*Structural Engineering and Mechanics, Vol. 12, No. 2 (2001) 215-230* 215
DOI: http://dx.doi.org/10.12989/sem.2001.12.2.215

# Vibrations of long repetitive structures by a double scale asymptotic method

E.M. Daya† and M. Potier-Ferry‡

*Laboratoire de Physique et Mécanique des Matériaux*
*UMR CNRS 7554, Institut Supérieur de Génie Mécanique et Productique*
*Université de Metz, Ile du Saulcy, 57045 Metz cedex 01, France*

**Abstract.** In this paper, an asymptotic two-scale method is developed for solving vibration problem of long periodic structures. Such eigenmodes appear as a slow modulations of a periodic one. For those, the present method splits the vibration problem into two small problems at each order. The first one is a periodic problem and is posed on a few basic cells. The second is an amplitude equation to be satisfied by the envelope of the eigenmode. In this way, one can avoid the discretisation of the whole structure. Applying the Floquet method, the boundary conditions of the global problem are determined for any order of the asymptotic expansions.

**Key words:** vibrations; periodic structures; asymptotic two-scale method; boundary layer; Floquet theory.

## 1. Introduction

Long flexible structures exhibiting a repetitive form are used in many domains. Aircraft fuselages, space and ship structures, parabolic antennas, etc. are typical examples of long periodic structures, the interior of which consists of identical cells connected in identical manner along the structure. If the periodic system has only one type of coupling between adjacent cells, it is called mono-coupled system, otherwise it is called multi-coupled. Such structures have a large number of vibration eigenfrequencies which are closely located in well separated bands (Fig. 3). Thus the periodic structure has the characteristic of a filter (Brillouin 1953, Mead 1970, Touratier 1986, and Lee *et al.* 1992). By using finite element method to compute the eigenmodes of a lattice structure, we note (Fig. 1) that some eigenmodes in the first packet are global modes, but other modes appear as slow modulations of a periodic one.

Many interesting investigations have been developed to simplify the analysis of these structures. So, periodic structure can be analysed by the homogeneization technique. This approach is to replace the actual lattice by a substitute continuum model that is equivalent to the original structure in some sense, by considering the constitutive relation, the strain energy, and/or kinetic energy (Noor and Anderson 1979, Moreau 1996). This method gives good approximations of the global modes. However, the general analysis of modulated modes has to take into account local deformation of eigenmodes in the homogeneization method (Hubert Palencia 1989, Daya 1994).

† Assistant Professor
‡ Professor

The wave propagation approach can be adopted to analyse the periodic structures (Mead 1970, Lee and Ke 1992). Generally, the wave propagation method is applied most simply to infinite or semi-infinite periodic structures.

Some analytical methods such as the difference equation approach (Lin 1962), the transfer matrix method (Lin and McDaniel 1969) or the propagation constant method (Gupta 1970) have been applied to study periodic structures. Moreover, for such complex structures (i.e. truss structures), it is more difficult to find analytical solutions. This is due to the existence of many types of wave motions. So, it has been suggested that a Timonshenko beam element (Flotow 1986) or other continuum elements (Noor and Anderson 1970) could be substituted for such a complicated structure cell in the analysis. However, the error involved in such a substitution may be quite significant.

Other contributions combined finite elements and transfer matrix method, mainly for cyclically symmetric structures or structures with a simple geometry (McDaniel and Chang 1980, Anderson *et al*. 1986, William 1986). For instance, vibrations of a mono-coupled system are very well described by considering the eigenvalues of the transfer matrix (Faulkner and Hongo 1985, Young and Lin 1989). The transfer matrix method has also been applied to study localization phenomena in nearly periodic systems. This phenomenon, whose dynamical effect could be dramatic, is governed by the existence of some irregularities in the periodic system. By using the Lyapunov exponents of a transfer matrix, Castanier and Pierre (1995) measure the degree of wave localization in multi-coupled nearly periodic systems.

In this paper, we present a two scale asymptotic method to account for a packet of modulated modes. In other words, this way is equivalent to a homogenization technique, but it is applied to another class of modes (i.e. modulated modes in Fig. 1). This method has the advantage to be easily applicable in the nonlinear range or for two-dimensional or three-dimensional arrays, what is well known in cellular stability analysis (Wesfreid and Zaleski 1984). Here we limit ourselves to evaluate the possibilities of the method, in the framework of a linear modal analysis and for a representative example of a one-dimensional periodic array. The boundary layer analysis is done with the help of the Floquet theory (Jordan and Smith 1987).

## 2. Representative example with modulated modes

In this section, we consider a periodic structure with modulated modes. This is a beam related to springs that are discretely and periodically distributed, as pictured in Fig. 2. In this paper, we limit our selves to bending motion of beam.

### *2.1 Equation of motion*

The formulation of natural vibrations of the representative structure is obtained from the virtual

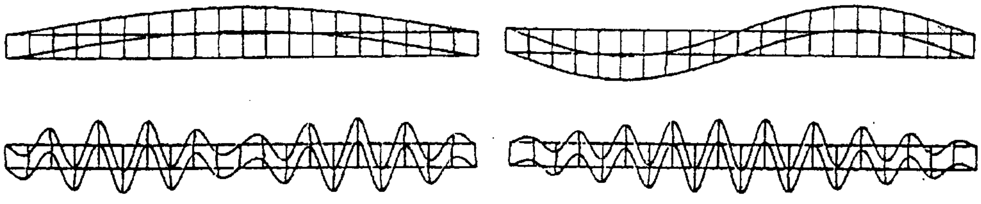

Fig. 1 First, second, 19th and 20th modes of a lattice structure with 20 basic cells

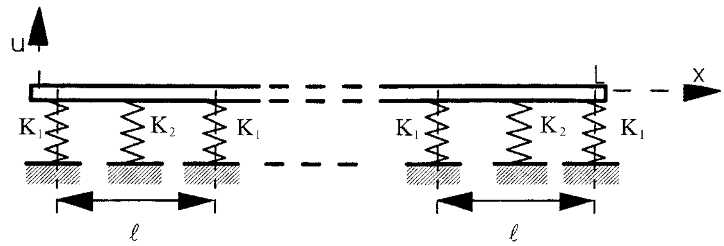


Fig. 2 The representative structure

work equation:

$$\begin{cases} \int_0^L \mathrm{EI}\dfrac{\partial^2 U(x,t)}{\partial x^2}\dfrac{\partial^2 \delta U(x,t)}{\partial x^2}dx+\sum_{i=0}^{N} K_1 U(il,t)\delta U(il,t) \\ +\sum_{i=1}^{N-1} K_2 U\left(il-\dfrac{l}{2},t\right)\delta U\left(il-\dfrac{l}{2},t\right)+\int_0^L \rho S\dfrac{\partial^2 U(x,t)}{\partial t^2}\delta U(x,t)dx=0 \end{cases} \tag{1}$$

where $U(x, t)$ is the deflection
$\delta U(x, t)$: the virtual deflection
$x$ : co-ordinate along the beam
$K_1$: stiffness of spring at the extremities of a cell
$K_2$: stiffness of spring at the middle of a cell
$E$: Young's modulus
$I$: second moment of cross section area
$S$: cross section area of beam
$\rho$: mass density

The total length is denoted by $L$, the length of the basic cell is $l$ and $N=L/l$ is the number of basic cells.

The displacement can be expressed as harmonic time function as given below:

$$U(x, t)=u(x)e^{i\omega t} \tag{2}$$

Where $i=\sqrt{-1}$ and $\omega$ is the natural frequency.

Combining Eqs. (1) and (2) and using two integrations by parts, one can rewrite the natural vibrations problem in the following form:

$$\begin{cases} \dfrac{d^4 u}{dx^4}=\lambda u \\ \left[\left[\dfrac{d^3 u}{dx^3}(il)\right]\right]=-k_1 u(il) \\ \left[\left[\dfrac{d^3 u}{dx^3}\left(il-\dfrac{l}{2}\right)\right]\right]=-k_2 u\left(il-\dfrac{l}{2}\right) \\ i=1\ldots N-1 \end{cases} \tag{3a-3c}$$

where $\lambda=\rho S\omega^2/EI$, $k_j=K_j/EI$ ( $j$=1, 2) and [[.]] is the usual jump symbol.

Thus, the Eqs. (3) represent an eigenvalues problem in which $\lambda$ and $u(x)$ are the unknowns. Of course, these equations have to be completed by boundary conditions at the ends of the beam.

As said in section 1, the eigenvalue problem (3) can be analyzed by different ways: direct computation by finite element method, homogeneization technique, wave propagation approach and transfer matrix or Floquet method.

The first method becomes very expensive when the structure is very large. The second one is only available for eigenmodes without local deformation (as first and second modes in Fig. 1). The other methods cannot be applied easily for periodic structures with complex geometry of cells.

The goal of this paper is to develop an asymptotic two-scale method which splits the Eq. (3) into two problems. The first one is posed on a few basic cells with periodicity conditions. The second is an amplitude equation to be satisfied by the envelope of the eigenmode. The product of the solutions of these problems yield approximation of solutions of Eq. (3). As in most analysis with the help of the asymptotic two-scale method, there exist boundary layers (Hubert and Palencia 1989). In the present method, the Floquet theory is used to satisfy accurately boundary conditions of the amplitude equation.

Now, we analyze direct computations of Eq. (3) by finite element method, we define the periodic modes of the representative structure and we recall the Floquet analysis of modes packet.

### *2.2 Direct computation of Eq. (3)*

For example, the eigenvalues spectrum and the corresponding eigenmodes have been computed by the finite element method for a clamped beam with 20 cells. The eigenvalues spectrum is plotted in Fig. 3 and some eigenmodes of the first packet are pictured in Fig. 4.

After a look at the computed modes, the following comments can be given:

a) the number of modes in one packet is exactly equal to the number of cells.

b) the first, second and third modes appear as slow modulations of the same periodic mode (periodicity=2$l$).

c) the last mode (mode number 20) in the first packet is periodic (periodicity=$l$).

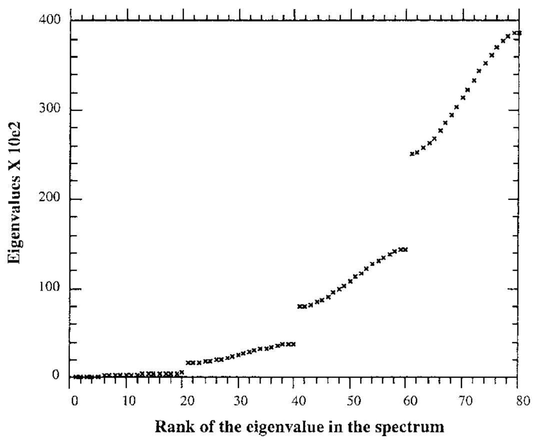


Fig. 3 Packets of eigenvalues obtained by direct numerical simulation (L=20, $l$=1, $k_1$=100000, $k_2$=0, clamped beam)

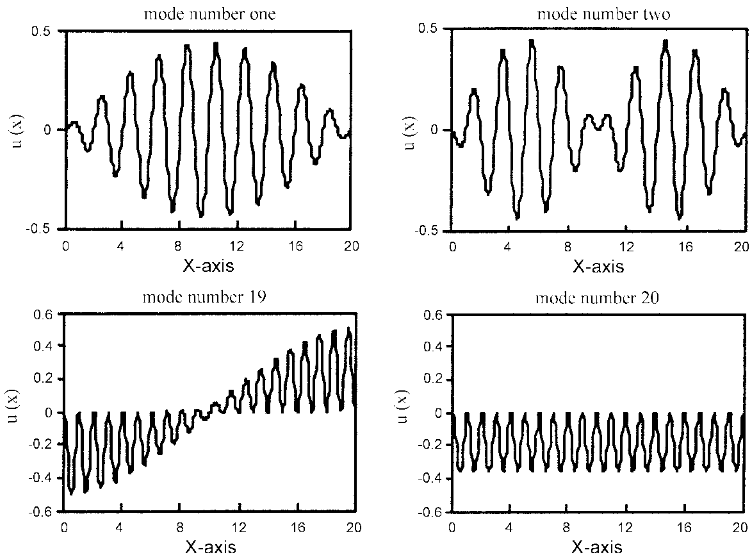


Fig. 4 Modulated shape of the eigenmodes at the beginning and at the end of the first packet (L=20, $l$=1, $k_1$=100000, $k_2$=0, clamped beam)

d) the mode number 19 appears as a slow modulation of the last periodic mode.
Similar remarks can be done for the others packets.

### 2.3 Definition of periodic modes

The periodic modes are defined as the eigenmodes of a vibration problem that is posed on a few basic cells with periodicity conditions. The periodic problem of the representative structure can be written in following form, where $p$ is an integer:

$$\begin{cases} \dfrac{d^4}{dx^4}u=\lambda u \\ \left[\left[\dfrac{d^3}{dx^3}u(il)\right]\right]=-k_1 u(il) \\ \left[\left[\dfrac{d^3}{dx^3}u\left(il-\dfrac{l}{2}\right)\right]\right]=-k_2 u\left(il-\dfrac{l}{2}\right) \\ u(x) \text{ is periodic of period } pl \end{cases} \qquad i=0...p-1 \tag{4}$$

In Fig. 5, we present the shape of some periodic modes of the representative structure. One denotes that the two first (respectively last) modes in Fig. 4 appear as slow modulations of the first (respectively second) mode that is plotted in Fig. 5.

### 2.4 Evolution of floquet multipliers in a mode packet

In this section, we recall the evolution of Floquet exponents of transfer matrix when the eigenvalue $\lambda$ takes its value in a mode packet. Let us rewrite the differential Eq. (3a) and the associated jump conditions (3b), (3c) in the form of a first order differential system:

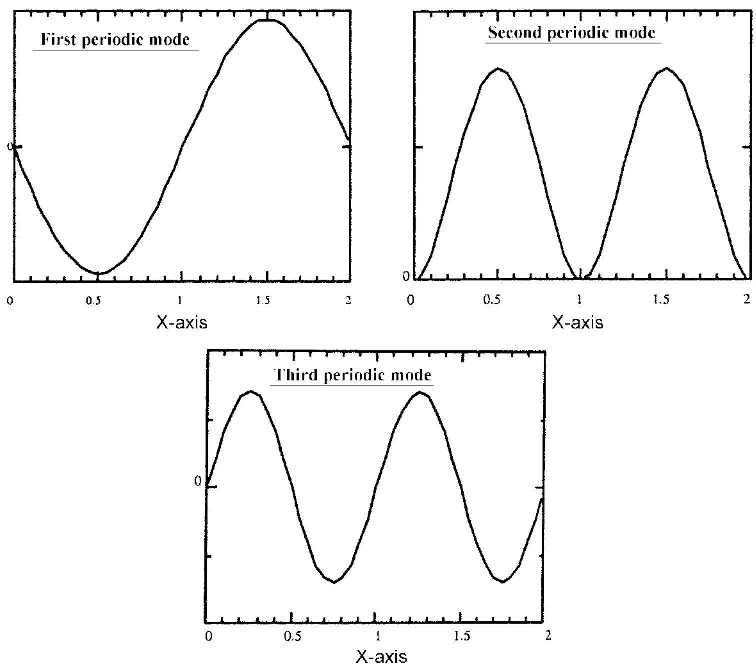


Fig. 5 The three lowest periodic modes. The first periodic mode is of period $2l$ and corresponds to the beginning of the first packet; The second periodic mode is of period $l$ and corresponds to the end of the first packet; The third periodic mode is of period $l$ and corresponds to the beginning of the second packet. ($l$=1, $k_1$=100000, $k_2$=0)

$$\frac{d}{dx}\boldsymbol{u}(x)=\Im(\lambda, x)\boldsymbol{u}(x) \tag{5}$$

where $\Im(\lambda, x)$ is $4 \times 4$ matrix whose coefficients are periodic with respect to $x$, and where the new unknown is $\boldsymbol{u}(x)=(u(x),u''(x),u'(x),u'''(x)) \in \mathrm{IR}^4$. In the case of Eq. (3), the operator $\Im(\lambda, x)$ involves Dirac function because of the jump conditions (3b) (3c). The Floquet theory, for instance (Lee and Ke 1992, William 1986), permits to characterise the general solution of the system (5), whatever initial and boundary conditions may be. First one defines the fundamental matrix $\Phi(x) \in \mathsf{L}(\mathrm{IR}^4,\mathrm{IR}^4)$ as follows:

$$\begin{cases} \frac{d}{dx}\Phi(x)=\Im(\lambda, x)\Phi(x) \\ \Phi(\mathbf{0})=\text{Identity} \end{cases}$$

The Floquet multipliers $\sigma$ are the eigenvalues of the fundamental matrix $\Phi(l)$. One can note that $\Phi(l)$ is also the transfer matrix of the periodic structure. A special solution of the system (5) is obviously associated with the corresponding eigenvectors. This special solution is decreasing if $|\sigma|$ is lower than 1, and it is increasing if $|\sigma|$ is greater than 1. If $\sigma$ is equal to 1 (respectively −1) the solution is $l$-periodic (respectively $2l$-periodic) and if $\sigma$ is equal to $p$th root of 1 the solution is $pl$-periodic.

In the case of Eqs. (3), the fundamental matrix involves, first the general solution $\Gamma(\lambda)$ of the differential Eq. (3a), second a matrix $\Psi(k)$ corresponding to shear forces discontinuity

$$\psi(k)=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -k & 0 & 0 & 1 \end{bmatrix}$$

So the fundamental matrix is given by:

$$\Phi(1)=\Psi\left(\frac{k_1}{2}\right)\Gamma(\lambda)\Psi(k_2)\Gamma(\lambda)\Psi\left(\frac{k_1}{2}\right)$$

The previously described properties of the vibration spectrum of long repetitive structure can be related to the evolution of the Floquet exponents as a function of the eigenvalue $\lambda$. This analysis is briefly recalled in what follows and it is well known in wave propagation domain (Brillouin 1953, Touratier 1986, Lee and Ke 1992).

There are generally four different Floquet multipliers, that are denoted by $\sigma_1$, $\sigma_2$, $\sigma_3$, $\sigma_4$. Because the system (5) has real coefficients, we get conjugate pairs of Floquet multipliers $\sigma$, $\bar{\sigma}$ except if $\sigma$ is real. Because the basic cell has symmetry properties, we get conjugate pairs of Floquet multipliers $\sigma$, $1/\sigma$ except if $\sigma=\pm 1$.

We have analysed the evolution of the Floquet multiplier in the case of the representative example, especially when $\lambda$ crosses $\lambda_0$ the smallest eigenvalue of the periodic problems. If $\lambda$ is smaller than $\lambda_0$, the four multipliers are real, as pictured in the Fig. 7. Two of them are inside the unit circle, which corresponds to two types of localised solutions near the left boundary $x$=0. Because of the symmetry, the two others are outside the unit circle. When $\lambda$ reaches $\lambda_0$, two eigenvalues $\sigma_2$, $\sigma_3$ become equal to $-1$. As it is generic, this eigenvalue is not semi-simple, which means that the corresponding eigenspace is of dimension 1 only. The eigenvector corresponds to the periodic solution of period 2$l$, that has been got numerically in section 2.2. The two eigenvalues $\sigma_4$, $\sigma_1$ characterise respectively an increasing and a decreasing solution of the system. When $\lambda$ exceeds $\lambda_0$, $\sigma_2$ becomes a complex number with unit modulus and $\sigma_3$ is its conjugate.

If one increases $\lambda$, the eigenvalues $\sigma_2=\bar{\sigma}_3$ move along the unit circle. They reach the value $\sigma_2=\bar{\sigma}_3=1$ for a $\lambda$ that corresponds exactly to the end of first mode packet and to the second eigenvalue of the periodic mode (Fig. 5).

The same evolution of Floquet multipliers in the other packets can be pointed out.

## 3. An asymptotic two-scale method

In this section, an asymptotic two-scale method is developed to determine the solutions of Eq. (3).

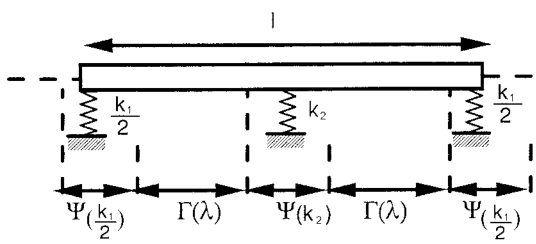


Fig. 6 Basic cell for fundamental matrix definition

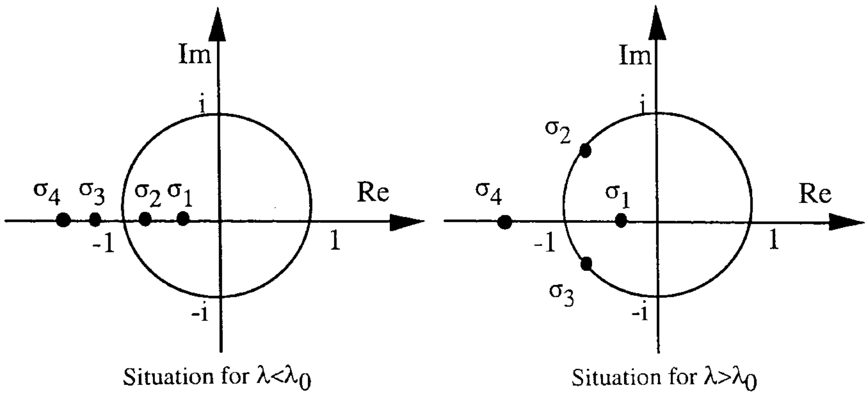


Fig. 7 Evolution of Floquet multipliers

### *3.1 Principle of the method*

As described in section 2, some modes at the beginning and at the end of the packet appears as slow modulations of periodic modes. An asymptotic two-scale method is developed from this observation. The principle of this method can be described as follows. A small parameter $\eta$ is introduced, for instance as the ratio between the length of the basic cell and the length of the complete structure. As it is classical (Hubert and Palencia 1989), $u$ and $\lambda$ are sought as integro-power series with respect to $\eta$:

$$\begin{cases} u = \sum_{i=0} \eta^i u_i(x,X) \\ \lambda = \sum_{i=0} \eta^i \lambda_i \end{cases} \tag{6a}$$

where $x$ is a local variable and $X=\eta x$ is a global variable that can describe the slow variation of the eigenmodes (Fig. 4). Furthermore, we supposed that the mode $u(x,X)$ is locally periodic, i.e. periodic with respect to the local variable $x$. The period is exactly the same as in section 2.2. Thus, if we consider the asymptotic development from the beginning (respectively end) of the first packet, the period is equal to $2l$ (respectively $l$) and $\lambda_0$ is the eigenvalue corresponding to the first (respectively second) periodic mode that is plotted in Fig. 5. After insertion of the series (6a) into (3) and application of the classical rules of the two-scale expansion method (Sanchez and Palencia 1989):

$$\frac{d}{dx} = \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial X} \tag{6b}$$

one gets linear equations to be satisfied by $u_i$ that are brought together in the following manner:

<u>First order problem</u>

$$\begin{cases} \dfrac{\partial^4}{\partial x^4}u_0 - \lambda_0 u_0 = 0 \\ \left[\!\left[ \dfrac{\partial^3}{\partial x^3}u_0(il) \right]\!\right] = -k_1 u_0(il) & i=1\dots p-1 \\ \left[\!\left[ \dfrac{\partial^3}{\partial x^3}u_0\left(il-\dfrac{l}{2}\right) \right]\!\right] = -k_2 u_0\left(il-\dfrac{l}{2}\right) \end{cases} \tag{7}$$

<u>Second order problem</u>

$$\begin{cases} \dfrac{\partial^4}{\partial x^4}u_1 - \lambda_0 u_1 = -4\dfrac{\partial^4}{\partial X \partial x^3}u_0 + \lambda_1 u_0 \\ \left[\!\left[ \dfrac{\partial^3}{\partial x^3}u_1(il) \right]\!\right] = -k_1 u_1(il) & i=1\dots p-1 \\ \left[\!\left[ \dfrac{\partial^3}{\partial x^3}u_1\left(il-\dfrac{l}{2}\right) \right]\!\right] = -k_2 u_1\left(il-\dfrac{l}{2}\right) \end{cases} \tag{8}$$

<u>Third order problem</u>

$$\begin{cases} \dfrac{\partial^4}{\partial x^4}u_2 - \lambda_0 u_2 = -6\dfrac{\partial^4}{\partial X^2 \partial x^2}u_0 + \lambda_2 u_0 - 4\dfrac{\partial^4}{\partial X \partial x^3}u_1 \\ \left[\!\left[ \dfrac{\partial^3}{\partial x^3}u_2(il) \right]\!\right] = -k_1 u_2(il) & i=1\dots p-1 \\ \left[\!\left[ \dfrac{\partial^3}{\partial x^3}u_2\left(il-\dfrac{l}{2}\right) \right]\!\right] = -k_2 u_1\left(il-\dfrac{l}{2}\right) \end{cases} \tag{9}$$

<u>(*k*+1)th order problem</u>

$$\begin{cases} \dfrac{\partial^4}{\partial x^4}u_k - \lambda_0 u_k = -4\dfrac{\partial^4}{\partial X \partial x^3}u_{k-3} - 6\dfrac{\partial^4}{\partial X^2 \partial x^2}u_{k-2} - 4\dfrac{\partial^4}{\partial X^3 \partial x}u_{k-1} - \dfrac{\partial^4}{\partial X^3}u_{k-4} + \displaystyle\sum_{j=0}^{k-1} \lambda_{k-j} u_j \\ \left[\!\left[ \dfrac{\partial^3}{\partial x^3}u_k(il) \right]\!\right] = -k_1 u_k(il) & i=1\dots p-1 \\ \left[\!\left[ \dfrac{\partial^3}{\partial x^3}u_k\left(il-\dfrac{l}{2}\right) \right]\!\right] = -k_2 u_k\left(il-\dfrac{l}{2}\right) \end{cases} \tag{10}$$

At each order, the problem has to be completed by periodicity conditions. The general solution of the first order problem (7) can be written in the following form:

$$u_0(x,X) = A_0(X) w_0(x)$$

where $w_0(x)$ is the periodic mode defined in section 2.2 and $A_0(X)$ is an amplitude function that can account for slow modulations of the modes. The other problems (8), (9), (10) have a solution if and only if the right-hand sides $F_i$ of these equations satisfy the following solvability condition:

$$\langle F_i,\, w_0 \rangle = 0 \tag{11}$$

where $\langle f(x),\, g(x)\rangle = \int_0^{pl} f(x)g(x)dx$

By this solvability condition, we shall find the equations governing the global evolution of the problem. In the following part, this treatment is presented in the case of the amplitude $A_0(X)$.

### 3.2 Amplitude equation (Global problem) and corresponding boundary conditions

In this part, we establish differential equations and boundary conditions in order to define the first or the last eigenpairs of a packet. In the case of a truss, the global modes, that correspond to the lowest eigenvalues can be obtained by an equivalent homogeneous beam (Noor and Anderson 1989, Noor and Nemoth 1980, John *et al.* 1985, Flotow 1986, Noor 1988, Caillerie *et al.* 1989). In what follows, we present a similar homogeneous model for the modulated modes (Figs. 5).

#### 3.2.1 Amplitude differential equation

The general solution of the Eqs. (7) and (8) can be written as :

$$\begin{cases} u_0(x,X) = A_0(X)w_0(x) \\ u_1(x,X) = A_1(X)w_0(x) + \dfrac{dA_0}{dX}w_1(x) \end{cases} \tag{12}$$

where $w_1$ is the $pl$-periodic function that satisfies:

$$\begin{cases} \dfrac{d^4}{dx^4}w_1(X,x) - \lambda_0 w_1(X,x) = -\dfrac{4d^3}{dx^3}w_0 \\ \left[\left[\dfrac{d^3}{dx^3}w_1(il)\right]\right] = -k_1 w_1(il) \qquad\qquad i=1\ldots p-1 \\ \left[\left[\dfrac{d^3}{dx^3}w_1\left(il-\dfrac{l}{2}\right)\right]\right] = -k_2 w_1\left(il-\dfrac{l}{2}\right) \end{cases}$$

By using the solvability condition for the problem at the second and third orders, we obtain $\lambda_1=0$ and an amplitude differential equation to be satisfied by the envelope $A_0(X)$. This equation is written in the form:

$$C\frac{d^2A_0}{dX^2} + \lambda_2 A_0 = 0 \tag{13}$$

where $C$ is a constant that is defined explicitly from the previously computed periodic functions (In the appendix, we present numerical method which permits us to obtain the periodic function $w_1$):

$$C = \frac{-6\langle w_0'', w_0\rangle - 4\langle w_1''', w_0\rangle}{\langle w_0, w_0\rangle}$$

The amplitude Eq. (13) is the sought "equivalent homogeneous model", that will permit us to characterise the eigenpairs close to $\lambda_0$. It is also an eigenvalue problem, where the unknowns are $A_0(X)$ and $\lambda_2$. To get a well-posed problem, it is necessary to complete the Eq. (13) by the boundary conditions.

### *3.2.2 How to get the boundary conditions for the global problem*

For example, let us consider a clamped beam. These conditions are written as:

$$\begin{cases} u(0)=\dfrac{du(0)}{dx}=0 \\ u(L)=\dfrac{du(L)}{dx}=0 \end{cases} \tag{14}$$

The two scale expansion method yields a rule for the expansion derivative (6b). This rule allows us to write the displacement and its derivative with respect to $x$ in the following asymptotic form.

$$\begin{cases} u=A_0(X)w_0(x)+\eta u_1(x,X)+\theta(\eta^2) \\ \dfrac{du}{dx}=A_0(X)\dfrac{dw_0}{dx}+\eta\left(\dfrac{dA_0}{dX}w_0(x)+\dfrac{d}{dx}u_1(x,X)\right)+\theta(\eta^2) \end{cases} \tag{15}$$

The boundary conditions for $A_0(X)$ are to be deduced from the exact boundary conditions (14) and from the estimate (15). By considering the shape of the mode in Fig. 4, one expects those boundary conditions to be different at the beginning and at the end of the packet. Two cases are considered.

a - Suppose that $w_0$ and $dw_0/dx$ are not simultaneously equal to zero at the ends of the beam, as it is the case for the first periodic mode in Fig. 5. Thus the boundary conditions for $A_0(X)$ are:

$$A_0(0)=0 \text{ and } A_0(L\eta)=0$$

b - Suppose that $w_0$ and $dw_0/dx$ are simultaneously equal to zero at the ends of the beam, as the second periodic mode in Fig. 5. Thus the boundary conditions for $A_0(X)$ are:

$$\frac{dA_0}{dX}(0)=0 \quad \text{and} \quad \frac{dA_0}{dX}(L\eta)=0$$

One remarks that these boundary conditions correspond to the observed shapes in Fig. 4. A more detailed analysis of the boundary conditions will be presented in section 3.3. Similar analysis can be applied to other boundary conditions.

### *3.2.3 A second order estimate of the spectrum*

The global problem can be completely solved with account of the latter boundary conditions. The eigenvalues of Eq. (3) can be obtained in an asymptotic form. For example, we present the amplitude equation corresponding to the first modes of the first packet for clamped beam:

$$\begin{cases} C\dfrac{d^2A_0}{dX}+\lambda_2 A_0=0 \\ A_0(0)=A_0(L\eta)=0 \end{cases}$$

Then, the solutions are in form:

$$\begin{cases} A_0(X)=a\ \sin\left(\dfrac{n\pi}{L\eta}X\right) \\ a \text{ is an arbitrary constant} \\ \lambda_2=C\dfrac{n^2\pi^2}{L^2\eta^2} \quad n=1,2,\ldots \end{cases}$$

Using the formula (6a) we obtain from an eigenvalue $\lambda_0$ (first eigenvalue of periodic problem defined in section 2.2), an asymptotic estimate of the beginning of the first packet in the form:

$$\begin{cases} \lambda(n)=\lambda_0+C\dfrac{n^2\pi^2}{L^2}+\theta(\eta^2) \\ n=1,2,... \end{cases} \tag{16}$$

At this stage of the asymptotic two scale analysis, one can note that the method permits to generate from a periodic mode *a* infinite number of eigenvalues of the initial structure.

### 3.3 Boundary layer analysis and fourth order estimate of the spectrum

From the previous analysis, one can guess the existence of boundary layers. The initial Eq. (3) is a fourth order one, that requires four boundary conditions. The global evolution is governed by the second order differential Eq. (13) that requires only two boundary conditions. Because of this lack of boundary conditions, it is not possible to satisfy the initial boundary conditions at any order, except by correcting the assumed expansions (6a) to account for boundary layers solutions. This correction is made from the boundary layer solution $w_{\mathrm{loc}}$ that we can compute by Floquet theory. Two asymptotic expansions have to be done, the first one being valid is close to $x$=0 and the second one is close to $x$=$L$. In the first case $x$=0, $w_{\mathrm{loc}}$ is the decreasing function that corresponds to the first Floquet exponent $\sigma_1$ ($|\sigma_1|<1$), see Fig. 7. In the second case $x$=$L$, $w_{\mathrm{loc}}$ is the increasing function that corresponds to the fourth Floquet exponent $\sigma_4$ ($|\sigma_4|>1$). This solution is introduced in the asymptotic expansion close to one end for instance $x$=0:

$$u=u_0(x,X)+\sum_{i=1}\eta^i(u_i(x,X)+\alpha_i w_{\mathrm{loc}}(x)) \tag{17}$$

Using the expansions (17) and from the boundary conditions (15), we get a simple system of two equations where the unknowns $\alpha_i$ are $A_i(0)$, what permits us to find exactly the global boundary conditions at any order. For example, this system for $x$=0 is in the following form.

$$\begin{cases} A_1(0)w_0(0)+\alpha_1 w_{\mathrm{loc}}(0)=-\dfrac{dA_0}{dX}(0)w_1(0) \\ A_1(0)\dfrac{dw_0}{dx}(0)+\alpha_1\dfrac{dw_{\mathrm{loc}}}{dx}(0)=-\dfrac{dA_0}{dX}(0)\left[w_0(0)+\dfrac{dw_1}{dx}(0)\right] \end{cases}$$

So, the boundary conditions for $A_1(X)$ at $x$=0 is:

$$A_1(0)=C_1\frac{dA_0}{dX}(0)$$

where $C_1$ is a constant defined from $w_0(0)$, $\dfrac{dw_0}{dx}(0)$, $w_1(0)$, $\dfrac{dw_1}{dx}(0)$, $w_{\mathrm{loc}}(0)$ and

$$\frac{dw_{\mathrm{loc}}}{dx}(0)$$

We can obtain a similar boundary condition at $x$=$L$. By this way, the boundary conditions for the envelope can be treated accurately at any order. Thus, we have got the following asymptotic eigenvalue spectrum at order $\eta^4$ for the case presented in section 4.2 (for the details, we refer to Daya 1994):

$$\begin{cases} \lambda(n)=\lambda_0+C\dfrac{n^2\pi^2}{L^2}+D\dfrac{n^4\pi^4}{L^4}+\theta(\eta^4) \\ n=1,2,\ \ldots \end{cases} \tag{18}$$

where $C$ and $D$ are determined from the periodic modes and from the function $w_{\mathrm{loc}}$. Similar analysis can be applied to find the asymptotic estimate of the spectrum at superior order.

## 4. Numerical results and discussion

### *4.1 Results at order 2*

In this section, we present comparison between the presented method at order 2 and the direct computation of Eq. (3) by finite element method. The solutions of periodic problems have also been obtained by finite element method.

In the following tables, we present the first nine eigenvalues of the first packet for different cases that mainly differ from one by another by the beam length, the values of $k_1$, $k_2$ and the boundary conditions.

From the previous results, one can note that the asymptotic two scale method at order two describes quite perfectly the eigenvalues near $\lambda_0$. The corresponding eigenmodes are not reported here, but we have observed that the first term of the expansions (6a) is a good approximation of the exact mode. Nevertheless, this method is not able to evaluate the number of modes in the packet. Indeed the mode number $n$ is not limited except by the assumption of two different length scales, which would lead to "n/N small".

Similar results have been obtained for the last nine eigenvalues of the first packet. The corresponding values at order 4 are presented in the following.

### *4.2 Results at fourth order and asymptotic estimate of the whole first packet*

In the following table, we present the first and last nine eigenvalues obtained at fourth order of

Table 1 Results for clamped beam ($k_1$=100000, $k_2$=10, $l$=10, $L$=200, $C$=2.368)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Proposed method | 0.1564 | 0.1582 | 0.1611 | 0.1652 | 0.1705 | 0.1769 | 0.1845 | 0.1932 | 0.2032 |
| Direct computation | 0.1564 | 0.1580 | 0.1606 | 0.1643 | 0.1691 | 0.1749 | 0.1817 | 0.1896 | 0.1985 |

Table 2 Results for clamped beam ($k_1$=100000, $k_2$=10, $l$=10, $L$=300, $C$=2.368)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Proposed method | 0.1559 | 0.1561 | 0.1569 | 0.1582 | 0.1600 | 0.1623 | 0.1655 | 0.1685 | 0.1725 |
| Direct computation | 0.1559 | 0.1561 | 0.1568 | 0.1580 | 0.1596 | 0.1617 | 0.1643 | 0.1674 | 0.1709 |

Table 3 Results for simply supported beam ($k_1$=100000, $k_2$=0, $l$=1, $L$=23, $C$=59.194)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Proposed method | 97.41 | 98.51 | 101.83 | 107.35 | 115.08 | 125.03 | 137.18 | 151.54 | 168.11 |
| Direct computation | 97.41 | 98.47 | 101.60 | 107.00 | 114.40 | 123.90 | 135.60 | 149.50 | 165.40 |

Table 4 Results for clamped-simply supported beam ($k_1$=100000, $k_2$=0, $l$=1, $L$=30, $C$=59.194)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Proposed method | 97.56 | 98.86 | 101.46 | 105.35 | 110.55 | 117.04 | 124.83 | 133.92 | 144.31 |
| Direct computation | 97.56 | 98.81 | 101.30 | 105.00 | 110.00 | 116.30 | 123.80 | 132.50 | 142.50 |

Table 5 First nine eigenvalues $\lambda$ ($k_1$=100000, $k_2$=0, $l$=1, $L$=20, $C$=59.194, $D$=−1.013)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Proposed method | 98.87 | 103.19 | 110.40 | 120.39 | 133.04 | 148.21 | 165.72 | 185.36 | 206.90 |
| Direct computation | 98.88 | 103.00 | 110.00 | 119.90 | 132.50 | 148.00 | 166.30 | 187.30 | 211.10 |

Table 6 Last nine eigenvalues $\lambda$ ($k_1$=100000, $k_2$=0, $l$=1, $L$=20, $C$=−209.94, $D$=63.73)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Proposed method | 326.70 | 338.63 | 363.10 | 394.04 | 426.35 | 455.82 | 479.20 | 494.15 | 499.3 |
| Direct computation | 297.70 | 329.4 | 362.28 | 396.00 | 427.70 | 456.10 | 478.90 | 493.80 | 498.9 |

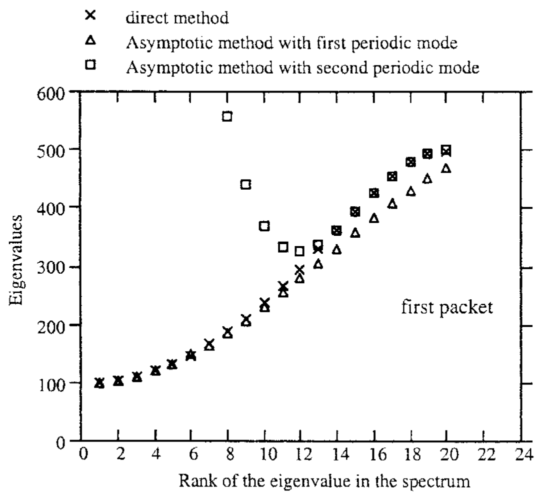


Fig. 8 First mode packet ($L$=20, $l$=1, $k_1$=100000, $k_2$=0, clamped beam)

the asymptotic method (formula 18) for clamped beam.

In Fig. 8, we present the eigenvalues of the first packet obtained by the two analyses. We can note that the two-scale method gives a good approximation of the beginning and end of the eigenvalues packet. Nevertheless, it is not possible to get the whole packet by the present method, because the assumption of two different length scales is not satisfied in the middle of the packet. This analysis also does not permit to predict the number of the modes that are in the packet.

## 5. Conclusions

In this paper, we have presented an asymptotic two scale method, which splits the original problem into periodic problems and global ones. The periodic problems are modal problems on few basic cells with periodicity conditions. The global problems are amplitude differential equations to be satisfied by the envelope of the mode. The boundary conditions can be treated accurately, the boundary layer solutions being derived from the Floquet theory. The larger the number of cells is, the more efficient the asymptotic method is. Unfortunately it is not possible to predict in that manner the number of modes that are in the packet. Moreover, it is not possible to get the whole packet. To achieve these goals it is necessary to account for the interaction between two periodic modes, a first approach having been proposed in Daya (1994).

The presented method could be interesting to simplify more complex problems like coupling between local and overall modes, non-linear resonance effects and general dynamic problems or to study two-dimensional or three-dimensional arrays, that are encountered in many applications (Bourgeois 1997).

## References

Anderson, M.S. and William, F.W. (1986), “Natural vibration and buckling of general periodic lattice structures”, *AIAA J.*, **24**(1), 163-169.

Bourgeois, S. (1997), “Modélisation numérique des panneaux structuraux legers”, Thesis of University of Aix-MarseilleII-France.

Brillouin, L. (1953), *Wave Propagation in Periodic Structures*, Dover, New York.

Caillerie, D. (1989), “Thin elastic and periodic plates”, *Math. Meth. in Appl. Sci*., **6**, 159-191.

Caillerie, D., Trompette, P. and Verna, P. (1989), “Homogeneization of periodic trusses”, Congress IASS, Madrid.

Castanier, M.P. and Pierre, C. (1995), “Lyapunov exponents and localization phenomena in multi-coupled nearly periodic systems”, *J. Sound and Vibration* **183**(3), 493-515.

Damil, N. and Potier-Ferry, M. (1990), “A new method to compute perturbed bifurcation: Application to the buckling of imperfect elastic structures”, *Int. J. of Eng. Sci.*, **28**(9), 943-957.

Daya, E.M. (1994), “Vibration et stabilité des longues structures flexibles à forme répétitive”, Thesis of University of Metz-France.

Faulkner, M.G. and Hongo, D.P. (1985), “Free vibrations of a mono-coupled periodic system”, *J. Sound and Vibration*, **99**(1), 29-42.

Flotow, A.H. von (1986), “Disturbance propagation in structural networks”, *J. Sound and Vibration*, **106**(3), 99-118.

John, O.W., Su, Z.W. and Feng, C.C. (1985), “Equivalent continuum representation of structures composed of repeated elements”, *AIAA J.* **23**(10), 1564-1569.

Jordan, D.W. and Smith, P. (1987), “Non-linear ordinary differential equations”, 2nd edn., Clarendon Press, Oxford.

Lee, S.Y. and Ke, H.Y. (1992), “Flexural wave propagation in an elastic beam with periodic structure”, *J. Appl. Mech.*, **57**, 779-783.

Lin, Y.L. (1962), “Free vibration of continous beam on elastic supports”, *Int. J. Mech. Sci.*, **4**, 409-423.

Lin, Y.L. and McDaniel, T.J. (1969), “Dynamics of Beam-type periodic structures”, *J. Eng. Industry*, **Nov**, 1133-1141.

McDaniel, T.J. and Chang, K.J. (1980), “Dynamics of rotationnally periodic large space structures”, *J. Sound and Vibration*, **66**, 351-368.

Mead, D.J. (1970), “Free wave propagation in periodically support infinite beam”, *J. Sound and Vibration*, **11**(2), 81-197.

Moreau, G. (1996), Thesis of university of Grenoble-France. Homogénéisation de structures discrètes en élasticité et en incrémental. Application aux modélisations continues linéaires et *non-linéaires de treillis quasi-périodiques*.

Noor, A.K. (1988), “Continuum modeling for repetitive lattice structures”, *Appl. Mech. Rev.*, **41**(7), 285-296.

Noor, A.K. and Andersen, C.M. (1979), “Analysis of beamlike lattice trusses”, *Comput. Meth. Appl. Mech. and Eng.*, **20**, 53-70.

Noor, A.K. and Anderson, M.S. (1970), “Continuum models for beam and plate-like lattice structures”, *American Inst. of Aeronaut. and Astronaut. J.* **16**, 1219-1228.

Noor, A.K. and Nemeth, M.P. (1980), “Analysis of spatial beamlike lattices with rigid joints”, *Computer. Meth. Appl. Mech. and Eng.*, **24**, 35-59.

Sanchez Hubert, J. and Sanchez Palencia, E. (1989), *Vibrating and Coupling Continuous Systems, Asymptotic* *methods*, Springer-Verlag, Berlin.

Sen Gupta, G. (1970), "Natural flexural waves and the normal modes of periodically-supported beams and plates", *J. Sound and Vibration* **13**(1), 89-101.

Touratier, M. (1986), "Floquet waves in a body with slender periodic structure", *Wave Motion*, **8**, 485-495.

Von Flotow, A.H. (1986), "Disturbance propagation in structural networks", *J. Sound and Vibration*, **106**(3), 433-450.

Wesfreid, J.E. and Zaleski, S. (1984), "Cellular structures in instability problems", Lecture Notes in Physics, Spring, Berlin.

William, F.W. (1986), "Exact eigenvalue calculations for structures with rotationally periodic substructures", *Int. J. Numer. Meth. Eng.*, **24**(4), 695-706.

Young, Y. and Lin, Y.L. (1989), "Propagation of decaying waves in periodic and piecewise periodic structures of finite length", *J. Sound and Vibration*, **129**(2), 99-118.

## Appendix A1

### *How to get the constant C*

After insertion of the general solutions (12) into the third problem (9) and using the solvability condition (11), we obtain the following equation:

$$-6\frac{d^2A_0}{dX^2}\langle w_0'', w_0\rangle - 4\frac{d^2A_0}{dX^2}\langle w_1''', w_0\rangle - 4\frac{dA_1}{dX}\langle w_0''', w_0\rangle + \lambda_2 A_0\langle w_0, w_0\rangle = 0$$

Because $w_0$ is *pl*-periodic function, we have the following property:

$$\langle w_0''', w_0\rangle = \int_0^{pl} w_0''' w_0 dx = 0$$

So, we get the amplitude differential equation to be satisfied by the envelope $A_0(X)$:

$$C\frac{d^2A_0}{dX^2} + \lambda_2 A_0 = 0$$

where $C = \dfrac{-6\langle w_0'', w_0\rangle - 4\langle w_1''', w_0\rangle}{\langle w_0, w_0\rangle}$

## Appendix A2

### *Numerical computation of the periodic functions $w_i$*

By using the finite element method, one finds that the nodal displacement vector $[w_i]$ of the periodic function $w_i$ satisfies an equation in the form:

$$\begin{cases} [A][w_i] = [F] \\ {}^t[w_i][w_0] \end{cases} \qquad i = 1, \ldots \tag{19}$$

where $[A] = [K] - \lambda_0[M]$

$[K]$ is the stiffness matrix of the basic cell

$[M]$ is the mass matrix of the basic cell

$[w_0]$ is the nodal displacement vector of the periodic problem (4)

$\lambda_0$ is the eigenvalue value of the problem (4)

$[F_i]$ depends only on the previously computed vectors $[w]$ and coefficients $\lambda_1$, $\lambda_2$, ....

We note that the matrix $[A]$ is singular. But, if we take into account the condition (19b) by means of a Lagrange multiplier $k$ as in Damil and Potier-Ferry (1990), we get the following equation that involves an invertible matrix

$$\begin{bmatrix} [A] & [w_0] \\ {}^t[w_0] & 0 \end{bmatrix} \begin{bmatrix} w_i \\ k \end{bmatrix} = \begin{bmatrix} F_i \\ 0 \end{bmatrix} \qquad i = 1, \ldots \tag{20}$$