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Stochastic finite element method homogenization of heat conduction problem in fiber composites

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Abstract. The main idea behind the paper is to present two alternative methods of homogenization of the heat conduction problem in composite materials, where the heat conductivity coefficients are assumed to be random variables. These two methods are the Monte-Carlo simulation (MCS) technique and the second order perturbation second probabilistic moment method, with its computational implementation known as the Stochastic Finite Element Method (SFEM). From the mathematical point of view, the deterministic homogenization method, being extended to probabilistic spaces, is based on the effective modules approach. Numerical results obtained in the paper allow to compare MCS against the SFEM and, on the other hand, to verify the sensitivity of effective heat conductivity probabilistic moments to the reinforcement ratio. These computational studies are provided in the range of up to fourth order probabilistic moments of effective conductivity coefficient and compared with probabilistic characteristics of the Voigt-Reuss bounds.

Key words: homogenization method; stochastic second order perturbation; stochastic finite element method; Monte-Carlo simulation; composites.

1. Introduction

Major problem with computational analysis of multicomponent (composite) media by the use of different discrete grid or non-grid methods is scale effect occurring in their structure (Christensen 1979, Furmański 1997, Sanchez-Palencia 1987, Schellekens 1992). The problem simplifies when composite considered appears to be periodic, which implies the existence of some geometrical cell (periodicity cell or representative volume element) that, due to geometric translation, can cover the whole structure. Considering the fact, that in most engineering problems the scale factor relating periodicity cell to the entire structure is very small, the discretization process can be very complicated. To solve this problem, the homogenization method is introduced, which makes it possible to replace original multicomponent composite with an equivalent medium, that can be characterized by homogeneous tensor of material properties. Thus, we can model composite structure without differentiating the regions belonging to different materials, which simplifies the meshing procedure significantly (Schellekens 1992).

Another engineering problem is how to use experimental data, described by mean values and standard deviations of material as well as physical parameters of the composite constituents, to evaluate the effective parameters and their probabilistic characteristics for the whole composite. Moreover, it is observed that in most composites microgeometry has generally a random character, which can be decisive for their overall macroscopic behavior. Considering these facts, the homo-

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genization method should contain the randomness of a composite occurring in its most constitutive parameters. The papers devoted to modern homogenization problems, especially in the context of thermal analysis, are collected and discussed in (Furmański 1997, Kamiński and Kleiber 2000).

The main goal of the paper is to formulate and solve the homogenization problem for heat conduction in *n*-component composites, where the heat conductivity coefficients are input random variables defined using the first two probabilistic moments. The micro as well as macro geometry of the composite is treated here as deterministic. However, it should be noted that the model presented allows generally to introduce some geometrical uncertainties, for instance in the form of randomly located microvoids within the components (Kamiński and Kleiber 1996). To calculate effective conductivity of the composite, the effective modules method is introduced. The temperature homogenization function, periodic on external boundary conditions of the periodicity cell, is applied there. The natural boundary conditions in the homogenization problem are taken in the form of differences between heat conductivity coefficients of neighboring components. To compute the expected values, variances and higher order probabilistic moments of effective conductivity, Monte-Carlo simulation technique (Hammersley and Handscomb 1964, Sab 1992, Hurtado and Barbat 1998) is used, consisting of random trials and statistical estimation procedure. The technique has been used widely in other mechanical and physical problems, including probabilistic approaches to the homogenization of elasticity tensor presented in Kamiński (1996). On the other hand, the second order perturbation and second probabilistic moment approach, with its computational implementation known as the stochastic finite element method, is used to compute the first two probabilistic moments. These two alternative methods are used to provide some comparative studies and, further, to eliminate their methodological and numerical disadvantages. The main computational problems that should be eliminated are: the accuracy of Monte-Carlo simulation technique (with respect to the random sample length) and, on the other hand, upper bounds on input random variables coefficients of variation (in the stochastic finite element approach), which cannot be greater than 0.1-0.2 (Kleiber and Hien 1992, Hien and Kleiber 1997). It should be mentioned that another possibility is to introduce the random uncertainty in the homogenization approach, using the stochastic spectral methods (Ghanem and Spanos 1997) or the stochastic weighted integral approach (Choi and Noh 1996).

All these numerical capabilities are introduced in the homogenization-oriented and FEM-based program MCCEFF (Kamiński 1996, 1999). Thanks to the implemented numerical algorithm, probabilistic moments of the effective heat conductivity coefficient are computed for the fiber-reinforced two-component composite. Moreover, the sensitivity of the probabilistic moments with respect to reinforcement ratio as well as to the total number of random trials performed (so-called numerical convergence verification) is verified numerically. Finally, it should be noticed that the mathematical homogenization method, together with its simulation or perturbation based probabilistic extensions, may be implemented numerically by the use of the Boundary Element Method (BEM) as well. Such an implementation may be most efficient in case of the composites, where some part of internal geometry (especially interfaces) of the RVE is defined probabilistically.

2. Periodic composite structure model

2.1 General remarks

The main problem presented in the paper is to find the probabilistic distribution of the effective heat conduction coefficient for the entire class of random composite structures, which are periodic

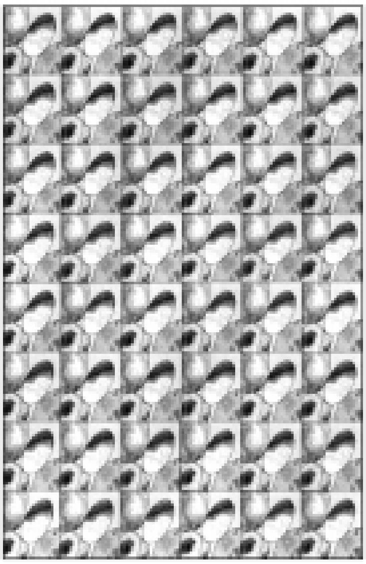

Fig. 1 Periodic composite structure Y

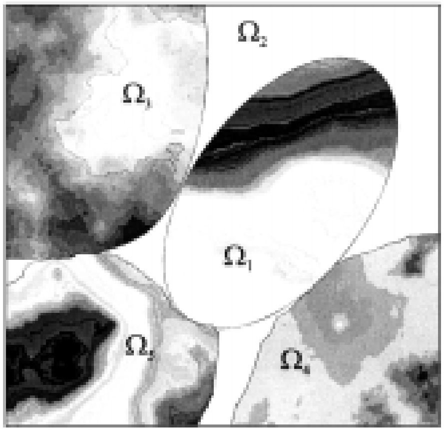

Fig. 2 The Representative volume element (RVE)

and built of n components. For this purpose, let us assume that the composite structure is periodic in the probabilistic sense, if for an additional ω belonging to a suitable probability space there exists such a translation of Ω, that covers the entire composite region. Next, let us assume that the section $Y \subset \Re^2$ of the composite with x_3=0 plane is constant along the x_3 axis. The section of the composite considered in the plane orthogonal to the longitudinal direction is shown in Figs. 1 and 2.

Further, let region Ω contain n perfectly bonded, coherent and disjoint subsets and let the scale between respective geometrical diameters of Ω and Y be described by a small parameter $\delta > 0$. Let $\partial\Omega$ denote external boundary of the Ω, while $\Gamma_{(a\text{-}1,a)}$ is the interface boundary between $\Omega_{a\text{-}1}$ and Ω_a regions. Moreover, let Ω_a for a=1, ..., n contain transversely isotropic material, where the heat conduction coefficient is the cut-off Gaussian random variable defined as follows:

$$0 < k(\boldsymbol{x};\ \omega) < \infty\,, \tag{1}$$

$$E\left[\, k(\boldsymbol{x};\ \omega)\,\right] = \chi_a(\boldsymbol{x}) E\left[k^{(a)}(\omega)\right];\ \ \boldsymbol{x} \in \Omega_a, \tag{2}$$

$$\mathrm{Var}\left(k(\boldsymbol{x};\ \omega)\,\right) = \chi_a(\boldsymbol{x}) \mathrm{Var}\left[k^{(a)}(\omega)\right];\ \ \boldsymbol{x} \in \Omega_a, \tag{3}$$

where $\chi_a(\boldsymbol{x})$ is a characteristic function given as follows:

$$\chi^{(a)}(\boldsymbol{x}) = \begin{cases} 1;\ \ \boldsymbol{x} \in \Omega_a \\ 0;\ \ \boldsymbol{x} \notin \Omega_a, \end{cases} \tag{4}$$

where the expected values $E\left[\, k(\boldsymbol{x};\ \omega)\,\right]$ and the variances $\mathrm{Var}\left(k(\boldsymbol{x};\ \omega)\,\right)$ are calculated by the use of the following classical definitions:

$$E\left[\, k^{(a)}(\boldsymbol{x};\ \omega)\,\right] = \int_{-\infty}^{+\infty} k^{(a)}(\boldsymbol{x};\ \omega) p\left(k^{(a)}(\boldsymbol{x};\ \omega)\right)\, dk(\boldsymbol{x};\ \omega)\,, \tag{5}$$

$$\mathrm{Var}\Big(k^{(a)}(\boldsymbol{x};\,\omega) \Big) = \int_{-\infty}^{+\infty} \Big(k^{(a)}(\boldsymbol{x};\,\omega) - E\Big[k^{(a)}(\boldsymbol{x};\,\omega) \Big] \Big)^2 \, p\Big(k^{(a)}(\boldsymbol{x};\,\omega) \Big)\, dk(\boldsymbol{x};\,\omega)\,. \tag{6}$$

It should be underlined that the zeroing of the off-diagonal terms of the heat conductivity covariance matrix follows the lack of experimental data describing probabilistic correlation of heat conductivities in the composite components. In the context of above definitions, the periodicity of the composite structure is in fact equivalent to the periodicity of probability density functions (PDFs) of the heat conductivity coefficients (or generally of any physical properties). Moreover, taking into account the assumption of Gaussian character of these variables, the periodicity of the first two probabilistic moments of heat conductivity coefficients is obtained.

Generally, the heat conduction problem consists in determining temperature field T by the use of the following differential equation:

$$(k_{ij}T_{,j})_{,i} - g = 0;\ \ x_i \in \Omega, \tag{7}$$

where k_{ij} is the heat conductivity tensor, while $g=g(T)$ is the rate of heat generated per unit volume and variable T denotes temperature field values. This equation should fulfill the following boundary conditions on the $\partial\Omega$:

1) temperature (essential) boundary conditions

$$T = \hat{T};\ \ x \in \partial\Omega_T, \tag{8}$$

2) heat flux (natural) boundary conditions

$$\frac{\partial T}{\partial n} = \hat{q};\ \ x \in \partial\Omega_q, \tag{9}$$

where $\partial\Omega_T \cup \partial\Omega_q = \partial\Omega$ and $\partial\Omega_T \cap \partial\Omega_q = \{\varnothing\}$. Further, let Ω contain n perfectly bonded, coherent and disjoint subregions Ω_a, fulfilling the following conditions:

$$\Omega = \bigcup_{\alpha=1}^{\nu} \Omega_a;\ \ \Omega_a \cap \Omega_b = \varnothing;\ \ a \neq b;\ \ 1 \leq a,\, b \leq n\,. \tag{10}$$

Considering the above, the formulation (7) may be rewritten as:

$$\chi_a \Big(\Big(k_{ij}^{(a)} T_{,j} \Big)_{,i} - g^{(a)} \Big) = 0;\ \boldsymbol{x} \in \Omega\,. \tag{11}$$

Multiplying Eq. (11) by the test function δT and integrating over the region Ω it is obtained

$$\int_\Omega \chi_a \Big(\Big(k_{ij}^{(a)} T_{,j} \Big)_{,i} - g^{(a)} \Big)\ \delta T\, d\Omega = 0;\ 1 \leq a \leq n;\ \ \boldsymbol{x} \in \Omega\,. \tag{12}$$

Introducing the boundary conditions (cf. Eqs. 8-9) and integrating by parts there holds

$$\int_\Omega \chi_a \Big(k_{ij}^{(a)} T_{,j} \delta T_{,i} - g^{(a)}\, \delta T \Big)\ d\Omega - \int_{\partial\Omega_q} \hat{q}\ \ \delta T\, d(\partial\Omega) = 0. \tag{13}$$

Eq. (13) is a transient formulation of virtual temperatures principle and is discretized by the use of a classical, deterministic (Pepper and Heinrich 1992, Krishnamoorthy 1994) as well as the stochastic finite element approach (Kleiber and Hien 1992, Hien and Kleiber 1997).

2.2 Second order second moment stochastic approach

The stochastic variational principle for linear transient heat transfer problems is formulated on the basis of Eq. (13) and is employed by the combination of the second-order perturbation technique and the stochastic second central moment analysis. To introduce the formulation, let us denote the input random variables vector of the heat conductivity as $\{k^r(x;\omega)\}$ and the ordinary as well as joint probability densities of its components by $g(k^r)$ and $g(k^r, k^s)$, respectively. Indices r, s run here over 1 to R, being the total number of input random vector components. The expected value of the vector $\{k^r(x;\omega)\}$ can be evaluated using expression (5), while the covariance - from Eq. (6). If, for example, the composite is built of two components, the vector of input random variables has two uncorrelated components in the form of $k_1(\omega)$ and $k_2(\omega)$.

Next, the Taylor series' stochastic expansion is used to rewrite variational principle of Eq. (13). Therefore

$$F(\boldsymbol{x};\omega)=F^0(\boldsymbol{x};\omega)+\sum_{n=1}^{N}\left\{\frac{\theta^n}{n!}F^{(n)}(\boldsymbol{x};\omega)\prod_{n=1}^{n}\Delta k^r(\omega)\right\}, \tag{14}$$

where θ denotes given small perturbation, $\theta\Delta k^r$ is the first order variation of Δk^r about its expected value $E[k^r]$ and $F^{(n)}(\boldsymbol{x};\omega)$ represents the n-order partial derivatives of the function considered with respect to the input random variables evaluated for their expected values. Considering the complexity of the model, the second order perturbation approach is used, where the random function $F(\boldsymbol{x};\omega)$ is extended as follows:

$$F(\boldsymbol{x};\omega)=F^0(\boldsymbol{x};\omega)+\theta F^{,r}(\boldsymbol{x};\omega)\Delta k^r+\frac{1}{2}\theta^2F^{,rs}(\boldsymbol{x};\omega)\Delta k^r\Delta k^s. \tag{15}$$

The first order variation of b_r about its expected value is equal to

$$\theta\Delta k^r=\delta k_r=\theta(k_r-k_r^0), \tag{16}$$

while the second variation is given as

$$\theta^2\Delta k^r\Delta k^s=\delta k_r\delta k_s=\theta^2(k_r-k_r^0)(k_s-k_s^0); \tag{17}$$

symbols $(.)^0$, $(.)^{,r}$ and $(.)^{,rs}$ represent zeroth, first and second order partial derivatives with respect to input random variables.

Due to the second-order perturbation technique, Eq. (15) is now inserted in the formulation (13). As the result, three sets of partial differential equations of 0th, 1st and 2nd order are obtained (Kleiber and Hien 1992, Hien and Kleiber 1997). Hence, there holds

· zeroth-order, one partial differential equation

$$\int_\Omega \chi_a\left(k_{ij}^{(a)0}T_{,j}^0\delta T_{,i}\right)d\Omega=\int_{\partial\Omega_q}\hat{q}^0\ \delta T\ d(\partial\Omega)+\int_\Omega\chi_a\ g^{(a)0}\delta T\ d\Omega, \tag{18}$$

· first-order, R partial differential equations

$$\int_\Omega \chi_a\left(k_{ij}^{(a)0}T_{,j}^{,r}\delta T_{,i}\right)d\Omega=\int_{\partial\Omega_q}\hat{q}^{,r}\ \delta T\ d(\partial\Omega)+\int_\Omega\chi_a\ g^{(a),r}\delta T\ d\Omega-\int_\Omega\chi_a\left(k_{ij}^{(a),r}T_{,j}^0\delta T_{,i}\right)d\Omega, \tag{19}$$

· second-order, one partial differential equation

$$\int_\Omega \chi_a\left(k_{ij}^{(a)0}T_{,j}^{(2)}\delta T_{,i}\right)d\Omega=\int_{\partial\Omega_q}\hat{q}^{(2)}\ \delta T\ d(\partial\Omega)$$
$$+\int_\Omega\chi_a\ g^{(a)(2)}\delta T\ d\Omega-\int_\Omega\left(k_{ij}^{(a),rs}T_{,j}^0+2k_{ij}^{(a),r}T_{,j}^{,s}\right)S^{rs}\ \delta T_{,i}\ d\Omega, \tag{20}$$

where symbols $(.)^{(2)}$ denote the product $(.)^{,rs}S^{rs}$. Finally, the expected values and covariances of the temperature field are derived as

$$E[T(\boldsymbol{x})]=T^0(\boldsymbol{x})+\frac{1}{2}T^{(2)}(\boldsymbol{x}), \tag{21}$$

$$\mathrm{Cov}\left(T(\boldsymbol{x}^{(1)}),\ T(\boldsymbol{x}^{(2)}\right)=T^{r}\left(\boldsymbol{x}^{(1)}\right)T^{s}\left(\boldsymbol{x}^{(2)}\right)S^{rs}, \tag{22}$$

which completes the second order and second moment approach to the heat conduction problem.

3. Homogenization problem

3.1 Deterministic formulation

To derive the effective conductivity coefficient $k^{(eff)}$, the Representative Volume Element (RVE) of the composite structure is indicated first. The RVE has minimal geometric dimensions and is so defined that, due to some homothety, can cover the entire composite structure (Sanchez-Palencia 1987, Rao *et al.* 1997). Next, it is assumed that the essential and natural boundary conditions on $\partial\Omega$ are periodic, which means that temperatures are equal on opposite boundaries, symmetrically to the horizontal or vertical axes provided through the center of the RVE, respectively. Finally, a periodic homogenization function χ is introduced, being some special temperature field. To determine the mathematical description of the problem, the heat conductivity coefficient of the composite is introduced as

$$k^{\delta}(\boldsymbol{x})=k(\boldsymbol{y}). \tag{23}$$

The linear heat conduction problem can be formulated as follows:

$$\begin{cases} k^{\delta}(\boldsymbol{x})T^{\delta}_{,i}-q^{\delta}_{i}=0 \\ q^{\delta}_{i,i}+f=0 \\ T^{\delta}=\hat{T};\ \boldsymbol{x}\in\ \partial\Omega_T \\ q^{\delta}=\hat{q};\ \boldsymbol{x}\in\ \partial\Omega_q. \end{cases} \tag{24}$$

The homogenization of heat conduction problem consists in deriving of such temperature field T^0, that is a limit of solution T^{δ} with $\delta\to 0$. It should be underlined here that there are some approaches, where δ is some small parameter different from 0 (Woźniak and Woźniak 1995), which enables direct introduction of the interrelations between macro and micro scales. To solve this problem, the following expansion is applied:

$$T^{\delta}(\boldsymbol{x})=\sum_{j=1}^{\infty}\delta^{j}T^{(j)}(\boldsymbol{x},\boldsymbol{y}), \tag{25}$$

$$q^{\delta}_{i}(\boldsymbol{x})=\sum_{j=-1}^{\infty}\delta^{j}q^{(j)}(\boldsymbol{x},\boldsymbol{y}), \tag{26}$$

where $T^{(j)}(\boldsymbol{x},\boldsymbol{y})$ and $q^{(j)}(\boldsymbol{x},\boldsymbol{y})$ are Ω-periodic functions. Introducing these expansions in governing equations of the problem it is obtained that

$$\sum_{j=-1}^{\infty} \delta^j q^{(j)} = \frac{1}{\delta} k \frac{\partial T^{(0)}}{\partial y_i} + k \sum_{j=0}^{\infty} \delta^j \left(\frac{\partial T^{(j)}}{\partial x_i} + \frac{\partial T^{(j+1)}}{\partial y_i} \right), \tag{27}$$

$$\frac{1}{\delta^2} \frac{\partial q_i^{-1}}{\partial y_i} + \sum_{j=0}^{\infty} \delta^{j-1} \left(\frac{\partial q_i^{j-1}}{\partial x_i} + \frac{\partial q_i^{j}}{\partial y_i} \right). \tag{28}$$

Therefore, the following pairs of equations are formed for the terms of the same order:

$$\begin{cases} \dfrac{\partial q_i^{-1}}{\partial y_i} = 0 \\ q_i^{-1}(\boldsymbol{x},\boldsymbol{y}) = k(\boldsymbol{y}) \dfrac{\partial T^{(0)}}{\partial x_i} \end{cases}, \tag{29}$$

$$\begin{cases} \dfrac{\partial q_i^{-1}}{\partial x_i} + \dfrac{\partial q_i^{0}}{\partial y_i} = 0 \\ q_i^{0}(\boldsymbol{x},\boldsymbol{y}) = k(\boldsymbol{y}) \left(\dfrac{\partial T^{(0)}}{\partial x_i} + \dfrac{\partial T^{(1)}}{\partial y_i} \right) \end{cases} \tag{30}$$

and

$$\begin{cases} \dfrac{\partial q_i^{0}}{\partial x_i} + \dfrac{\partial q_i^{1}}{\partial y_i} = 0 \\ q_i^{1}(\boldsymbol{x},\boldsymbol{y}) = k(\boldsymbol{y}) \left(\dfrac{\partial T^{(1)}}{\partial x_i} + \dfrac{\partial T^{(2)}}{\partial y_i} \right), \end{cases} \tag{31}$$

The solutions $T^{(0)}$, $T^{(1)}$ and $T^{(2)}$ are determined recurrently from above equations. The first result is

$$T^{(0)}(\boldsymbol{x},\boldsymbol{y}) = T(\boldsymbol{x}). \tag{32}$$

Taking into account periodicity conditions on $T^{(1)}$ and $q^{(0)}$, the first order terms are obtained as

$$\begin{cases} \dfrac{\partial q_i^{0}}{\partial y_i} = 0 \\ q_i^{0}(\boldsymbol{x},\boldsymbol{y}) = k(\boldsymbol{y}) \left(\dfrac{\partial T^{(0)}}{\partial x_i} + \dfrac{\partial T^{(1)}}{\partial y_i} \right), \end{cases} \tag{33}$$

Then, $T^{(1)}$ may be rewritten as

$$T^{(1)}(\boldsymbol{x},\boldsymbol{y}) = \chi_j(\boldsymbol{y}) \frac{\partial T^{0}}{\partial x_j} . \tag{34}$$

Hence, there holds

$$q_i^0(\boldsymbol{x},\boldsymbol{y})=k(\boldsymbol{y})\left(\delta_{ij}+\frac{\partial\chi_j}{\partial y_i}\right)\frac{\partial T^{(0)}}{\partial x_j}, \tag{35}$$

where $\chi_j(\boldsymbol{y})$ is a homogenization function being $\boldsymbol{\Omega}$-periodic and being a solution for the following partial differential equation, known as a local problem

$$\frac{\partial}{\partial y_i}\left[k(\boldsymbol{y})\left(\delta_{ij}+\frac{\partial\chi_j}{\partial y_i}\right)\right]=0;\ \ \boldsymbol{x}\in\Omega\,. \tag{36}$$

Applying averaging operator to the first expression of Eq. (24) one can obtain that

$$\frac{\partial\bar{q}_i^0}{\partial x_i}+f=0\ . \tag{37}$$

where

$$\bar{q}_i^0=\frac{1}{|\Omega|}\left[\int_\Omega k(\boldsymbol{y})\left(\delta_{ij}+\frac{\partial\chi_j}{\partial y_i}\right)d\Omega\right]\frac{\partial T^{(0)}}{\partial x_j}\,. \tag{38}$$

Thus, the effective conductivity tensor can be defined as

$$k_{ij}^{(eff)}=\frac{1}{|\Omega|}\int_\Omega k(\boldsymbol{y})\left(\delta_{ij}+\frac{\partial\chi_j}{\partial y_i}\right)d\Omega\,. \tag{39}$$

To derive the weak formulation of the homogenization problem, let us introduce the general variational formulation of the heat conduction problem in the following form:

$$\begin{cases}-\int_Y q_i^\delta\frac{\partial\xi}{\partial x_i}d\boldsymbol{x}+\int_Y f\xi d\boldsymbol{s}=0\\ \int_Y k^\delta(\boldsymbol{x})\frac{\partial T^\varepsilon}{\partial x_i}\frac{\partial\xi}{\partial x_i}d\boldsymbol{x}=\int_Y f\xi d\boldsymbol{s}\ ,\end{cases} \tag{40}$$

where Y denotes the region occupied by the composite considered. Next, let us introduce the following Sobolev space

$$H_0^1(Y)=\left\{\xi\in L^2(Y)\,|\,\frac{\partial\xi}{\partial x_i}\in L^2(Y),\ \ \xi\in 0:\ \ \boldsymbol{x}\in\partial Y\right\} \tag{41}$$

with the following norm:

$$\|\xi\|=\left[\int_Y\left(\frac{\partial\xi}{\partial x_i}\right)^2dx\right]^{1/2}. \tag{42}$$

Further, let us define bilinear and continuous form $a(T,\ \xi)$ on $H_0^1(Y)$ and the linear one $L(\xi)$ as

$$a(T,\ \xi)=\int_Y k^\delta(\boldsymbol{x})\frac{\partial T}{\partial x_i}\frac{\partial\xi}{\partial x_i}d\boldsymbol{x}\ . \tag{43}$$

The weak formulation of heat conduction problem is to find $T^{\delta} \in H_0^1(Y)$, such that for any $\xi \in H_0^1(Y)$

$$a(T^{\delta}, \xi)=L(\xi) . \tag{45}$$

To solve this problem, the following periodic Sobolev space on the composite RVE is introduced:

$$H_{\text{per}}^1(Y)=\left\{\zeta \in L^2(\Omega) | \frac{\partial \zeta}{\partial x_i} \in L^2(\Omega),\ \zeta \text{ is periodic}\right\} \tag{46}$$

with the norm

$$\|\zeta\|=\left[\int_Y \zeta^2 dy+\int_Y \left(\frac{\partial \zeta}{\partial y_i}\right)^2 dy\right]^{1/2}, \tag{47}$$

where the bilinear $a_y(\chi, \zeta)$ and linear $L_j(\zeta)$ forms are defined as follows

$$a(\chi,\xi)=\int_\Omega k(y)\frac{\partial \chi}{\partial y_i}\frac{\partial \zeta}{\partial y_i}dy, \tag{48}$$

$$L_j(\zeta)=-\int_\Omega k(y)\frac{\partial \zeta}{\partial y_j}dy. \tag{49}$$

By the analogy to the expression (46), the weak formulation of the heat conduction homogenization problem is introduced, which consists in determination of $\chi \in H_{per}^1(Y)$ such that for any $\zeta \in H_{per}^1(Y)$

$$a_y(\chi_j, \zeta)=L_j(\zeta). \tag{50}$$

Due to the assumption about the piecewise constant character of coefficient $k(\boldsymbol{y})$, the R.H.S. linear operator can be rewritten as

$$L_j(\zeta)=-\sum_{a=2}^{n} \int_{\Gamma_{(a-1,a)}} [k]\zeta n_j dy \tag{51}$$

for fiber-like composite (see Fig. 2) and for general n-component composite with m interfaces between its constituents as follows it can be expressed in the following form:

$$L_j(\zeta)=-\sum_{r=1}^{n} \int_{\Gamma r} [k]\zeta n_j dy, \tag{52}$$

where n_j denotes the components of a vector normal to the considered interface, directed out of the RVE.

Having computed the effective heat conductivity coefficient, its value can be compared with the upper and lower bounds in a well-known Voigt-Reuss form (Christensen 1979)

$$\sup k=\frac{1}{|\Omega|}\sum_{a=1}^{n} \Omega_a k_a, \tag{53}$$

$$\inf k=\left[\frac{1}{|\Omega|}\sum_{a=1}^{n} \frac{\Omega_a}{k_a}\right]^{-1}, \tag{54}$$

which completes deterministic characterization of the effective conductivity tensor.

3.2 Stochastic second order second moment perturbation approach

Rewriting partial differential Eqs. (18-20) in a conjunction with Eqs. (50-51) one can obtain
· zeroth-order term, one partial differential equation

$$\sum_{a=1}^{n} \int_{\Omega_a} k_{ij}^{0}\chi_{,j}^{0}\delta T_{,j}d\Omega = \sum_{a=2}^{n} \delta T_{,i}[k_{ij}^{0}]\big|_{\Gamma_{(a-1,a)}} n_j d(\partial\Omega), \tag{55}$$

· first-order term, R partial differential equations (r =1,..., R)

$$\sum_{a=1}^{n} \int_{\Omega_a} k_{ij}^{0}\chi_{,j}^{,r}\delta T_{,j}d\Omega = \sum_{a=2}^{n} \delta T_{,i}[k_{ij}^{,r}]\big|_{\Gamma_{(a-1,a)}} n_j d(\partial\Omega) - \sum_{a=1}^{n} \int_{\Omega_a} k_{ij}^{,r}\chi_{,j}^{0}\delta T_{,j}d\Omega, \tag{56}$$

· second-order term, one partial differential equation

$$\sum_{a=1}^{n} \int_{\Omega_a} k_{ij}^{0}\chi_{,j}^{(2)}\delta T_{,j}d\Omega = \sum_{a=2}^{n} \delta T_{,i}\left[k_{ij}^{(2)}\right]\big|_{\Gamma_{(a-1,a)}} n_j d(\partial\Omega)$$

$$-\left\{\sum_{a=1}^{n} \int_{\Omega_a} \left(k_{ij}^{,rs}\chi_{,j}^{0} + 2k_{ij}^{,r}\chi_{,j}^{,r}\right)\delta T_{,j}d\Omega\right\}\mathrm{Cov}(k^r,\, k^s). \tag{57}$$

Considering the fact that the second partial derivatives of the first component of the R.H.S. with respect to the input random variable are equal to 0, we can arrive at

$$\sum_{a=1}^{n} \int_{\Omega_a} k_{ij}^{0}\chi_{,j}^{(2)}\delta T_{,j}d\Omega = -2\sum_{a=1}^{n} \int_{\Omega_a} k_{ij}^{,r}\chi_{,j}^{,s}\delta T_{,j}d\Omega \; \mathrm{Cov}(k^r,\, k^s). \tag{58}$$

It is observed that to get the formulation for general composites, the R.H.S. summation should be carried out over all interfaces in the RVE. Then, solving Eqs. (12-15) for $\chi_{,j}^{0}$, $\chi_{,j}^{,r}$, and $\chi_{,j}^{,rs}$ successively, the expected values of the homogenization function can be derived as

$$E[\chi] = \chi_{,j}^{0} + \frac{1}{2}\chi_{,j}^{,rs}\mathrm{Cov}(k^r,\, k^s), \tag{59}$$

whereas the covariances are determined as

$$\mathrm{Cov}(\chi_{,i(\alpha)},\, \chi_{,j(\beta)}) = \chi_{,i(\alpha)}^{,r}\chi_{,j(\beta)}^{,s}\mathrm{Cov}(k^r,\, k^s),\; \alpha,\, \beta = 1,\, \ldots,\, N, \tag{60}$$

where N denotes the total number of degrees of freedom introduced in the RVE. Next, the first two probabilistic moments of the effective heat conductivity coefficient are determined. The expected values is obtained as

$$E[k^{(eff)}] = \frac{1}{|\Omega|}\int_{\Omega} E\left[k(\boldsymbol{y})\right]d\Omega + \frac{1}{|\Omega|}\int_{\Omega} E\left[k(\boldsymbol{y})\chi_{,j}\right]n_j d\Omega. \tag{61}$$

The second component of Eq. (61) can be rewritten as

$$E[k(\boldsymbol{y})\chi_{,j}] = \int_{-\infty}^{+\infty}\left(k^{0}(\boldsymbol{y}) + \Delta k^r k^{,r}(\boldsymbol{y}) + \frac{1}{2}\Delta k^r\Delta k^s k^{,rs}(\boldsymbol{y})\right) p_R\left(k(\boldsymbol{y})\right)d\boldsymbol{k}$$

$$\times\int_{-\infty}^{+\infty}\left(\chi_{,j}^{0}(\boldsymbol{y}) + \Delta k^u\chi_{,j}^{,u}(\boldsymbol{y}) + \frac{1}{2}\Delta k^u\Delta k^v\chi_{,j}^{,uv}(\boldsymbol{y})\right) p_R\left(k(\boldsymbol{y})\right)d\boldsymbol{k}. \tag{62}$$

By observing that

$$\int_{-\infty}^{+\infty} p_R\left(k(\boldsymbol{y})\right) d\boldsymbol{k}=1, \tag{63}$$

$$\int_{-\infty}^{+\infty} \Delta k^r p_R\left(k(\boldsymbol{y})\right) d\boldsymbol{k}=0 \tag{64}$$

and

$$\int_{-\infty}^{+\infty} \Delta k^r \Delta k^s k^{,rs}(\boldsymbol{y}) p_R(k(\boldsymbol{y})) d\boldsymbol{k}=\mathrm{Cov}(k^r,k^s);\ 1 \le r,s \le R, \tag{65}$$

we finally arrive at

$$\begin{aligned} E[k(\boldsymbol{y})\chi_{,j}] &= \int_{-\infty}^{+\infty} k^0(\boldsymbol{y})\chi_{,j}^0(\boldsymbol{y}) p_R(k(\boldsymbol{y})) d\boldsymbol{k} + \int_{-\infty}^{+\infty} \Delta k^r k^{,r}(\boldsymbol{y}) \Delta k^u \chi_{,j}^{,u}(\boldsymbol{y}) p_R(k(\boldsymbol{y})) d\boldsymbol{k} \\ &+ \frac{1}{2}\int_{-\infty}^{+\infty} k^0(\boldsymbol{y}) \Delta k^u \Delta k^v \chi_{,j}^{,uv}(\boldsymbol{y}) p_R(k(\boldsymbol{y})) d\boldsymbol{k} \\ &= k^0(\boldsymbol{y})\chi_{,j}^0(\boldsymbol{y}) + \left\{ k^{,r}(\boldsymbol{y})\chi_{,j}^{,u}(\boldsymbol{y}) + \frac{1}{2} k^0(\boldsymbol{y})\chi_{,j}^{,uv}(\boldsymbol{y}) \right\} \mathrm{Cov}(k^r,k^s)\,. \end{aligned} \tag{66}$$

Further, the variances of effective conductivity coefficient $\mathrm{Var}\left(k^{(eff)}\right)$ are determined. There holds

$$\mathrm{Var}\left(k^{(eff)}\right) = \frac{1}{|\Omega|^2}\int_\Omega \left(\mathrm{Var}\left(k(\boldsymbol{y})\right) + \mathrm{Var}\left(k(\boldsymbol{y})\chi_{,j}\right) n_j\right) d\Omega. \tag{67}$$

It can be derived from the definition of the variance that

$$\mathrm{Var}\left(k^{(eff)}\right) = \frac{1}{|\Omega|^2}\int_\Omega \left(k(\boldsymbol{y}) + k(\boldsymbol{y})\chi_{,j} n_j - E[k(\boldsymbol{y}) + k(\boldsymbol{y})\chi_{,j} n_j]\right)^2 d\Omega\, p_R(k(\boldsymbol{y})) d\boldsymbol{k}. \tag{68}$$

Therefore

$$\mathrm{Var}\left(k^{(eff)}\right) = \frac{1}{|\Omega|^2}\int_\Omega \int_{-\infty}^{+\infty} \left(\Delta k^r k^{,r} + \Delta k^r k^0 \chi_{,j}^{,r} n_j + \Delta k^r k^{,r} \chi_{,j}^0 n_j\right)^2 p_k(k(\boldsymbol{y})) d\boldsymbol{k} d\Omega. \tag{69}$$

Observing that

$$\mathrm{Var}(k) = \int_{-\infty}^{+\infty} (\Delta k^r)^2 p_k(k(\boldsymbol{y})) d\boldsymbol{k}, \tag{70}$$

the variance of the effective heat conductivity coefficients is derived as

$$\begin{aligned} \mathrm{Var}\left(k^{(eff)}\right) &= \frac{1}{|\Omega|^2}\int_\Omega \mathrm{Var}(k)\left(\left(k^{,r}\right)^2 + \left(k^0\chi_{,j}^{,r} n_j\right)^2 + \left(k^{,r}\chi_{,j}^0 n_j\right)^2\right) d\Omega \\ &+ \frac{1}{|\Omega|^2}\int_\Omega \mathrm{Var}(k)\left(2k^{,r}k^0\chi_{,j}^{,r} n_j + 2(k^{,r})^2 \chi_{,j}^0 n_j + 2k^{,r}\chi_{,j}^0 n_j k^0 \chi_{,j}^{,r} n_j\right) d\Omega. \end{aligned} \tag{71}$$

(no sum over repeated indices on the R.H.S.)

Alternatively to the presented perturbation approach, the probabilistic moments of effective heat conductivity coefficient can be evaluated using the Monte-Carlo simulation technique from Eqs. (55-58). Moreover, it can be observed that in case of randomly defined conductivity coefficients of composite components, the expected values and variances of the upper bounds (as well as any higher order probabilistic moments) can be derived as

$$E[\sup k_{ij}]=\frac{|\Omega_a|}{|\Omega|}\sum_{a=1}^{n}E\left[k_{ij}^{(a)}\right] \tag{72}$$

and for the variance

$$\mathrm{Var}(\sup k_{ij})=\left(\frac{|\Omega_a|}{|\Omega|}\right)^2\sum_{a=1}^{n}\mathrm{Var}\left(k_{ij}^{(a)}\right). \tag{73}$$

4. Finite element implementation

4.1 Deterministic problem discretization

Let us assume that the region Ω is discretized by a set of E finite elements and the homogenization temperature field χ is described by the nodal temperatures vector Ψ_α as (Krishnamoorthy 1994, Bathe 1996)

$$\chi(\boldsymbol{x})=H_\alpha(\boldsymbol{x})\Psi_\alpha, \tag{74}$$

where N is the total number of degrees of freedom within the region Ω. It follows that:

$$\chi_{,i}=H_{\alpha,i}\Psi_\alpha. \tag{75}$$

Then, the heat conductivity matrix $K_{\alpha\beta}$ and the R.H.S. vector P_α can be expressed as follows:

$$K_{\alpha\beta}=\int_\Omega k_{ij}H_{\alpha,i}H_{\beta,j}d\Omega, \tag{76}$$

$$P_\alpha=\int_\Omega gH_\alpha d\Omega+\int_{\partial\Omega}\hat{q}H_\alpha d\Omega. \tag{77}$$

As a result it is obtained that

$$K_{\alpha\beta}\Psi_\beta=P_\alpha. \tag{78}$$

Solving this equation for Ψ_β we compute discretized values of the homogenization function and, finally, the effective thermal conductivity coefficient given by Eq. (39). This expression can be discretized as follows:

$$k_{ij}^{(eff)}=\frac{1}{|\Omega|}\int_\Omega k_{ij}d\Omega+\frac{1}{|\Omega|}\int_\Omega k_{ij}1_m H_{\alpha,m}\Psi_\alpha d\Omega. \tag{79}$$

4.2 Probabilistic approach

Analogously to the classical finite element approach (Pepper and Heinrich 1992, Krishnamoorthy 1994), the following systems of algebraic equations can be introduced, which is the second-order stochastic perturbation formulation of the heat conduction problem:

· zeroth-order, one system of N ordinary differential equations

$$K_{\alpha\beta}^0\Psi_\beta^0=P_\alpha^0 \tag{80}$$

· first-order, R systems of N ordinary differential equations

$$K^0_{\alpha\beta}\Psi^{,r}_{\beta}=P^{,r}_{\alpha};-K^{,r}_{\alpha\beta}\Psi^0_{\beta}; \tag{81}$$

· second-order, one system of N ordinary differential equations

$$K^0_{\alpha\beta}\Psi^{,(2)}_{\beta}=[P^{,rs}_{\alpha}-2K^{,r}_{\alpha\beta}\Psi^{s}_{\beta}-K^{,rs}_{\alpha\beta}\Psi^0_{\beta}]S^{rs}. \tag{82}$$

$K^{(.)}_{\alpha\beta}$ and $P^{(.)}_{\alpha}$ denote here the heat conductivity matrix and the R.H.S. vector derivatives with respect to input random variables. The expected values and covariances of homogenization function are computed using the following equations:

$$E[\Psi_{\alpha}]=\Psi^{\alpha}_0+\frac{1}{2}\Psi^{(2)}_{\alpha} , \tag{83}$$

$$\mathrm{Cov}(\Psi_{\alpha}(x),\ \Psi_{\beta}(x))=\Psi^{,r}_{\alpha}(x)\Psi^{,s}_{\beta}(x)S^{rs} . \tag{84}$$

Having computed the second order probabilistic characterization of the homogenization function, Eqs. (18, 34) are applied to describe the expected value of effective conductivity coefficient as

$$\begin{aligned}E[k^{(eff)}]=&\frac{1}{|\Omega|}\int_{\Omega} E[k(\boldsymbol{y})]d\Omega+\\&+\frac{1}{|\Omega|}\int_{\Omega}\left\{k^0(y)1_jH_{\alpha,j}\Psi^0_{\alpha}+\left[k^{,r}(y)1_jH_{\alpha,j}\Psi^{,s}_{\alpha}+\frac{1}{2}k^0(y)1_jH_{\alpha,j}\Psi^{,rs}_{\alpha}\right]\mathrm{Cov}(k^r,k^s)\right\}n_j d\Omega\end{aligned} \tag{85}$$

Analogously, the stochastic finite element description of the effective conductivity variance can be obtained.

On the contrary, using the simulation technique, a large set of purely deterministic solutions is obtained with the heat conductivity coefficients generated randomly (Boswell *et al.* 1991) due to the input probability space. Defining necessary estimators (Bendat and Piersol 1971) of the effective heat conductivity coefficient, the expected value of the effective conductivity coefficient is approximated as

$$\mathrm{E}\left[k^{(eff)}\right]=\frac{1}{M}\sum_{j=1}^{M}k^{(eff)(j)}=\frac{1}{M}\sum_{j=1}^{M}\langle k\rangle^{(j)}_{\Omega}-\frac{1}{M}\sum_{j=1}^{M}\langle k1_i\Psi_{,i}\rangle^{(j)}_{\Omega} , \tag{86}$$

where M is the total number of samples that should be optimized, taking into account the convergence of estimators (see Figs. 7-12). Another probabilistic characteristics can be computed as follows:

· variance and standard deviation

$$\mathrm{Var}\left(k^{(eff)}\right)=\frac{1}{M-1}\sum_{j=1}^{M}\left(k^{(eff)(j)}-E\left[k^{(eff)}\right]\right)^2,\ \sigma\left(k^{(eff)}\right)=\sqrt{\mathrm{Var}(k^{(eff)})} , \tag{87}$$

· ordinary probabilistic moment (OPM) of the n-th order estimator

$$m_n(k^{(eff)})=\frac{1}{M}\sum_{i=1}^{M}(k_i^{(eff)})^n , \tag{88}$$

· central probabilistic moment (CPM) of the k-th order estimator

$$\mu_k\left(k^{(eff)}(\omega)\right)=m_k\left[\left(k^{(eff)}(\omega)\right)-m_1\left(k^{(eff)}(\omega)\right)\right]. \tag{89}$$

In case of Gaussian random variables, any central moments of the odd order are equal to 0, while the first three of even order can be described as follows:

$$\mu_2(k^{(eff)})=\frac{\sigma^2(k^{(eff)})}{E[k^{(eff)}]},\ \mu_4(k^{(eff)})=\frac{3\sigma^4(k^{(eff)})}{E^2[k^{(eff)}]},\ \mu_6(k^{(eff)})=\frac{15\sigma^6(k^{(eff)})}{E^3[k^{(eff)}]}. \tag{90}$$

Starting from these estimators the coefficients of variation, skewness and concentration can be estimated as

$$\alpha(k^{(eff)})=\frac{\sigma(k^{(eff)})}{E[k^{(eff)}]},\ \beta(k^{(eff)})=\frac{\mu_3(k^{(eff)})}{\sigma^3[k^{(eff)}]},\ \gamma(k^{(eff)})=\frac{\mu_4(k^{(eff)})}{\sigma^4[k^{(eff)}]}\ . \tag{91}$$

Taking into account $M \to \infty$ and following the Central Limit Theorem, there holds for Gaussian variables

$$\lim_{n\to\infty} \beta^{(n)}=0,\quad \lim_{n\to\infty} \gamma^{(n)}=3. \tag{92}$$

Further, considering the fact that there is no way to verify whether the probability density function computed is Gaussian or not, this fact can be verified numerically using up to 4th order probabilistic moments. If Eqs. (92) are fulfilled, the tested PDF can be treated as Gaussian with a relatively small error and the first two moments are necessary in further analysis. To arrive at this conclusion the output PDF estimator can be estimated too, however such a possibility is very costly from the point of view of the computational time. The conclusion on Gaussian character is important, considering the fact that in case of homogenization procedure we cannot prove this fact mathematically and, on the other hand, the second order perturbation second probabilistic moment method can be used with any other further extension to compute the first two moments of the effective conductivity for the composites.

Moreover, it is essential to underline that, contrary to another probabilistic approaches, the simulation technique assures existence and uniqueness of the effective conductivity coefficients probabilistic characteristics, which follows deterministic results and the nature of the statistical estimation methods. Further, it can be seen that the accuracy of the estimation results depends on the total number of random trials performed, denoted in equations posed above by M while it does not depend at all on the input random variables coefficients of variance. Finally, it can be underlined that the applied technique is difficult to use to large scale systems considering an increase of simulation time with a higher total number of degrees of freedom and technical problems caused by data storage.

5. Computational experiments

The computational experiments are performed using the FEM-based homogenization-oriented program MCCEFF (Kamiński 1996). This program enables computations of the composite materials effective characteristics and their upper and lower bounds together with the respective probabilistic moments for linear elastostatics and heat conduction problems. Generally, *n*-component composite materials may be homogenized in elasticity problems, while at present two-component for heat

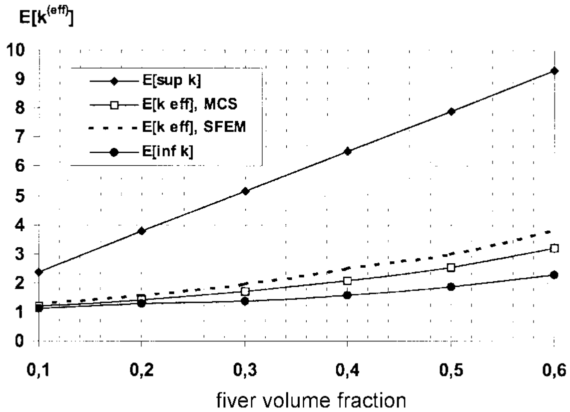

Fig. 3 Expected values of the effective heat conductivity

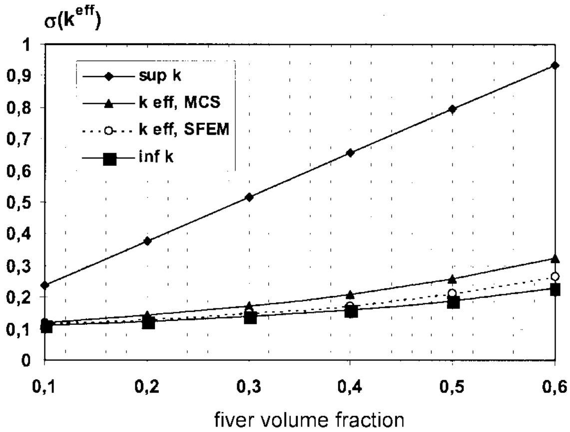

Fig. 4 Standard deviations of the effective heat conductivity

conduction, however upper and lower bounds for effective tensors can be calculated automatically for all cases. The two-component composite under consideration has rectangular RVE, centrally located reinforcement with round section and is built at the fiber (k_1=14.8) and matrix (k_2=1.0). The coefficients of variation are taken as equal to 0.1, while the reinforcement volume fraction is taken from the interval (0.1, 0.6).

First, the sensitivity of effective heat conductivity coefficient probabilistic moments is verified with respect to the composite reinforcement ratio in comparison with the expected values and standard deviations for the MCS and the SFEM implementations. The results of the analyses are collected in Figs. 3-6 shown below as the functions of the fiber volume fraction within the RVE.

The first figure (Fig. 3) presents the expected values of effective values and their bounds, the next one (Fig. 4) shows the standard deviations of these characteristics. Fig. 5 illustrates the third-order central moment in the function of the fiber volume fraction, while in Fig. 6 the fourth-order central

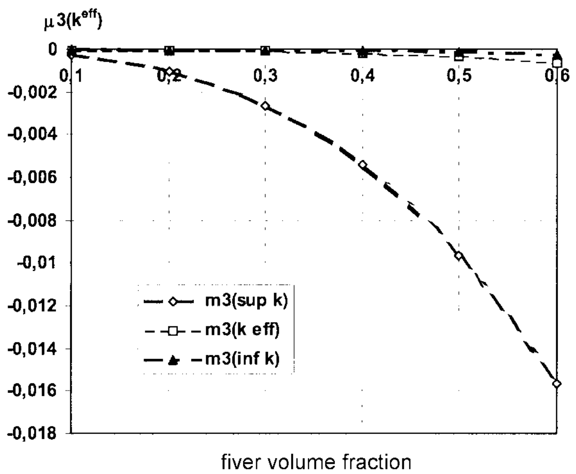

Fig. 5 3rd order CPM of the effective conductivity

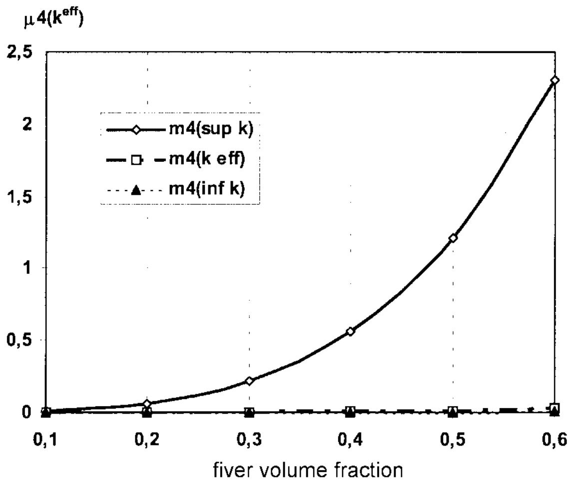

Fig. 6 4th order CPM of the effective conductivity probabilistic moments are presented. Generally, it is visible on all these figures that the moments of effective conductivity upper and lower bounds bound the moments of effective heat conductivity coefficient very well. Taking into account the interrelations between all these probabilistic characteristics, the approximation of the effective composite conductivity moments by the respective values of lower bound can be proposed. It is very important, considering shortening of computational time, since lower bounds are obtained from simple algebraic equation simulation, while the effective heat conductivity coefficient must be calculated by FEM solution of some heat conduction boundary value problem. Observing the first two figures, it should be noted that expected values and standard deviations of upper bounds change linearly, while the changes of the first two moments of other effective parameters and their lower bounds have quite nonlinear character.

Considering the comparison between the expected values and standard deviations, resulting from the MCS and SFEM analyses, it can be seen that the first probabilistic moments of effective conductivity are generally greater for the stochastic perturbation approach than those obtained in simulations. It may be caused by the fact that the SFEM expectations are calculated as the sum of zeroth and second order terms respectively, while the MCS results usually tend to their deterministic equivalents. The reverse observation may be done in case of standard deviations. The main reason for such a relation is that the SFEM second probabilistic moments include the second order terms only, while the higher order terms are neglected. To verify this fact precisely, the sensitivity of both these methods with respect to input coefficients of variation is to be carried out.

Fluctuations in 3rd and 4th order probabilistic moments have nonlinear character for all effective characteristics, but the differences between the values of probabilistic moments increase, together with increase of an order of the moment being analyzed and for data collected in Fig. 6 it is even more than 10 times between upper and lower bounds for effective conductivity coefficient. Considering these dependencies, lower bounds should be used for approximation of the effective behavior of the composite, while for some reasons it is impossible to compute effective quantities

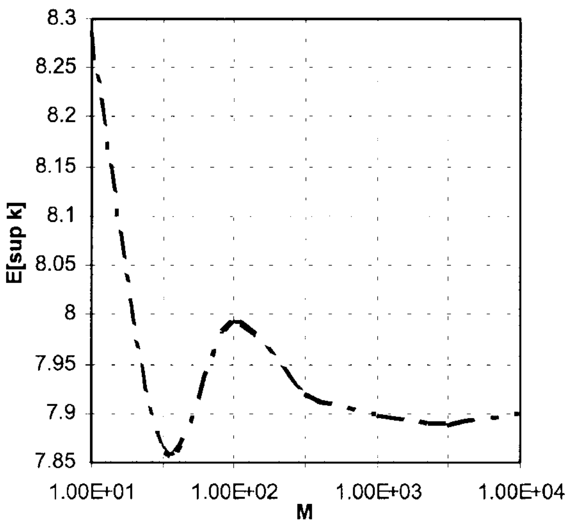

Fig. 7 Convergence of the upper bound expected value

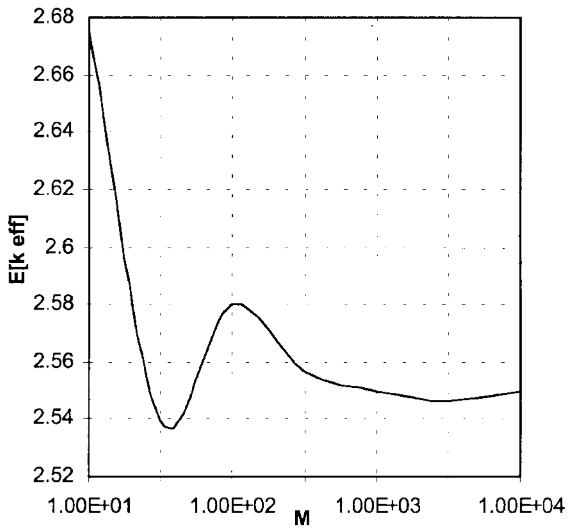

Fig. 8 Convergence of the effective conductivity expected value estimator

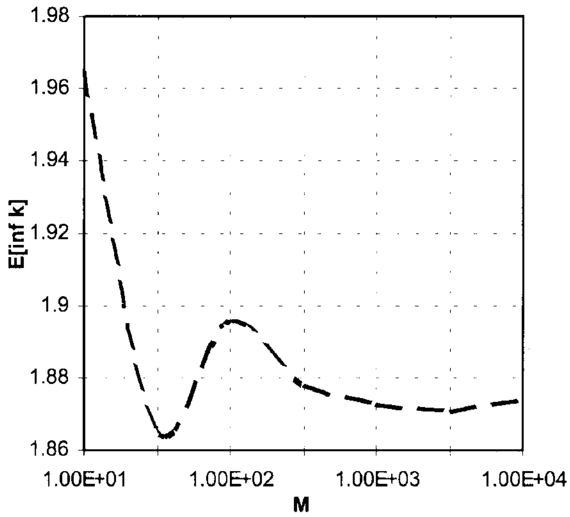

Fig. 9 Convergence of the lower bound expected value estimator

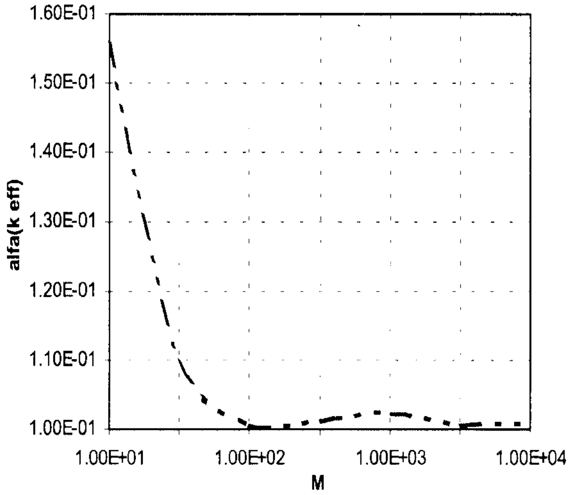

Fig. 10 Convergence of the effective conductivity coefficient of variation

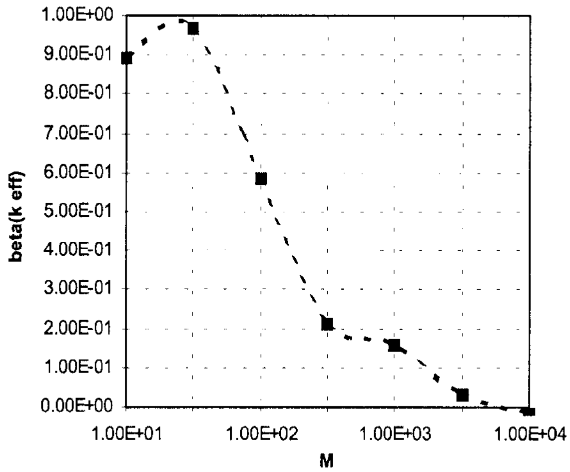

Fig. 11 Convergence of the effective conductivity coefficient of skewness

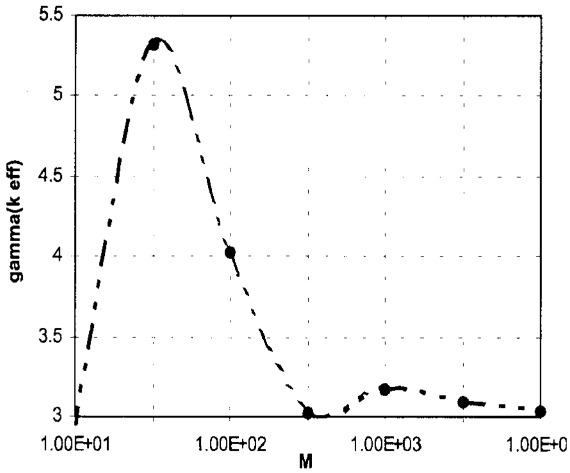

Fig. 12 Convergence of the effective conductivity coefficient of concentration

due to the homogenization method introduced.

Next, convergence of probabilistic moments of effective conductivity coefficients has been verified. The main purpose of these experiments was to establish an optimal number of random trials for a probabilistic simulation, necessary to obtain the computed moments with a relatively small numerical error. The results of the analysis have been presented in Figs. 7-12. The respective estimators are marked on the vertical axes of the graphs, while the total number of random trials on the horizontal ones.

First, it should be noted that the convergence of the expected values estimators have analogical character for upper and lower bounds as well as for effective conductivity coefficient, see Figs. 7-9.

The value of estimator decreases rapidly from maximum reached for 10 random trials to minimum for about 50 iterations. Next, with inverse tendency, it increases to 100 trials and asymptotically converges to be stable for about 10^4 iterations. The character of variation coefficient convergence presented in Fig. 10 is quite similar to the one discussed above, however asymptotic changes are smoother than for expected values shown on the previous figure. The coefficient of asymmetry (which should be equal to 0 for Gaussian deviates) decreases from maximum reached at the 10 iterations to 0 for 10^4 samples without any asymptotic fluctuations (observed for effective elasticity tensor components estimators, too). The coefficient of concentration (which should be equal to 3 for Gaussian random variables) converges analogously as first and second order characteristics to 3 for 10^4 random iterations in simulation. Analyzing these figures, it is clear that the probability density function of the effective conductivity and its bounds are very close, with a relatively small error, to the corresponding Gaussian PDF. This remark makes it possible to define uniquely their PDFs using the first two probabilistic moments only.

6. Conclusions

The formulation presented and discussed above describes a homogenization method for the *n*-component composite materials with randomly defined heat conductivity. The proposed model makes it possible to compute expected values and variances of the effective conductivity by using FEM-based Monte-Carlo simulation analysis or the Stochastic Finite Element approach. Numerical implementations, which seem to be effective and easy to provide, appear to be efficient tools in stochastic sensitivity studies of effective conductivity to the composite components volume fraction. In the same time it should be noticed that Monte-Carlo simulation technique can be successfully implemented in any commercial FEM (ABAQUS, for instance) or any other computational discrete method based package. Usually there is no need to have direct access to the source code of the program extended, contrary to the SFEM technique implementation.

Computational experiments performed show that all probabilistic moments of $k^{(eff)}$ are well bounded by the respective characteristics of their upper and especially lower bounds. Moreover, it can be seen that the expected values and variances of lower bound give quite effective approximation of effective conductivity moments, which can be useful in further computational modeling of non-periodic composites where direct FEM-based homogenization is too complicated. Taking into account simulational aspect of the method, the most recommended minimal number of random trials has been determined as about 10^4. Considering the fact that homogenization presented is in fact equivalent to the solution of a boundary value problem, this conclusion deals as well with any Monte-Carlo simulated heat conduction problem, where heat conductivity coefficients are treated as random variables.

It should be noticed that the probabilistic homogenization procedure involved may be applied for seepage, torsion, irrotational and imcompressible flow, film lubrication, acoustic vibration as well as for the electric conduction, electrostatic field, electromagnetic waves and all field problems with statistically defined physical characteristics. Next, using proposed homogenization procedure, the sensitivity of probabilistic moments of $k^{(eff)}$ with respect to the expected values and higher order probabilistic moments interrelations for composite components heat conductivities may be verified in further computational tests. On the other hand, it is relatively easy to extend the presented model on homogenization of *n*-component random composites as well as on the periodic heterogeneous media with stochastic structural defects introduced in (Kamiński and Kleiber 1996). Finally, probabilistic (homogenization-based) reliability of the composite structures with parameter sensitivity studies (probabilistic characteristics of material parameters) may be carried out on the basis of the presented model.

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