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Evaluation of vertical dynamic characteristics of cantilevered tall structures

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Abstract. In this paper, cantilevered tall structures are treated as cantilever bars with varying cross-section for the analysis of their free longitudinal (or axial) vibrations. Using appropriate transformations, exact analytical solutions to determine the longitudinal natural frequencies and mode shapes for a one step non-uniform bar are derived by selecting suitable expressions, such as exponential functions, for the distributions of mass and axial stiffness. The frequency equation of a multi-step bar is established using the approach that combines the transfer matrix procedure or the recurrence formula and the closed-form solutions of one step bars, leading to a single frequency equation for any number of steps. The Ritz method is also applied to determine the natural frequencies and mode shapes in the vertical direction for cantilevered tall structures with variably distributed stiffness and mass. The formulae proposed in this paper are simple and convenient for engineering applications. Numerical example shows that the fundamental longitudinal natural frequency and mode shape of a 27-storey building determined by the proposed methods are in good agreement with the corresponding measured data. It is also shown that the selected expressions are suitable for describing the distributions of axial stiffness and mass of typical tall buildings.

Key words: longitudinal vibration; tall buildings; high-rise structures; natural frequencies; mode shapes.

1. Introduction

It has been recognised that the magnitude of the vertical component of earthquake ground motion is often about one-third of the horizontal component, and the vertical component of ground motions has a significant effect on earthquake-induced responses of structures (e.g., Wang 1978, Li *et al*. 1994). Thus, it is required to determine the natural frequencies and mode shapes in the vertical direction for high-rise structures in design stage for certain cases (Li *et al.* 1998). In analysing free vibrations of high-rise structures, for the sake of simplicity, it is possible to regard such structures as a cantilever bar (Wang 1978, Li *et al.* 1994, 1996, 2000). However, in general, it is not possible or, at least, very difficult to get the exact analytical solutions of differential equations for free vibrations

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of bars with variably distributed mass and stiffness. These exact bar solutions are available only for certain bar shapes and boundary conditions. Tuma and Cheng (1983) found the analytical solutions for the free longitudinal vibrations of a straight bar with uniformly distributed stiffness and mass. Wang (1978) derived the closed form solutions for the free longitudinal vibration of a cantilever bar with variably distributed stiffness and mass. Bapat (1995) obtained exact solutions for the free longitudinal vibration of exponential and catenoidal rods. Abrate (1995) derived closed-form solutions for the free longitudinal vibration of rods whose cross-section varies as $A(x)=A_0[1+a(x/L)]^2$. Kumar *et al*. (1997) found exact solutions for the free longitudinal vibration of non-uniform rods whose cross-section varies as $A(x)=(a+bx)^n$ and $A(x)=A_0\sin^2(ax+b)$. The natural frequencies of such rods for various end conditions were calculated, and their dependence on taper was discussed.

In the previous studies mentioned above, it is usually assumed that the mass of a bar (rod) is proportional to its stiffness (e.g., Wang 1978, Abrate 1995, Bapat 1995, Kumar *et al*. 1997). This calculation model is reasonable for a part of high-rise structures such as chimney, T.V. tower etc, but it is not suitable for tall buildings and many high-rise structures. This is because that the mass of floors is 80% or even more of the total mass of a tall building and the variation of mass at different floors is not significant, the mass distribution with height is almost constant for many cases. This suggests that the value of mass of a tall building is not necessarily proportional to its stiffness. This is confirmed by a series of shaking tests on buildings of various types in which the mass and stiffness of individual buildings have been measured and reported (Jeary and Sparks 1977, Ellis and Jeary 1980). In this paper, exact analytical solutions for free longitudinal vibrations of bars with variably distributed mass and stiffness, in which the value of mass is not necessarily proportional to its stiffness, are proposed.

The free longitudinal vibration of a multi-step bar with varying cross-section is a complex problem, and the exact solution of this problem has not previously been obtained. Use of the exact solution of a one-step bar together with a transfer matrix procedure and a recurrence formula is presented in this paper in order to resolve this problem.

2. Free longitudinal vibrations of one-step cantilever bars

The differential equation for longitudinal (or axial) vibration of a bar with varying cross-section (Fig. 1) is

$$\frac{\partial}{\partial x}\left(K_x\frac{\partial y}{\partial x}\right)-\overline{m}_x\frac{\partial^2 y}{\partial t^2}=0 \tag{1}$$

in which y, K_x and $\overline{m}_x$ are the displacement in the longitudinal direction (vertical direction), axial stiffness and mass per unit length, respectively, at section x.

Using the method of separation of variables

$$y(x,\ t)=X(x)e^{i\omega t}$$

obtains the longitudinal vibration mode function, $X(x)$, as follows

$$K_x\frac{d^2X}{dx^2}+\frac{dK_x}{dx}\frac{dX}{dx}+\overline{m}_x\omega^2X=0 \tag{2}$$

where ω is the circular natural frequency.

As found by Li *et al*. (1994, 1996) and Li (1999) that in many cases, the variations of mass and stiffness of typical high-rise structures can be described by the following exponential functions:

$$K_x = \alpha e^{-\beta\frac{x}{L}}, \quad m_x = a e^{-b\frac{x}{L}} \tag{3}$$

The parameters α, β, a, b can be determined by

$$\alpha = K_0,\ \beta = Ln(K_0) - Ln(K_L),\ a = m_0,\ b = Ln(m_0) - Ln(m_L) \tag{4}$$

where m_0, K_0, m_L, K_L are the mass intensity and the axial stiffness, respectively, at x=0 and L. L is the height of the structure considered (Fig. 1).

Substituting Eq. (3) into Eq. (2) and setting

$$\left.\begin{aligned} &X = \xi^{\nu} Z \\ &\xi = e^{\frac{(\beta - b)x}{2L}} \\ &\nu = \frac{\beta}{\beta - b} \\ &\lambda^2 = \frac{4a\omega^2 L^2}{\alpha(\beta - b)^2} \end{aligned}\right\} \tag{5}$$

lead to

$$\frac{d^2 Z}{d\xi^2} + \frac{1}{\xi}\frac{dZ}{d\xi} + \left(\lambda^2 - \frac{\nu^2}{\xi^2}\right) Z = 0 \tag{6}$$

Eq. (6) is a Bessel's equation of the ν-th order. For a non-integral ν, the vibration mode shape function can be expressed as

$$X = \xi^{\nu}[C_1 J_{\nu}(\lambda\xi) + C_2 J_{-\nu}(\lambda\xi)] \tag{7}$$

where $J_{\nu}(\lambda\xi)$ is the Bessel function of the first kind of order ν.

The boundary conditions of a cantilever bar (Fig. 1) are as follows

$$\left.\begin{aligned} &x = 0,\ X(0) = 0 \\ &x = L,\ K_L\left(\frac{dX}{dx}\right)_{x=L} = m\omega^2 X(L) \end{aligned}\right\} \tag{8}$$

Substituting the above boundary conditions into Eq. (7) obtains the following frequency equation

$$J_{-\nu}(\lambda)\left[\frac{(\beta - b)\lambda m}{2aL} J_{\nu}(\lambda A) + A e^{-\beta} J_{\nu-1}(\lambda A)\right] = J_{\nu}(\lambda)\left[\frac{(\beta - b)\lambda m}{2aL} J_{-\nu}(\lambda A) - A e^{-\beta} J_{-(\nu-1)}(\lambda A)\right] \tag{9}$$

in which m is the lumped mass attached to the top of the cantilever bar with varying cross-section (Fig. 1).

$$A = e^{\frac{\beta - b}{2}} \tag{10}$$

If there is no lumped mass at the top of the bar, that is, m=0, then the frequency equation for this case is given by

$$J_{-\nu}(\lambda)J_{\nu-1}(\lambda A) = -J_{\nu}(\lambda)J_{-(\nu-1)}(\lambda A) \tag{11}$$

If ν is an integral, the functions $J_{-\nu}$, $J_{-(\nu-1)}$ should be replaced by Y_ν and $Y_{\nu-1}$, respectively, and the negative signs in Eqs. (9) and (11) should be changed to the positive signs, where Y_ν is the Bessel function of the second kind of order ν.

The procedure presented above for determining the longitudinal natural frequencies and mode shapes in the vertical direction for a cantilever bar with variably distributed stiffness and mass is called the analytical method in this paper.

3. Free longitudinal vibrations of multi-step cantilever bars

Although the general solutions derived above for one-step bars with varying cross-section can be used to determine the longitudinal natural frequencies and mode shapes of many structures, there are two problems to be solved. First, some structures consist of several steps (see Fig. 2). Second, the distributions of stiffness and mass of some structures may not obey the assumed expressions given in the previous section. Such structures can be treated as multi-step bars for free vibration analysis. If the steps are divided appropriately, the distributions of stiffness and mass in each step may match accurately or approximately the expressions described above. The exact solution of a one-step bar with varying cross-section can be used to derive the general solution and the frequency equation of a multi-step bar using the following method.

The general solution of the longitudinal vibration mode shape function of the i-th step of a multi-step cantilever bar (Fig. 2) can be expressed as

$$X_i(x) = C_{i1}S_{i1}(x) + C_{i2}S_{i2}(x) \qquad (i=1, 2, 3, ..., n) \tag{12}$$

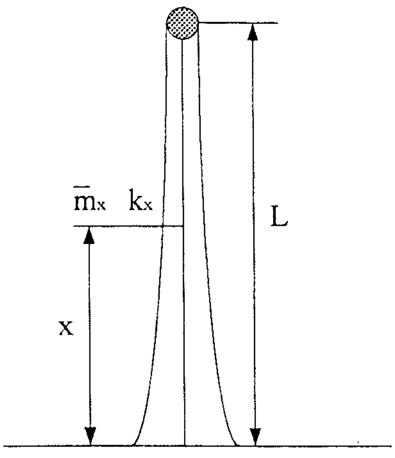

Fig. 1 A Cantilever bar with varying cross-section

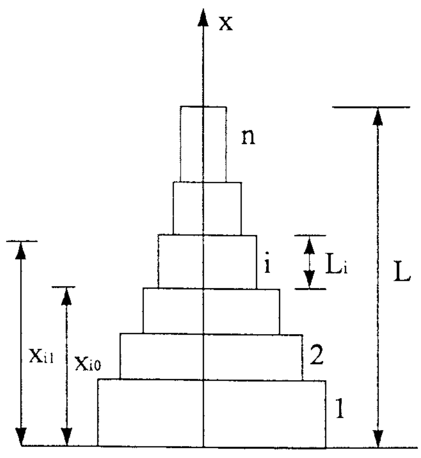

Fig. 2 A multi-step cantilever bar

If the axial stiffness and mass of the ith step are described by the exponential functions, Eq. (3), then $S_{i1}(x)$ and $S_{i2}(x)$ can be found from Eq. (7). If the stiffness and mass of the ith step are constants, then $S_{i1}(x)$ and $S_{i2}(x)$ can be expressed as

$$\left.\begin{aligned} S_{i1}(x) &= \sin\lambda_i x \\ S_{i2}(x) &= \cos\lambda_i x \end{aligned}\right\} \tag{13}$$

in which

$$\lambda_i = \omega\sqrt{\frac{\overline{m}_i}{K_i}} \tag{14}$$

$\overline{m}_i$, K_i are the mass intensity and axial stiffness of the i-th step, respectively.

A transfer matrix procedure is introduced herein to establish the mode shape equation and the frequency equation of the multi-step bar as shown in Fig. 2.

The mode shape $X_i(x)$ and the axial force $N_i(x)$ of the i-th step can be expressed as a matrix as follows

$$\begin{bmatrix} X_i(x) \\ N_i(x) \end{bmatrix} = [A_i(x)] \begin{bmatrix} C_{i1} \\ C_{i2} \end{bmatrix} \tag{15}$$

in which

$$[A_i(x)] = \begin{bmatrix} S_{i1}(x) & S_{i2}(x) \\ K_{x1}\dfrac{dS_{i1}(x)}{dx} & K_{x2}\dfrac{dS_{i2}(x)}{dx} \end{bmatrix} \tag{16}$$

The parameters, C_{i1} and C_{i2}, can be found by

$$\begin{bmatrix} C_{i1} \\ C_{i2} \end{bmatrix} = [A_i(x)]^{-1} \begin{bmatrix} X_i(x) \\ N_i(x) \end{bmatrix} \tag{17}$$

The relationship between the parameters, X_{i1}, N_{i1}, at the end 1 and X_{i0}, N_{i0}, at the end 0 of the i-th step (Fig. 2) can be obtained by using Eq. (17) as follows

$$\begin{bmatrix} X_{i1} \\ N_{i1} \end{bmatrix} = [T_i] \begin{bmatrix} X_{i0} \\ N_{i0} \end{bmatrix} \tag{18}$$

in which

$$[T_i] = [A_i(x_{i1})][A_i(x_{i0})]^{-1} \tag{19}$$

$[T_i]$ is called the transfer matrix because it transfers the parameters at the end 0 to those at the end 1 of a step. According to Eq. (18), the parameters, X_{i1}, N_{i1}, at the end 1 of the i-th step can be represented by the parameters, $X_{i-1,0}, N_{i-1,0}$ at the end 0 of the (i–1)th step.

$$\begin{bmatrix} X_{i1} \\ N_{i1} \end{bmatrix} = [T_i][T_{i-1}] \begin{bmatrix} X_{i-1,0} \\ N_{i-1,0} \end{bmatrix} \tag{20}$$

The relationship of the parameters, X_{n1}, N_{n1}, at the end 1 of the top step (Fig. 2) and the parameters, X_{10}, N_{10} at the end 0 of the first step can be established by using Eq. (18) repeatedly.

$$\begin{bmatrix} X_{n_1} \\ N_{n_1} \end{bmatrix} = [T] \begin{bmatrix} X_{10} \\ N_{10} \end{bmatrix} \tag{21}$$

in which

$$[T] = [T_n][T_{n-1}]\cdots[T_1] \tag{22}$$

$[T]$ is a matrix which can be expressed as

$$[T] = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \tag{23}$$

The frequency equation can be established according to the boundary conditions. For example, high-rise structures can be treated as a cantilever bar for free vibration analysis, the boundary conditions are

$$\left.\begin{array}{c} x=0, \ X(x)=0 \\ x=L, \ N_{n1}=K\dfrac{dX(x)}{dx}\Big|_{x=L}=0, \ \text{i.e.,} \ \dfrac{dX(x)}{dx}\Big|_{x=L}=0 \end{array}\right\} \tag{24}$$

According to Eqs. (21)-(24), we have

$$\begin{bmatrix} X_{n1} \\ N_{n1} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} 0 \\ N_{10} \end{bmatrix} \tag{25}$$

From Eq. (25),

$$N_{n1} = T_{22} N_{10} = 0 \tag{26}$$

Because of $N_{10} \neq 0$, we have

$$T_{22} = 0 \tag{27}$$

This is the frequency equation of a multi-step cantilever bar.

If there is a lumped mass, m_i, attached to the end 1 of the i-th step, then, the transfer matrix $[T_i]$ should be replaced by $[T_{mi}]$ as follows

$$[T_{mi}] = \begin{bmatrix} 1 & 0 \\ m_i\omega^2 & 1 \end{bmatrix} [T_i] \tag{28}$$

The natural frequencies ω_j(j=1, 2,...) can be found by solving Eq. (27), then, the mode shape functions can be determined by use of Eq. (18). First, set X_{10}=0, N_{10} can take any value. For example, setting N_{10}=1, then,

$$X_{11} = T_{12}^{(1)}, \ X_{21} = T_{12}^{(2)}, \ \ldots, \ X_{n1} = T_{12}^{(n)} \tag{29}$$

where $T_{12}^{(i)}$ is the element of the matrix $[T_i][T_{i-1}]\cdots[T_1]$

The frequency equation of a multi-step cantilever bar can be also established by using the recurrence formula to be presented below. First, the special case that each step has constant parameters (mass and axial stiffness) is studied in order to explain the proposed procedure. The mode shape function of the *i*-th step (Fig. 2) is expressed as follows

$$X_i(x)=D_{i0}\cos\lambda_i(x-x_{i0})+\frac{D_{i1}-D_{i0}\cos\lambda_iL_i}{\sin\lambda_iL_i}\sin\lambda(x-x_{i0}) \tag{30}$$

in which D_{i0} and D_{i1} are the amplitude at end 0 and 1 of the *i*-th step, respectively, L_i is the length of the *i*-th step (Fig. 2).

The harmonious condition of the connecting section between the *i*-th step and the (*i*+1)th step is as follows

$$K_i\frac{dX_i}{dx}\bigg|_{x=x_{i1}}=K_{i+1}\frac{dX_{i+1}}{dx}\bigg|_{x=x_{i+1,0}} \tag{31}$$

According to Eq. (31), we obtain the relationship among D_{i0}, D_{i1} and $D_{i+1,1}$ as follows

$$D_{i+1,1}=P_{i1}D_{i1}-Q_{i0}D_{i0},\qquad i=1,\,2,\,...,\,n \tag{32}$$

in which

$$\left.\begin{aligned}
&P_{i1}=\frac{p_{i1}}{R_{i+1}}\\
&Q_{i0}=\frac{q_{i0}}{R_{i+1}}\\
&p_{i1}=K_i\frac{\lambda_i}{\lambda_{i+1}}\text{cotan}\lambda_iL_i+\text{cotan}\lambda_{i+1}L_{i+1}\\
&q_{i0}=\frac{K_i}{\sin\lambda_iL_i}\\
&R_{i+1}=\frac{K_{i+1}}{\sin\lambda_{i+1}L_{i+1}}
\end{aligned}\right\} \tag{33}$$

Eq. (32) is a recurrence formula. Because D_{10}=0 and D_{11} can take any value, we set D_{11}=1, then,

$$\left.\begin{aligned}
&D_{21}=P_{11}\\
&D_{31}=P_{21}P_{11}-Q_{20}\\
&D_{41}=P_{31}P_{21}P_{11}-P_{31}Q_{20}-Q_{30}P_{11}\\
&D_{51}=P_{41}P_{31}P_{21}P_{11}-P_{41}P_{31}Q_{20}-P_{41}P_{11}Q_{30}-P_{21}P_{11}Q_{40}-Q_{20}Q_{40}\\
&\cdots\cdots
\end{aligned}\right\} \tag{34}$$

According to the following boundary condition

$$\frac{dX_n(x)}{dx}\bigg|_{x=L}=0 \tag{35}$$

we obtain the frequency equation as follows

$$D_{n0}=D_{n1}\cos\lambda_n L_n \tag{36}$$

Solving Eq. (36) obtains ω_j(j=1, 2, ...), and substituting ω_j into Eq. (34) obtains the j-th mode shape.

If each step of a multi-step cantilever bar has variably distributed mass and stiffness that are described by Eq. (3), the recurrence formula which is similar to Eq. (33) can be established in terms of the same procedure presented above

$$D'_{i+1,1}=P_{i1}D'_{i1}-Q_{i0}D'_{i0} \tag{37}$$

in which

$$D'_{i1}=D_{i1}e^{-f_i\nu_i x_{i1}},\ \ D'_{i0}=D_0 e^{-f_i\nu_i x_{i0}},\ \ D'_{i+1,1}=D_{i+1,1}e^{-f_{i+1}\nu_{i+1}x_{i+1,1}}$$

$$f_i=\frac{\beta_i-b_i}{2x_{i1}},\qquad \nu_i=\frac{\beta_i}{\beta_i-b_i}$$

β_i, b_i are the parameters in Eq. (3) for the i-th step, the origin of coordinate is set at the base of the step.

$$P_{i1}=\frac{\theta_{i+1,1}}{\theta_{i+1,0}}+\frac{\Omega_{i0}}{\Omega_{i1}}S_{i-1}$$

$$\theta_{i+1,1}=e^{f_{i+1}x_{i+1,1}},\quad \theta_{i+1,0}=e^{f_{i+1}x_{i+1,0}}$$

$$\Omega_{i0}=J_{-\nu_i}(\lambda_i\theta_{i0})J_{\nu_i-1}(\lambda_i\theta_{i1})+J_{\nu_i}(\lambda_i\theta_{i0})J_{-(\nu_i-1)}(\lambda_i\theta_{i1})$$

$$\Omega_{i1}=J_{-\nu_i}(\lambda_i\theta_{i1})J_{\nu_i-1}(\lambda_i\theta_{i1})+J_{\nu_i}(\lambda_i\theta_{i1})J_{-(\nu_i-1)}(\lambda_i\theta_{i1})$$

$$S_{i-1}=\frac{\Delta_i}{\Delta_{i-1}}\cdot\frac{\Omega_{i-1,0}}{\Omega_{i0}}\cdot\frac{K_{i-1,1}\lambda_{i-1}f_{i-1}(\theta_{i-1,1})^{\nu_{i-1}+1}}{K_{i0}\lambda_i f_i(\theta_{i1})^{\nu_1+1}}$$

$$\Delta_i=J_{\nu_i}(\lambda_i\theta_{i0})J_{-\nu_i}(\lambda_i\theta_{i1})-J_{-\nu_i}(\lambda_i\theta_{i0})J_{\nu_i}(\lambda_i\theta_{i1})$$

$$\lambda_i=\frac{2\omega L}{f_i}\sqrt{\frac{a_i}{\alpha_i}}$$

For the first step, it is easy to obtain the following equations

$$\left.\begin{aligned}&D'_{10}=0\\&D'_{i1}=J_{\nu_1}(\lambda_1\theta_{11})-\frac{J_{\nu_1}(\lambda_1)}{J_{-\nu_1}(\lambda_1)}J_{-\nu_1}(\lambda_1\theta_{11})\end{aligned}\right\} \tag{38}$$

The parameters, D'_{i1}, D'_{i0} can be determined by using Eq. (37), and the frequency equation can be established in terms of Eq. (35) as follows

$$\frac{J_{\nu_n-1}(\lambda_n\theta_{n1})}{J_{-(\nu_n-1)}(\lambda_n\theta_{n1})}=\frac{D'_{n1}J_{\nu_n}(\lambda_n\theta_{n0})-D'_{n0}J_{\nu_n}(\lambda_n\theta_{n1})}{D'_{n0}J_{-\nu_n}(\lambda_n\theta_{n1})-D'_{n1}J_{-\nu_n}(\lambda_n\theta_{n0})} \tag{39}$$

The formulas presented above are derived for ν = a non-integer only. If ν = an integer, then $J_{-\nu}$ appeared in the above equations should be replaced by Y_ν, and the positive signs in the expressions of Ω_{i0} and Ω_{i1} should be changed to the negative signs.

4. Use of the Ritz method for free longitudinal vibration analysis

A multi-step bar, in which each step has constant parameters (mass and stiffness) or variable parameters, may be simplified as a one-step bar with continuously varying parameters for free vibration analysis (Wang 1978, Li *et al.* 1994). Free longitudinal vibration analysis of cantilever bars with variably distributed stiffness and mass as well as lumped masses will be presented herein in terms of the Ritz method.

In general, the fundamental circular frequency of a cantilever bar with variably distributed stiffness and mass as well as lumped masses (Fig. 3) can be determined in terms of the energy method as follows

$$\omega^2 = \frac{\int_0^L K_x \left(\frac{dX}{dx}\right)^2 dx}{\int_0^L \bar{m}_x X^2(x) dx + \sum_{\zeta=1}^{m} m_\zeta X^2(x_\zeta)} \tag{40}$$

where K_x and $\bar{m}_x$ are the axial stiffness and mass per unit length, respectively, at section x, m_ζ(ζ =1, 2, ..., m) are the lumped masses at section x_ζ.

Because the value of ω obtained from Eq. (40), in general, is greater than the real one, it is assumed that

$$X(x) = \sum_{i=1}^{n} a_i \psi_i(x) \tag{41}$$

in which $\psi_i(x)$ (i=1, 2, ..., n) is the i-th pre-chosen function called the coordinate function, and a_i is the i-th unknown parameter to be determined. n is the number of different $\psi_i(x)$ functions considered.

As indicated by Clough and Penzien (1993) among others, any assumed shape $X(x)$ leads to a calculated frequency which is higher than the true frequency, and so the best approximation of the

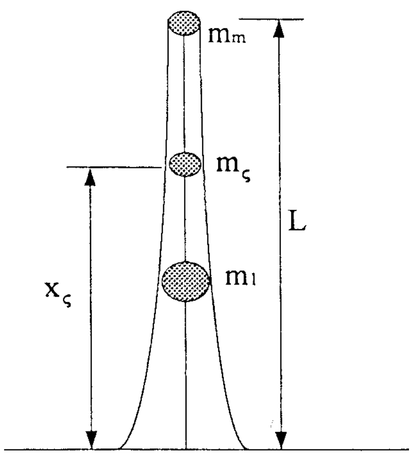

Fig. 3 A Cantilever bar with variable parameters and lumped masses

shape, that is, the best choice of a_i, will minimise the frequency. Thus differentiating the frequency expression with respect to any one of a_i and equating to zero gives

$$\frac{\partial}{\partial a_i}\left\{\frac{\int_0^L K_x\left(\frac{dX}{dx}\right)^2 dx}{\int_0^L \overline{m}_x X^2(x)dx+\sum_{\zeta=1}^{m} m_\zeta X^2(x_\zeta)}\right\}=0 \tag{42}$$

A set of equations can be obtained from Eq. (42) as follows

$$\left.\begin{array}{c}(u_{11}-\omega^2 v_{11})a_1+(u_{12}-\omega^2 v_{12})a_2+\cdots+(u_{1n}-\omega^2 v_{1n})a_n=0\\ (u_{21}-\omega^2 v_{21})a_1+(u_{22}-\omega^2 v_{22})a_2+\cdots+(u_{2n}-\omega^2 v_{2n})a_n=0\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ (u_{n1}-\omega^2 v_{n1})a_1+(u_{n2}-\omega^2 v_{n2})a_2+\cdots+(u_{nn}-\omega^2 v_{nn})a_n=0\end{array}\right\} \tag{43}$$

in which

$$u_{ik}=u_{ki}=\int_0^L K_x\frac{d\Psi_i}{dx}\frac{d\Psi_k}{dx}dx \tag{44}$$

$$v_{ik}=v_{ki}=v_{ik}^{(1)}+v_{ik}^{(2)}=v_{ki}^{(1)}+v_{ki}^{(2)} \tag{45}$$

$$v_{ik}^{(1)}=v_{ki}^{(1)}=\int_0^L \overline{m}_x\Psi_i\Psi_k dx \tag{46}$$

$$v_{ik}^{(2)}=v_{ki}^{(2)}=\sum_{\zeta=1}^{m} m_\zeta\Psi_i(x_\zeta)\Psi_k(x_\zeta) \tag{47}$$

The frequency equation can be found from Eq. (43) as follows:

$$\left\{\begin{array}{cccc}(u_{11}-\omega^2 v_{11}) & (u_{12}-\omega^2 v_{12}) & \cdots & (u_{1n}-\omega^2 v_{1n})\\ (u_{21}-\omega^2 v_{21}) & (u_{22}-\omega^2 v_{22}) & \cdots & (u_{2n}-\omega^2 v_{2n})\\ \cdots & \cdots & \cdots & \cdots\\ (u_{n1}-\omega^2 v_{n1}) & (u_{n2}-\omega^2 v_{n2}) & \cdots & (u_{nn}-\omega^2 v_{nn})\end{array}\right\}=0 \tag{48}$$

If n is a large number, it is better to rewrite Eq. (43) as

$$([U]-\omega^2[V])\{a\}=0 \tag{49}$$

or

$$[V]\{a\}=\bar{\lambda}[U]\{a\} \tag{50}$$

where

$$\bar{\lambda}=\frac{1}{\omega^2} \tag{51}$$

The elements of $[U]$ and $[V]$ are u_{ik} and v_{ik}, respectively. Because $[U]$ is a symmetric matrix and it is also a positive definition matrix, it is possible to find a matrix $[R]$ to make $[U]$ as the product:

$$[U]=[R][R]^T \tag{52}$$

in which $[R]$ is a lower triangular matrix.

Substituting Eq. (52) into Eq. (50) gives

$$[D]\{A\}=\bar{\lambda}\{A\} \tag{53}$$

in which

$$[D]=[R]^{-1}[V]([R]^{-1})^T \tag{54}$$

Because $[V]$ is a symmetric matrix, $[D]$ is also a symmetric matrix. $\bar{\lambda}_j$ and $\{A\}_j$ can be found by using the well-known QR method. The j-th circular natural frequency and the corresponding mode shape are

$$\left.\begin{array}{l} \omega_j=\sqrt{\dfrac{1}{\bar{\lambda}_j}} \\ \{a\}_j=([R]^T)^{-1}\{A\}_j \end{array}\right\} \tag{55}$$

As discussed above, in many cases, the variations of mass and stiffness of high-rise structures can be described by Eq. (3), and if the coordinate function $\psi_i(x)$ (i=1, 2, ..., n) is selected as

$$\psi_i=e^{\beta\frac{x}{L}}\sin\frac{(2i-1)\pi x}{2L} \tag{56}$$

then substituting Eqs. (3) and (56) into Eqs. (44)-(47), we obtain

$$u_{ik}=u_{ki}=(-1)^{i+k}\frac{K_0\beta}{2L} \tag{57}$$

$$u_{kk}=\frac{K_0}{8L}[\beta^2+(2k-1)^2\pi^2+4\beta] \tag{58}$$

$$v_{ik}^{(1)}=v_{ki}^{(1)}=\frac{1}{2}\bar{m}_0L(\beta-b)\left[\frac{(-1)^{i+k}e^{\beta-b}+1}{(\beta-b)^2+(i+k-1)^2\pi^2}+\frac{(-1)^{i+k}e^{\beta-b}-1}{(\beta-b)^2+(i-k)^2\pi^2}\right] \tag{59}$$

$v_{kk}^{(1)}$ can be determined by setting i=k in Eq. (59).

$$v_{ik}^{(2)}=v_{ki}^{(2)}=\frac{1}{2}\sum_{\zeta=1}^{m}e^{b\frac{x_\zeta}{L}}m_\zeta\left[\cos\frac{(i-k)\pi x_\zeta}{L}-\cos\frac{(i+k-1)\pi x_\zeta}{L}\right] \tag{60}$$

$v_{kk}^{(2)}$ can be determined by setting i=k in Eq. (60).

5. Numerical example

The Guangzhou Hotel Building (27 stories) is a shear-wall structure with a variable cross-section. Based on the full-scale measurement of free vibration of this building (Li *et al.* 1994), the Guangzhou Hotel Building can be treated as a cantilever bar with a variable cross-section (Fig. 1) for free vibration analysis. The procedures for determining the dynamic characteristics of this tall building in the vertical direction by using the methods proposed in this paper are as follows:

5.1 Ritz method

(1) Determination of the mass per unit length (Fig. 4)

The axial stiffness and mass per unit length (Figs. 4 and 5) of the Guangzhou Hotel Building vary with height. For simplicity, the building is treated as a variable cross-section cantilever bar, as shown in Fig. 1. Because the variation of the mass per unit length and the lumped mass attached at the top of the building are comparatively small, it is reasonable to assume that the mass is uniformly distributed along the height of the building (Fig. 4).

The mass per unit length, $\bar{m}$, is found as: $\bar{m}$=38,014.2 kg/m. The evaluated distribution of mass (dotted line) is plotted in Fig. 4.

(2) Evaluation of the axial stiffness, K_x, (Fig. 5)

The axial stiffness, K_x, is assumed as: $K_x = \alpha e^{-\beta \frac{x}{L}}$.

According to the following information for the axial stiffness of this building:

at $x=0$, $EF_0=133.14\times10^9$ N

$x=L$, $EF_L=69.27\times10^9$ N

The parameters, α, β are determined as

$\alpha=EF_0=133.14\times10^9$ N

$\beta=Ln(K_0)-Ln(K_L)=0.6534$

The evaluated distribution of axial stiffness (dotted line) is shown in Fig. 5.

(3) Evaluation of the natural frequencies

The formulae derived based on the Ritz method are used to determine the longitudinal natural frequencies of this tall building. Using Eqs. (57)-(59) obtains

$u_{11}=2.82707\times10^9$

$u_{12}=u_{21}=-5.7233\times10^8$

$u_{22}=2.01171\times10^{10}$

$v_{11}=2.30636\times10^6$

$v_{12}=v_{21}=-2.89667\times10^5$

$v_{22}=2.06941\times10^5$

Substituting u_{jk} and v_{ik} into Eq. (48) gives

$\omega^4-11072.14721\omega^2+12059322.85=0$

Solving the above equation obtains

$\omega_1=34.9887$ rad/sec, $f_1=5.5686$ Hz

$\omega_2=99.2305$ rad/sec, $f_2=15.7930$ Hz

If the first four terms in Eq. (41) are considered, we obtain

$\omega_1=34.9251$ rad/sec, $f_1=5.5585$ Hz

$\omega_2=96.4763$ rad/sec, $f_2=15.1346$ Hz

The longitudinal fundamental frequency obtained by the full-scale measurement (Li *et al.* 1994) is 5.47 Hz. It is clear that the computed value in terms of the proposed procedure corresponds closely to the measured one.

If the lumped mass (M=30612.2 kg) attached to the top of the building is considered, then, $v_{ik}^{(2)}$ must be included in v_{ik}, and the calculated longitudinal fundamental frequency is 5.5597 Hz which is in good agreement with the measured data.

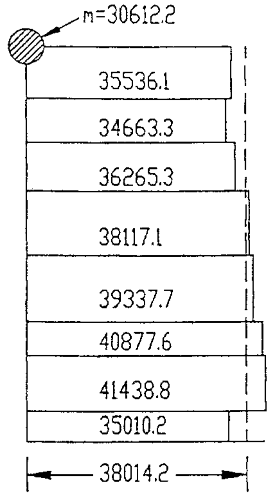

Fig. 4 Mass distribution of the tall buidling

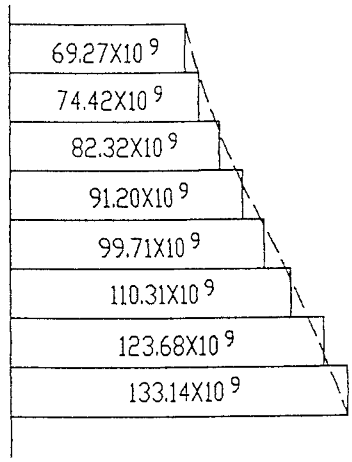

Fig. 5 Stiffness distribution of the tall buidling

(4) Calculation of the longitudinal fundamental vibration mode shape

After computing the first natural frequency f_1, the first mode shape, $X_1(x)$, can be determined from Eq. (43). The calculated results are shown in Table 1 and Fig. 6.

5.2 The analytical method

When the distributions of mass intensity and axial stiffness are described by Eq. (3), the analytical solutions can be expressed in terms of the Bessel functions.

According to the values of mass and axial stiffness obtained above, we have

$a=\overline{m}=38014.2$ kg/m, $b=0$, $\beta=0.6534$, $L=76$m

In this case, the value of v is given by

$$v=\frac{\beta-b}{\beta}=1$$

and the frequency equation can be established from Eq. (11) as follows

$$Y_1(\lambda)J_0(\lambda A)=J_1(\lambda)Y_0(\lambda A)$$

in which $A=e^{\frac{\beta}{2}}=1.3864$

Solving the frequency equation obtains

$\lambda_1=4.3417$, $\lambda_2=11.9333$

The natural frequencies can be found from Eq. (5) as

$\omega_1=34.9248$ rad/sec, $f_1=5.5584$ Hz

$\omega_2=96.0014$ rad/sec, $f_2=15.2791$ Hz

It can be seen that the first two natural frequencies determined by the Ritz method are very close to those calculated by the analytical method if the first four terms in Eq. (41) are considered.

If the lumped mass (M=30612.2 kg) attached to the top of the building is considered, then, the frequency equation is Eq. (9), in which

$v=1$, $m=30612.2$

The calculated longitudinal fundamental natural frequency for this case is 5.55 Hz which is also in good agreement with the measured data. The corresponding mode shape can be determined from Eq. (7), i.e.,

$$X=\xi\left[J_1(\lambda\xi)-\frac{J_1(\lambda)}{Y_1(\lambda)}Y_1(\lambda\xi)\right]$$

The values of the first mode shape are calculated and shown in Table 1 and Fig. 6. The natural frequency of the second mode is found as 15.1937 Hz.

5.3 The transfer matrix procedure

This tall building can be simplified as an 8-step cantilever bar, each step has constant mass and stiffness as shown in Fig. 4 and Fig. 5, respectively. The special solutions, $S_{i1}(x)$ and $S_{i2}(x)$, for the ith step are given by Eq. (13). The transfer matrix is as follows

$$[T]=[T_1][T_2]\cdots[T_8]$$

in which

$$[T_i]=\begin{bmatrix}\sin\lambda_i x_{i1} & \cos\lambda_i x_{i1}\\ \lambda_i K_i\cos\lambda_i x_{i1} & -\lambda_i K_i\sin\lambda_i x_{i1}\end{bmatrix}\begin{bmatrix}\sin\lambda_i x_{i0} & \cos\lambda_i x_{i0}\\ \lambda_i K_i\cos\lambda_i x_{i0} & -\lambda_i K_i\sin\lambda_i x_{i0}\end{bmatrix}^{-1}$$

Table 1 Longitudinal fundamental mode shape of Guangzhou hotel building

$x(m)$	0	5.35	15.25	21.25	33.85	43.15	52.45	61.75	76
$X_1(x)$ computed	[0] 0 (0)	[0.103] 0.0631 (0.102)	[0.268] 0.1922 (0.268)	[0.426] 0.3242 (0.427)	[0.566] 0.4613 (0.567)	[0.712] 0.5984 (0.712)	[0.838] 0.7375 (0.842)	[0.930] 0.8501 (0.935)	1
$X_1(x)$ measured	0	0.100	0.257	0.417	0.560	0.710	0.837	0.929	1

Note: The values in parentheses and square brackets are those calculated by the analytical method and the transfer matrix procedure, respectively.

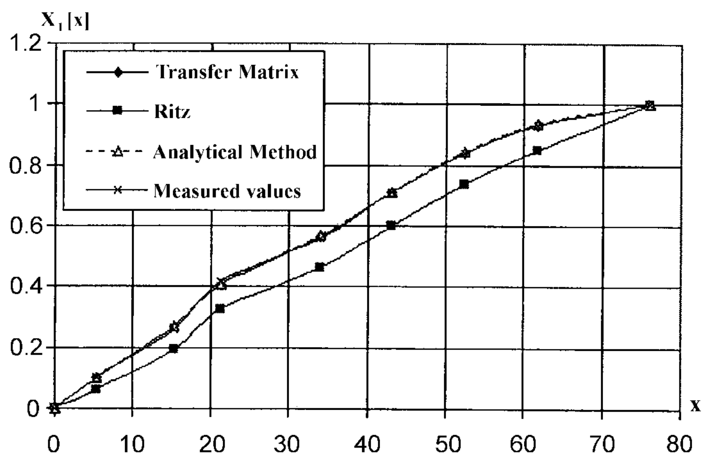

Fig. 6 The fundamental mode shape

As mentioned above, $[T]$ has the form as Eq. (22), and the frequency equation is Eq. (27), solving the equation obtains f_1=5.5643 Hz. It is obvious that the difference between the result calculated by use of the step varying distributions of stiffness and mass and that obtained based on the model of a one-step cantilever bar with continuously varying stiffness and mass is so small that it can be neglected. This suggests that it is reasonable to simplify a multi-step bar with step varying distributions of stiffness and mass as a one-step bar with continuously distributed stiffness and mass when the number of steps is large.

Setting N_{10}=1 and using Eq. (29) obtain the values of the first mode shape which are almost the same as those determined by the analytical method (see Table 1).

Because the transfer matrix procedure and the recurrence formula procedure are the exact approaches, the results obtained by the two procedures should be the same. Thus, it is not necessary to present the results by the recurrence formula procedure herein.

The fundamental mode shape measured by Li *et al*. (1994) is also presented in Table 1 and shown in Fig. 6 for comparison purposes. The values of the fundamental mode shape calculated by the proposed methods are in good agreement with the measured data. It should be noted that using the aforementioned procedures, the higher mode shapes could be also determined.

The numerical results show that a multi-step bar may be simplified as a one-step bar with continuously varying cross-section for free longitudinal vibration analysis when the number of steps is large. The natural frequencies computed by the Ritz method are in good agreement with the measured data and are very close to those calculated by the analytical method and the transfer matrix procedure, but the fundamental mode shape computed by the Ritz method is not very close to the measured one. In order to improve the calculation accuracy of the Ritz method, it is necessary to take more terms in Eq. (41). For example, if i=4 in Eq. (41), then the first mode shape obtained by the Ritz method is very close to the measured one and the first two natural frequencies are almost the same as those obtained by the analytical method. It is found in the present study that in general, if the first r mode shapes are required to be determined, then i (the number of terms in Eq. 41) should be greater than $(r+3)$.

6. Conclusions

Several approaches to evaluate the natural frequencies and mode shapes in the vertical direction for cantilevered tall structures which are treated as one-step cantilever bars or multi-step cantilever bars with varying cross-section have been proposed in this paper. Using appropriate transformations, exact analytical solutions to determine the longitudinal natural frequencies and mode shapes for a one step non-uniform bar are derived by selecting suitable expressions, such as exponential functions, for the distributions of mass and axial stiffness. The frequency equation of a multi-step bar is established using the approach that combines the transfer matrix procedure or the recurrence formula and the closed-form solutions of one step bars, leading to a single frequency equation for any number of steps. The Ritz method is also applied to determine the natural frequencies and mode shapes in the vertical direction for cantilevered tall structures with variably distributed stiffness and mass. All formulae proposed in this paper for determining the free longitudinal vibrations of cantilevered tall structures are simple and convenient for engineering applications. The numerical example shows that the difference between the results calculated by use of the step varying distributions of stiffness and mass and those obtained based on the model of a one-step cantilever bar with continuously varying stiffness and mass is so small that it can be neglected. This suggests that it is reasonable to simplify a multi-step bar with step varying distributions of stiffness and mass as a one-step bar with continuously distributed stiffness and mass when the number of steps is large. It is shown that the calculated fundamental natural frequency and mode shape of the Guangzhou Hotel Building are very close to the full scale measured data, suggesting that the calculation methods proposed in this paper are applicable to engineering application and practice. The example also demonstrates that the selected expressions are suitable for describing the distributions of mass and axial stiffness of typical tall buildings, and the selected coordinate functions for the Ritz method make the computing process to converge rapidly. It is found that if the Ritz method is used to determine the first r mode shapes, in general, the number of terms in Eq. (41) should be greater than (r+3). However, if only the first two natural frequencies are required to be calculated, taking the first four terms in Eq. (41) could provide accurate results for practical applications.

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References

Abrate, S. (1995), “Vibration of rods with uniformly tapered sections”, *J. of Sound and Vib.*, **185**, 703-716.

Bapat, C.N. (1995), “Vibration of rods with uniformly tapered sections”, *J. of Sound and Vib.*, **185**(1), 185-189.

Clough, R.W., and Penzien, J. (1993), *Dynamics of Structures*, McGraw-Hill Book Company.

Ellis, B.R., and Jeary, A.P. (1980), “A review of recent work on the dynamic behaviour of tall buildings at various amplitudes”, *Proc. of the 7th World Conf. on Earthquake Eng.*, Istanbul.

Jeary, A.P., and Sparks, P.R. (1977), “Some observations on the dynamic sway characteristics of concrete structures”, *Proc. of Symposium on Vibrations in Concrete Struct.*, New Orleans.

Kumar, B.M., and Sujith, R.I. (1997), “Exact solutions for the longitudinal vibration of non-unformed rods”, *J. of Sound and Vib.*, **207**(5), 721-729.

Li, Q.S. (1999), “Flexural free vibration of cantilevered structures of variable stiffness and mass”, *Struct. Eng. and Mech., An Int J.*, **8**(3), 243-256.

Li, Q.S. (2000), “Exact solutions for free longitudinal vibrations of non-uniform rods”, *J. of Sound and Vibration*, **234**(1), 1-19.

Li, Q.S., Cao, H., and Li, G. (1994), “Analysis of free vibrations of tall buildings”, *J. of Eng. Mech.*, ASCE, **120**(9), 1861-1876.

Li, Q.S., Cao, H., and Li, G. (196), “Static and dynamic analysis of straight bars with variable cross-section”, *Comput. & Struct.*, **59**(6), 1185-1191.

Li, Q.S., Fang, J.Q., and Jeary, A.P. (1998), “Calculation of vertical dynamic characteristics of tall buildings with viscous damping”, *J. of Solid & Struct.*, **35**(24), 3165-3176.

Li, Q.S., Fang, J.Q., and Jeary, A.P. (2000), “Free vibration analysis of cantilevered tall structures under various axial loads”, *Eng. Struct.*, **22**(5), 525-534.

Tuma, J.J., and Cheng, F.Y. (1983), *Dynamic Structural Analysis*, McGraw-Hill Book Company.

Wang, G.Y. (1978), *Vibration of Building and Structures*, Science and Technology Press, Beijing, 168-178.