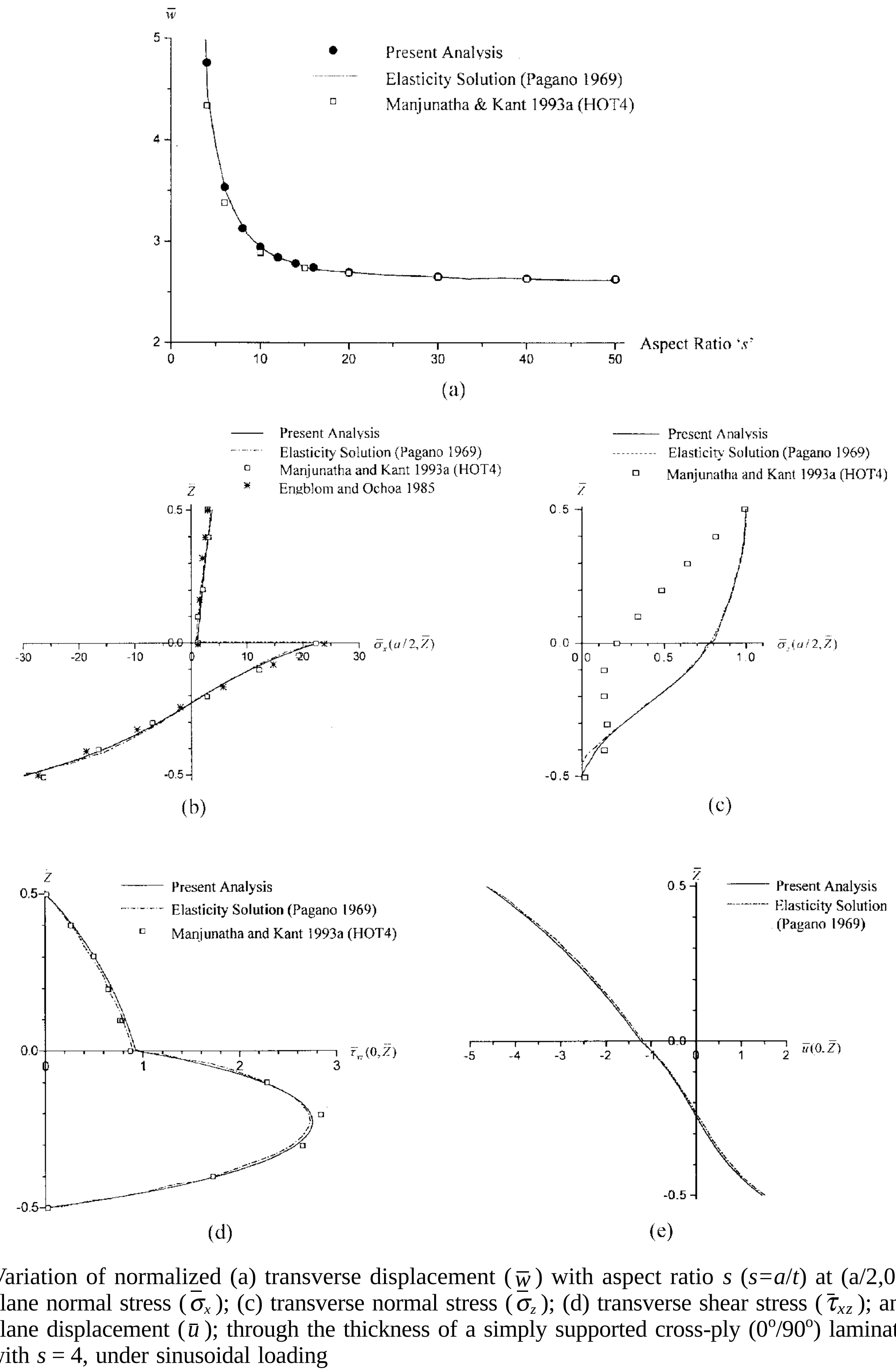


Fig. 3 Variation of normalized (a) transverse displacement ($\bar{w}$) with aspect ratio s ($s=a/t$) at (a/2,0); (b) in-plane normal stress ($\bar{\sigma}_x$); (c) transverse normal stress ($\bar{\sigma}_z$); (d) transverse shear stress ($\bar{\tau}_{xz}$); and (e) in-plane displacement ($\bar{u}$); through the thickness of a simply supported cross-ply ($0^o/90^o$) laminated beam with s = 4, under sinusoidal loading

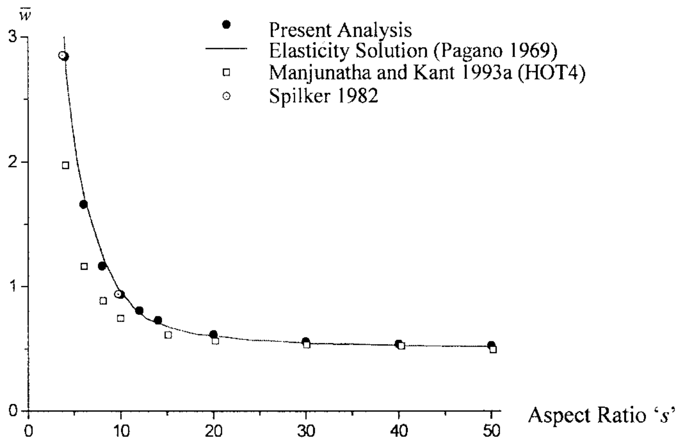
Table 4 Comparison of maximum transverse displacement, in-plane stress and transverse shear stress for simply supported laminated beam under sinusoial loading (lamination scheme: 0°/90°/0°)

Aspect Ratio '*s*'	Stress/ Displace-ment	Present Analysis	Elasticity Solution [Pagano 1969]	Lo *et al.* 1978	Spilker 1982	Engblom and Ochoa 1985	Toledano & Murakami 1987	Manjunatha & Kant 1993b (Direct Integration)	
								HOSTB3	HOSTB4
4	$\bar{\sigma}_x$ $(a/2,t/2)$	18.7523	18.6102	–	–	10.0090	–	13.8900	13.9400
	$\bar{\sigma}_x$ $(a/2,-t/2)$	-18.0497	-18.0023	–	–	-10.1081	–	-13.8900	-13.9600
	$\bar{\tau}_{xz}$ $(max.)$	1.6000	1.5974	1.5555	1.5636	1.7734	–	1.6630	1.6620
	$\bar{w}$ $(a/2,0)$	2.8400	2.8600	–	2.8410	–	2.8810	1.9705	1.9602
10	$\bar{\sigma}_x$ $(a/2,t/2)$	73.4453	73.6000	–	–	63.7344	–	67.4000	67.4100
	$\bar{\sigma}_x$ $(a/2,-t/2)$	-73.4042	-73.2000	–	–	-63.4025	–	-67.4000	-67.4200
	$\bar{\tau}_{xz}$ $(max.)$	4.2510	4.2346	–	4.5292	4.4590	–	4.3950	4.3950
	$\bar{w}$ $(a/2,0)$	0.9336	0.9568	–	0.9312	-	–	0.7491	0.7479

Note: '–' represents result not available

thickness of the beam with $s = 4$, on the other hand, have been shown in Figs. 3b through 3e. Almost all the results have been found to match very well with the elastic solutions given by Pagano (1969). The variation of transverse normal stress ($\bar{\sigma}_z$) in Fig. 3c clearly shows that the equivalent single layer based displacement formulation by Manjunatha and Kant (1993a) under-predicts the stress, whereas the present model yields results matching the elastic solution. Thus, it can be concluded that the present formulation is well able to handle such beams.

Example 3. A static analysis of a three-layered cross-ply (0°/90°/0°) laminated, simply supported beam, with equal thickness of each layer, has been considered in this example. The beam is subjected to sinusoidal loading (Eq. 25) applied at the top edge. The normalized maximum transverse displacement ($\bar{w}$), in-plane normal stress ($\bar{\sigma}_x$) and transverse shear stress ($\bar{\tau}_{xz}$) for the



(a)

Fig. 4 (a) Variation of normalized transverse displacement ($\bar{w}$) with aspect ratio s ($s = a/t$) at (a/2,0) for a simply supported cross-ply (0°/90°/0°) laminated beam under sinusoidal loading

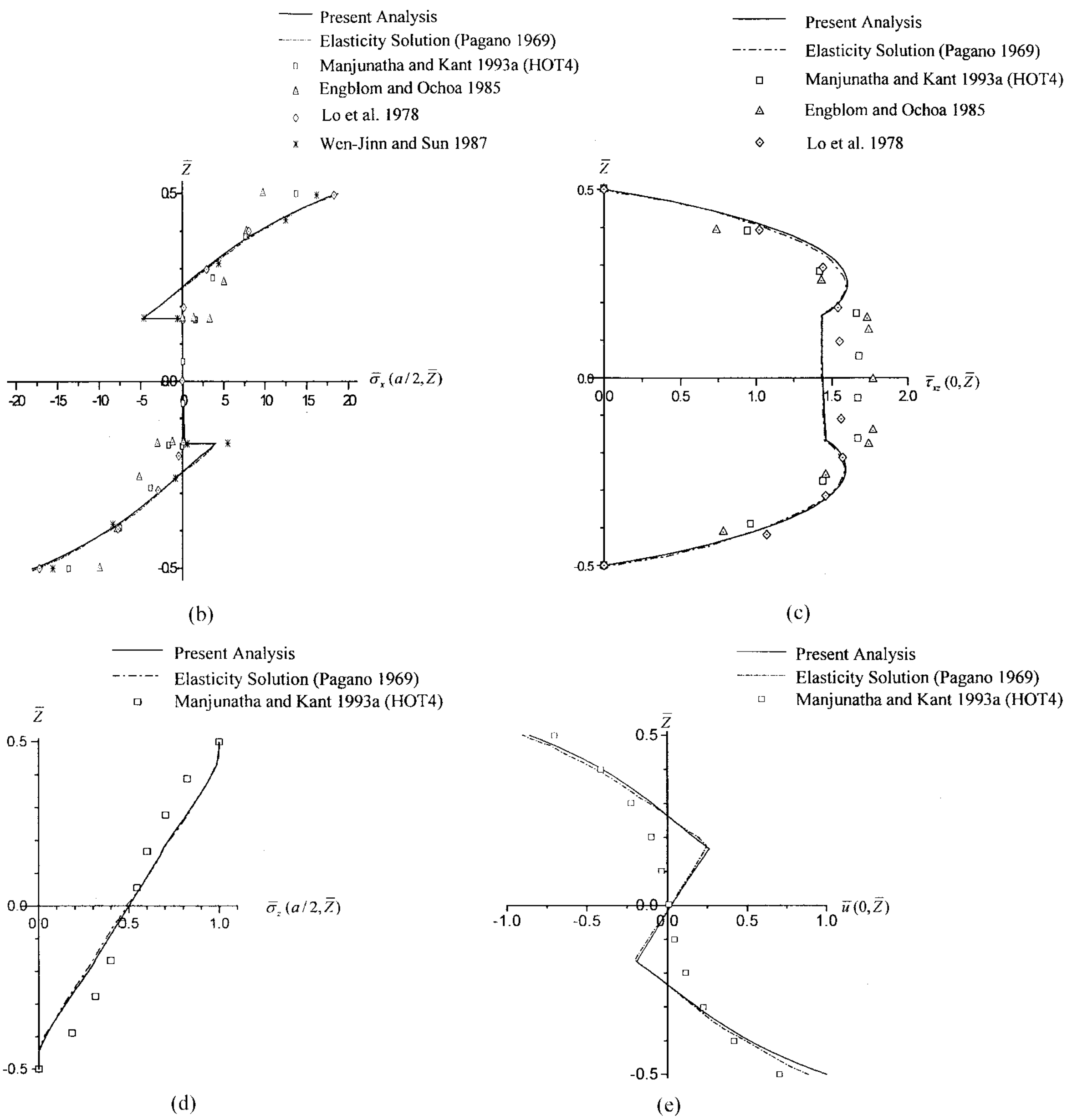
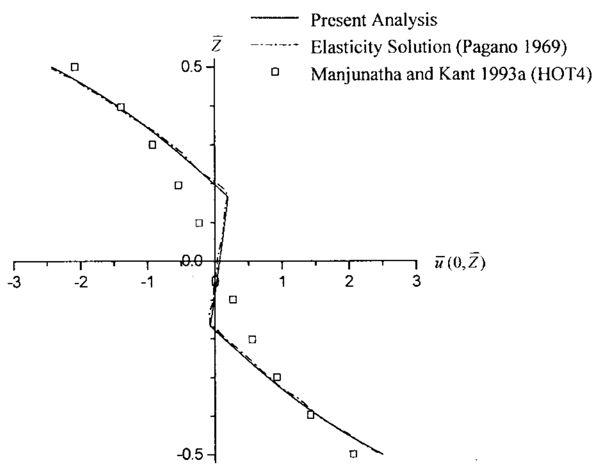
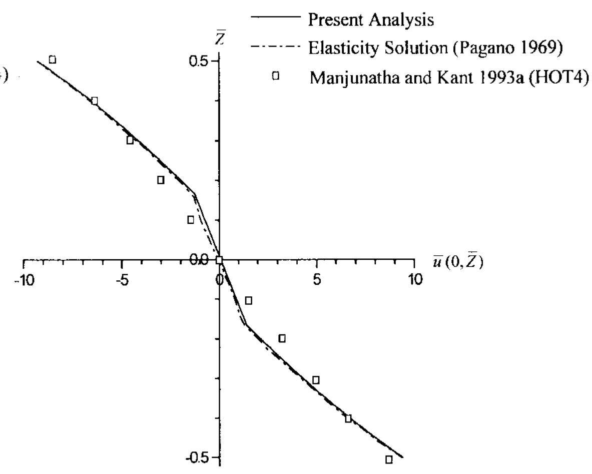


Fig. 4 Variation of normalized (b) in-plane normal stress ($\bar{\sigma}_x$); (c) transverse shear stress ($\bar{\tau}_{xz}$); (d) transverse normal stress ($\bar{\sigma}_z$); and (e) in-plane displacement ($\bar{u}$); through the thickness of a simply supported cross-ply (0°/90°/0°) laminated beam (s = 4) under sinusoidal loading

simply supported laminated beam (with s = 4, 10) have been presented in Table 4, where the results obtained through the present analysis have been compared with elastic solutions by Pagano (1969) and various analytical/FE solutions given by Lo *et al.* (1978), Spilker (1982), Engblom and Ochoa (1985), Toledano and Murakami (1987), and Manjunatha and Kant (1993b). The present results



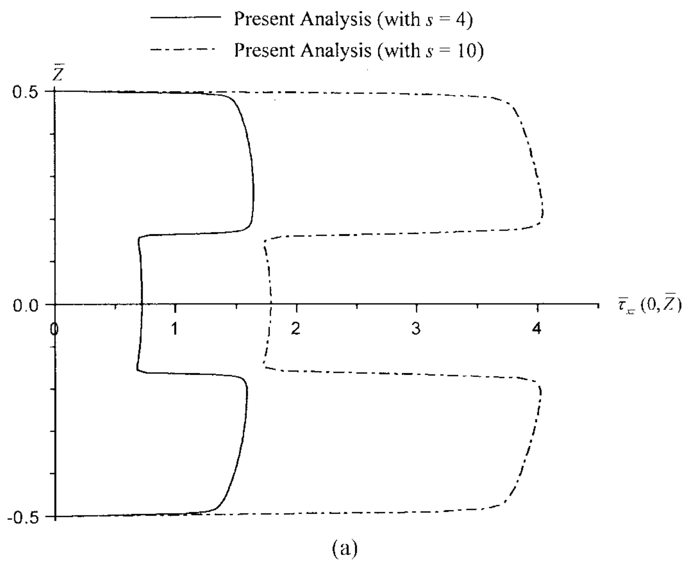
(f)



(g)

Fig. 4 (f, g) Variation of normalized (f) in-plane displacement ($\bar{u}$) with $s = 6$; and (g) in-plane displacement ($\bar{u}$) with $s = 10$; through the thickness of a simply supported cross-ply (0°/90°/0°) laminated beam under sinusoidal loading

show the closest match with the elastic solutions, which underlines the superiority of the present model over the previous models. The variation of normalized transverse displacement ($\bar{w}$) with different aspect ratios has been shown in Fig. 4a while the variation of normalized in-plane normal stress ($\bar{\sigma}_x$) and transverse normal and shear stresses ($\bar{\sigma}_z$ and $\bar{\tau}_{xz}$) through the thickness of the beam with $s = 4$, have been presented in Figs. 4b through 4d. Moreover, the variation of the in-plane displacement field ($\bar{u}$) through the thickness of laminated beams with aspect ratios 4, 6 and 10,



(a)

Fig. 5 (a) Variation of normalized (a) transverse shear stress ($\bar{\tau}_{xz}$) through the thickness of cross-ply (0°/90°/0°) laminated beams subjected to sinusoidal loading and supported on fixed supports at the ends

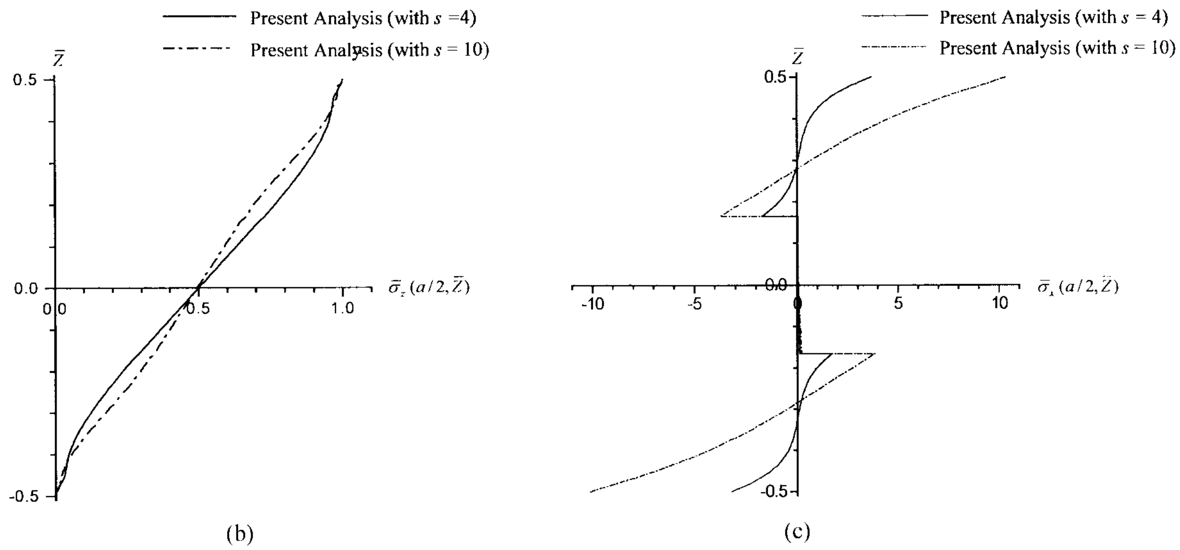


Fig. 5 (b, c) Variation of normalized (b) transverse normal stress ($\bar{\sigma}_z$); and (c) in-plane normal stress ($\bar{\sigma}_x$); through the thickness of cross-ply (0°/90°/0°) laminated beams subjected to sinusoidal loading and supported on fixed supports at the ends

respectively, have been presented in Figs. 4e, 4f and 4g. It is shown that the results from the present theory are in very good agreement with the elastic solution (Pagano 1969). It is also noted that the hybrid stress model of Wen-Jinn and Sun (1987) has yielded good results (Fig. 4b), but the displacement-based models (Lo *et al.* 1978, Engblom and Ochoa 1985, Manjunatha and Kant 1993a) have not been able to predict the in-plane and the transverse stresses (Figs. 4b, 4c, and 4d) as well as the variation of in-plane displacement field $\bar{u}$. This highlights the superiority of mixed/hybrid models over displacement-based models for the analysis of laminated structures.

Example 4. A fixed-supported cross-ply (0°/90°/0°) laminated beam with equal thickness of each layer has been considered to provide new benchmark solutions and also to show the ability of the present formulation to handle the problems with high stress gradients. It has been observed that 6 to 8 elements along *X*-direction of the beam, along with the L_x/L_z ratio in an element to be between 40 to 50 was found suitable for convergence of the results. Variations of transverse shear and normal stresses ($\bar{\tau}_{xz}$ and $\bar{\sigma}_z$), and in-plane normal stress ($\bar{\sigma}_x$) through the thickness of the beams (with s = 4 and 10) have been presented in Figs. 5(a-c). A very high stress gradient at the layer interface can be observed under the fixed-support condition for the transverse shear stress ($\bar{\tau}_{xz}$) (Fig. 5a).

4. Conclusions

A novel methodology for the formulation of a mixed finite element model has been proposed in this paper. It ensures the fundamental elasticity relationship between stress, strain and displacement fields within the elastic continuum. To the authors' knowledge, it is the first of its kind of mixed finite element model, which has been developed by using minimum potential energy principle. Because a layer-wise mixed finite element formulation has been developed with the transverse stress components as the primary variables, both the primary requirements of the continuity of transverse stress and layer-wise continuity of displacement fields, have been satisfied. Excellent agreement of the results with the elasticity solutions suggests that the formulation is well able to deal with laminated composite beam problems under different support and loading conditions. One such example of a cross-ply beam under fixed support condition depicts the ability of the formulation to deal with high stress gradients in the variation of transverse stresses.

Acknowledgements

Constructive comments of the reviewers are gratefully acknowledged.

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Notation

1, 2, 3	: Local co-ordinate system of lamina (material principal directions)
X, Z	: Global co-ordinate system of beam
x, z	: Local co-ordinate system of plane stress element
ξ, η	: x/L_x and z/L_z respectively
α	: Inclination of fibers of a lamina with the positive direction of laminate X-axis, measured in anticlockwise direction
a, b, t	: Length, width and depth (thickness) of beam, respectively
N	: Number of layers with different material properties, which constitute a beam
$2L_x$, $2L_z$	: Length and depth of an element, respectively
E_1, E_2, E_3	: Young's modulii of lamina in the material principal directions
G_{12}, G_{13}, G_{23}	: Shear modulii of lamina in the three orthogonal planes
D_{ij}	: Coefficients of constitutive matrix with reference to beam's and element's reference axes (i.e., X, Z and x, z)
u, w	: Displacement components along element's reference direction x and z respectively at a point (x, z)
σ_x, σ_z, τ_{xz}	: Components of stress at a point with reference to element's reference axes
$\in_x$, $\in_z$, γ_{xz}	: Components of strain at a point with reference to element's reference axes
Π^e, Π	: Total potential energy of an element and beam respectively
$[K]^e$, $[K]$	: Element property matrix and global property matrix respectively
$\{f\}^e$, $[F]$	: Element influence vector and global influence vector respectively
$\{q\}$, $\{Q\}$	: Element's and global nodal degrees-of-freedom vector respectively
$\hat{q}$, q_0	: Distributed transverse load and intensity of transverse loading
s	: Aspect ratio (i.e., a/t)
$\bar{Z}$	: Non-dimensionlized thickness of beam/ plate (i.e., Z/t)
$\bar{u}$, $\bar{w}$	: Non-dimensionlized displacement fields
$\bar{\sigma}_x$, $\bar{\sigma}_z$, $\bar{\tau}_{xz}$	: Non-dimensionlized stress components