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Postbuckling strength of an axially compressed elastic circular cylinder with all symmetry broken

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Abstract. Axially compressed circular cylinders repeat symmetry-breaking bifurcation in the postbuckling region. There exist stable equilibria with all symmetry broken in the buckled configuration, and the minimum postbuckling strength is attained at the deep bottom of closely spaced equilibrium branches. The load level corresponding to such postbuckling stable solutions is usually much lower than the initial buckling load and may serve as a strength limit in shell stability design. The primary concern in the present paper is to compute these possible postbuckling stable solutions at the deep bottom of the postbuckling region. Two computational approaches are used for this purpose. One is the application of individual procedures in computational bifurcation theory. Path-tracing, pinpointing bifurcation points and (local) branch-switching are all applied to follow carefully the postbuckling branches with the decreasing load in order to attain the target at the bottom of the postbuckling region. The buckled shell configuration loses its symmetry stepwise after each (local) branch-switching procedure. The other is to introduce the idea of path jumping (namely, generalized global branch-switching) with static imperfection. The static response of the cylinder under two-parameter loading is computed to enable a direct access to postbuckling equilibria from the prebuckling state. In the numerical example of an elastic perfect circular cylinder, stable postbuckling solutions are computed in these two approaches. It is demonstrated that a direct path jump from the undeformed state to postbuckling stable equilibria is possible for an appropriate choice of static perturbations.

Key words: circular cylindrical shell; symmetry-breaking bifurcation; branch-switching; path jump; stable postbuckling solution.

1. Introduction

There has been a discrepancy between theory and measurement for stability behaviour of thin shells, and this has been a strong motivation for research efforts in shell buckling since the work of Donnell (1934), Kármán and Tsien (1941) and Koiter (1945). Indeed, the asymptotic theory of Koiter (1945) provides mathematically a clear insight into the initial postbuckling behaviour of shells. One major reason for the concern in the postbuckling behaviour is to see how the buckled shell remains in equilibrium and also to find possible stable equilibria at a load level much lower than the initial buckling load.

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The present attention is therefore limited to the postbuckling analysis of axially compressed circular cylinders. Almroth (1963) increased stepwise the number of free variables involved in the displacement functions for the total potential energy to estimate the magnitude of the minimum postbuckling load of a cylinder. Hoff *et al.* (1966) also considered a large number of terms in the double Fourier series to represent accurately the radial displacement field of the buckled shell configuration. An increase in the number of terms in the trigonometric series lowered significantly the equilibrium load-shortening curve below that presented by Almroth (1963). Maewal and Nachbar (1977) computed in a finite element approach the postbuckling equilibrium paths of a clamped circular cylindrical shell under axial compression and attained a stable equilibrium state at the deep bottom of the postcritical region. The computed contour maps of radial displacement were qualitatively in good agreement with experimental observations by Yamaki (1984).

Regardless of these excellent works, however, the postbuckling behaviour of axially compressed thin-walled circular cylindrical shells is still complicated due to an infinite number of closely spaced postbuckling branches and bifurcation points. In laboratory tests by Yamaki (1984), the chaotic postbuckling behaviour frequently manifests itself in a dynamic mode jump. The equilibrium state dynamically moves from one stable branch to another with a sudden change in the number of buckles and tiers in the deformed configuration (see Fig. 1).

From the computational aspect, dynamic analysis has been well established for simulating the dynamic buckling process. Riks *et al.* (1996), for example, investigated mode jumping phenomena by direct time integration of the discretized equations of motion. Choong and Ramm (1996, 1998) also studied the buckling process of shells by computing the time history of structural response and simulated the propagation process of buckles in a long shallow cylindrical panel.

In static postbuckling analysis, however, there is little control on the computation in postbuckling. All the difficulties, such as closely spaced postbuckling branches and bifurcation points, make it difficult to attain the lowest possible stable equilibrium solution (point S in Fig. 1) in the deep postbuckling region and it still remains an active research area in computational mechanics.

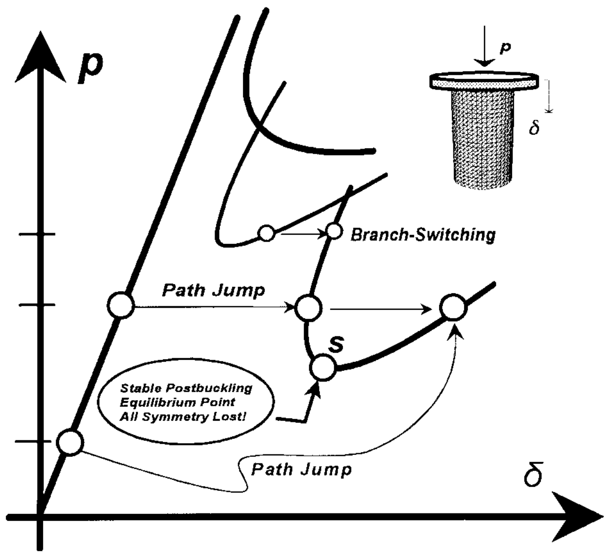

Fig. 1 Postbuckling behavior of a compressed circular cylinder

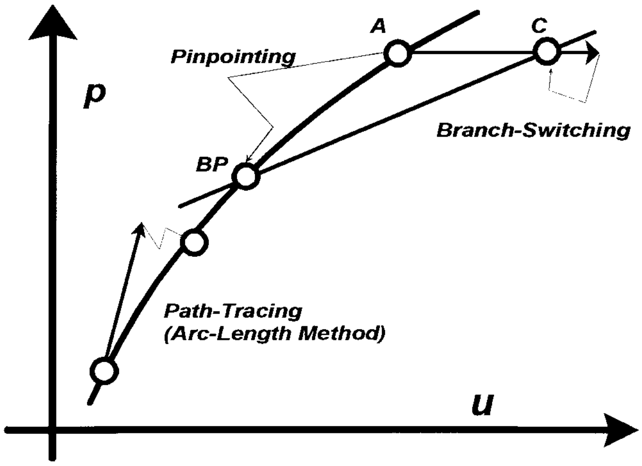

Fig. 2 Path-tracing, pinpointing and branch-switching

In the last decade, computational techniques for postbuckling analysis have been highly developed (Bushnell 1989). Among them, path-tracing based upon arc-length method (for example, Crisfield 1979 and Ramm 1980), is the most fundamental procedure for studying the postbuckling behaviour of nonlinear shell structures (see Fig. 2). The extended system (Wriggers *et al*. 1987) and branch-switching technique (Wagner and Wriggers 1988, Fujii and Choong 1992, Fujii and Ramm 1997, and Fujii and Okazawa 1997a, 1997b) explored a powerful computational tool for static stability behaviour of axially compressed cylinders. All these computational techniques are useful in static bifurcation analysis to surmount the obstacles in the postbuckling region.

Wohlever and Healey (1995) proposed a group theoretic approach to systematically compute the postbuckling equilibrium paths of a compressed cylindrical shell. The subsequent diagnosis for symmetry-breaking bifurcation enables the block-diagonalization of the stiffness matrix. Deml and Wunderlich (1997) established a nonlinear procedure to find the worst imperfection shape for elastic shells. The imperfections were introduced as free variables and the load level of the limit point was minimized to obtain the lowest possible load level in the deep postcritical range. Except these works, there seems to exist no other static buckling analysis to compute the stable postbuckling solutions at the bottom of the postbuckling region.

The primary concern in this present paper is to compute postbuckling equilibrium solutions, which are not only stable with the positive definite stiffness matrix, but also have all symmetry broken in the deformed configuration to exclude further symmetry-breaking bifurcation. To reach the target (point *S* in Fig. 1) at the bottom of the postbuckling region, two different computational approaches are employed.

The first is to apply the procedures in computational bifurcation theory (Fujii and Choong 1992, Fujii and Ramm 1997, and Fujii and Okazawa 1997a, 1997b), namely path-tracing, pinpointing bifurcation points and (local) branch-switching (see Fig. 2). For this purpose, unstable equilibrium branches, which are usually of no interest in engineering reality, have to be computed, because they may be connected to the target solution. The closeness of an infinite number of postbuckling branches and bifurcation points may make it difficult to attain the lowest possible stable equilibrium solution in the deep postbuckling region, so that careful treatments including restart are needed in dialog-style computation. The second is to introduce a direct path jump (Fujii *et al*. 1998, 2000, and

Fujii and Noguchi 1999), triggered by static perturbations, from the prebuckling state to postbuckling equilibria. With an appropriate choice for the load perturbation, it is possible to access the target via a shortcut jump (see Fig. 1). All the computational procedures are applied to the numerical example of a perfect circular cylinder.

2. Computational bifurcation theory

In this section, the governing equations in computational bifurcation theory are briefly outlined. For the linearized equations and individual computational steps, the previous work of the writers (Fujii and Choong 1992, Fujii and Ramm 1997, and Fujii and Okazawa 1997a) may be referred to.

2.1 Path-Tracing

For an elastic cylinder with N nodal degrees-of-freedom $\boldsymbol{u}$ subjected to axial compression represented by a load parameter p, the nonlinear equilibrium equations may be written in the form

$$\boldsymbol{E}(\boldsymbol{u}, p) = \boldsymbol{0} \tag{1}$$

Here, $\boldsymbol{E}$ is the out-of-balance load vector given by

$$\boldsymbol{E}(\boldsymbol{u}, p) = \boldsymbol{R}(\boldsymbol{u}) - p\boldsymbol{e} \tag{2}$$

where $\boldsymbol{R}(\boldsymbol{u})$ and $\boldsymbol{e}$ are the internal resistance and the reference load vector, respectively. The arc-length method is a well-known continuation scheme to trace the equilibrium path in R^{N+1} defined by Eq. (1). A large number of papers on the arc-length method (for example, Crisfield 1979, and Ramm 1980) are already available, and the path-tracing procedure is very popular in computational mechanics. Therefore, no more detailed information are necessary here for predictor and corrector steps during path-tracing.

2.2 Pinpointing

The stability point on the equilibrium path may be identified by the set of equations

$$\begin{cases} \boldsymbol{E}(\boldsymbol{u}, p) = \boldsymbol{0} \\ \lambda = 0 \end{cases} \tag{3)-(4}$$

where λ is the eigenvalue of the standard eigenproblem

$$\boldsymbol{K}\theta = \lambda\theta \tag{5}$$

with the corresponding eigenvector (buckling mode) θ and tangent stiffnss matrix $\boldsymbol{K}$ such that

$$\boldsymbol{K} = \frac{\partial \boldsymbol{R}}{\partial \boldsymbol{u}} \tag{6}$$

The pinpointing iteration (see Fig. 2) is nothing but the Newton-Raphson method to repeat the solution of the linearized equations

$$\begin{cases} \boldsymbol{K\delta u} - \boldsymbol{\delta p e} = -\boldsymbol{E} \\ \boldsymbol{\delta\lambda} = -\boldsymbol{\lambda} \end{cases} \tag{7)-(8}$$

for ($\delta \boldsymbol{u}$, δp). The corrector $\delta\lambda$ of the eigenvalue λ may be computed by

$$\boldsymbol{\delta\lambda} = \boldsymbol{\theta}^T \Delta\boldsymbol{K}\boldsymbol{\theta} \tag{9}$$

with the eigenvector θ updated at each iteration point. A finite difference method is useful for computing the small change $\Delta\boldsymbol{K}$ of the stiffness matrix due to $\delta\boldsymbol{u}$ such that

$$\delta\boldsymbol{u} = \delta\boldsymbol{u}_E + \delta p\boldsymbol{u}_e \tag{10}$$

with

$$\delta\boldsymbol{u}_E = \boldsymbol{K}^{-1}(-\boldsymbol{E}) \tag{11}$$

and

$$\boldsymbol{u}_e = \boldsymbol{K}^{-1}(+\boldsymbol{e}) \tag{12}$$

from Eq. (7).

The pinpointing iteration described above and the extended system (Wriggers *et al*. 1987) are applicable to simple bifurcation, for which there exists the unique critical eigenpair (λ, θ) during pinpointing iteration. However, in the case of multiple (double) bifurcation, the critical eigenvectors corresponding to two equal eigenvalues may not be discerned and some innovative idea is needed to separate the multiple eigenvalues during pinpointing iteration. One idea to pinpoint the multiple bifurcation point has recently been proposed by Fujii *et al*. (2001) and will be used in the numerical example of the present study for multiple bifurcation.

2.3 Branch-switching

It is well known that the symmetric bifurcation path exactly branches in the direction of the critical eigenvector θ (Fujii and Okazawa 1997b). When the higher-order terms of the equilibrium equations are not available in bifurcation analysis, there is no way to identify symmetric or

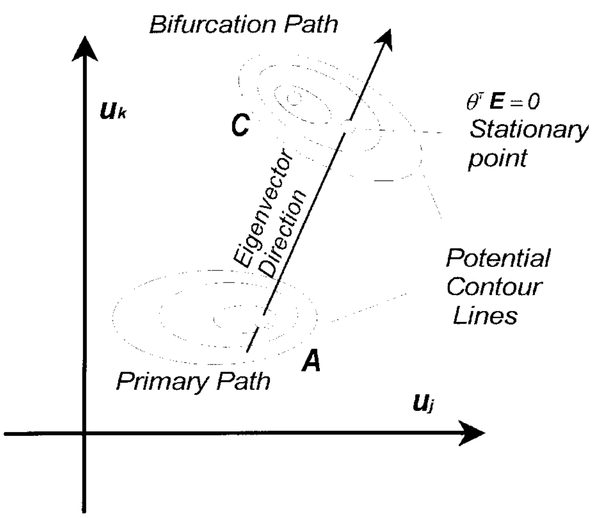

Fig. 3 Line search for a stationary point

asymmetric bifurcation. It is, therefore, usual in computational practice to first initiate branch-switching at a regular equilibrium point at $p=p_A$ near the pinpointed bifurcation point (see Fig. 2). The attained bifurcation path is then traced, and the configuration of the computed bifurcation paths will show that it is symmetric or asymmetric bifurcation.

For local branch-switching at a fixed load level $p=p_A$ ($A \rightarrow C$ in Fig. 2), line search is recommended (Fujii and Choong 1992, and Fujii and Ramm 1997) to find a stationary point of the potential in the direction of the critical eigenvector θ (see Fig. 3).

$$\boldsymbol{u}(q) = \boldsymbol{u}_A + q\theta \tag{13}$$

The magnitude q is determined by the following termination criterion:

$$\theta^T \boldsymbol{E}(\boldsymbol{u}, p_A) = 0 \tag{14}$$

For global branch-switching in a highly complex situation of equilibrium branches very close together, a more robust idea is to trace a bypass defined by

$$\boldsymbol{E}(\boldsymbol{u}, p_A) - \boldsymbol{q}\theta = \boldsymbol{0} \tag{15}$$

in R^{N+1} for $\boldsymbol{u}$ and q may well work with the expectation that the defined bypass intersects the bifurcation paths ($\boldsymbol{E}=\boldsymbol{0}$) at several points, when $q=0$ (see Fig. 4). For multiple bifurcation, θ may be a linear combination of the critical eigenmodes (Fujii and Ramm 1997, Fujii and Okazawa 1997a). The path-tracing procedure for the bypass defined by Eq. (15) differs from that for the equilibrium path defined by Eq. (1) only in loading parameter (q or p).

3. Dynamic path jump

The equations of motion for describing dynamic mode jumping (Riks *et al.* 1996) or dynamic buckling (Choong and Ramm 1996, 1998) are

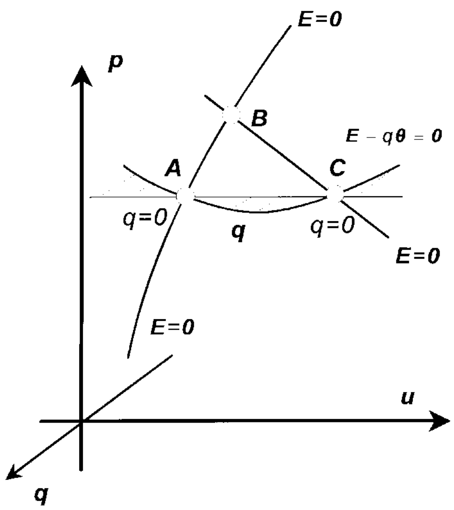

Fig. 4 Local branch-switching

$$\boldsymbol{M}\ddot{\boldsymbol{u}} + \boldsymbol{C}\dot{\boldsymbol{u}} + \boldsymbol{R}(\boldsymbol{u}) - p\boldsymbol{e} = \boldsymbol{0} \tag{16}$$

with mass matrix $\boldsymbol{M}$, damping matrix $\boldsymbol{C}$, nodal accelerations $\ddot{\boldsymbol{u}}$ and nodal velocities $\dot{\boldsymbol{u}}$. The time-dependent terms may be put together into the brackets { } to give

$$\boldsymbol{R}(\boldsymbol{u}) - p\boldsymbol{e} - \{-\boldsymbol{M}\ddot{\boldsymbol{u}} - \boldsymbol{C}\dot{\boldsymbol{u}}\} = \boldsymbol{0} \tag{17}$$

The terms in the brackets { } represent dynamic effects (inertia and damping) and may not be directly controlled during time integration. All these dynamic disturbances disappear at the end of the transient response ($t \to \infty$), so that a static stable equilibrium solution, $\boldsymbol{u}$ and p, satisfying Eq. (1) is attained (Lyapunov's stability theorem) after dynamic buckling and mode jumping.

4. Static path jump

In the present paper, the dynamic disturbance in Eq. (17) is simply replaced by a static perturbation vector $\boldsymbol{f}$ multiplied with a load parameter q, which acts on the structure during jumping. The static response of the structure subjected to the two-parameter loading (p and q) is then described by

$$\boldsymbol{R}(\boldsymbol{u}) - p\boldsymbol{e} - q\boldsymbol{f} = \boldsymbol{0} \tag{18}$$

These N equations and one more subsidiary equation ($h=0$) define a path in R^{N+2} for (N+2) variables, $\boldsymbol{u}$, p and q, guiding the current equilibrium solution to other equilibria.

$$\begin{cases} \boldsymbol{R}(\boldsymbol{u}) - p\boldsymbol{e} - q\boldsymbol{f} = \boldsymbol{0} \\ h(\boldsymbol{u}, p, q) = 0 \end{cases} \tag{19)-(20}$$

Here, $h(\boldsymbol{u},p,q)$ is a scalar function, representing an additional constraint, and may be nonlinear in $\boldsymbol{u}$, p and q. For engineering purposes, however, the simplest choice of the function is

$$h(\boldsymbol{u}, p, q) = p - p_A \tag{21}$$

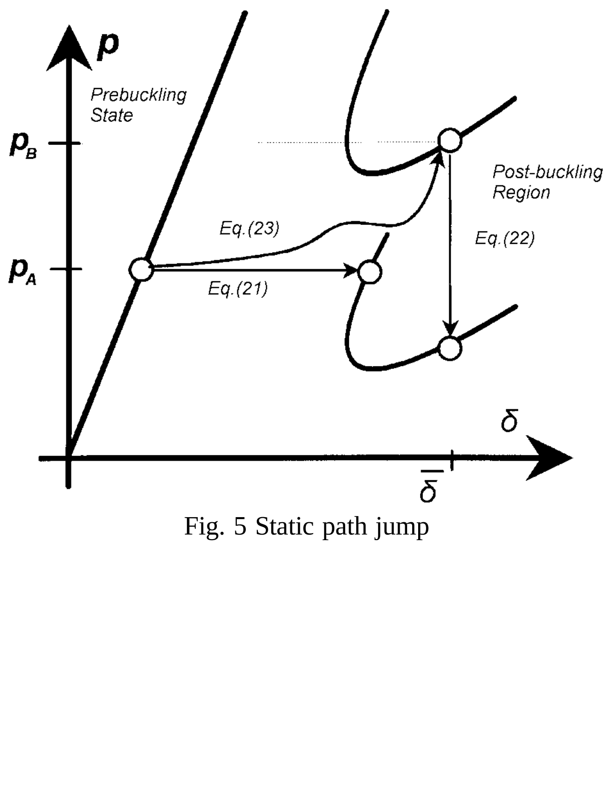

Fig. 5 Static path jump

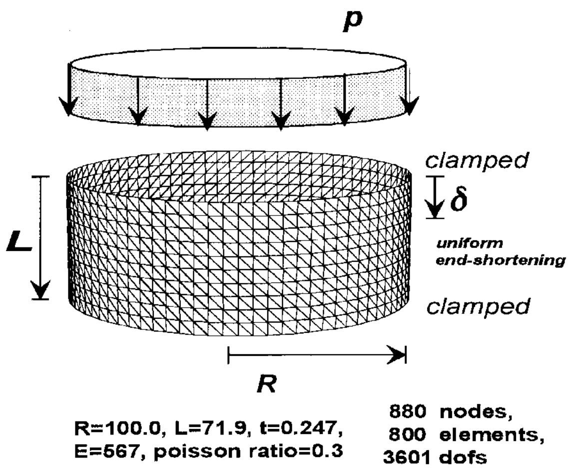

Fig. 6 Perfect circular cylindrical shell

for a static jump at the fixed load level $p{=}p_A$ (see Fig. 5). This is equivalent to Eq. (15), when $\boldsymbol{f} = \boldsymbol{\theta}$. The end-shortening δ of the cylindrical shell may be fixed to $\overline{\delta}$ as during jumping and

$$h(\boldsymbol{u}, p, q) = \delta - \overline{\delta} \tag{22}$$

may be chosen (see Fig. 5). One more possible choice for constraints may be a parabolic constraint imposed on p and q

$$h(\boldsymbol{u}, p, q) = q + q_{\max}\frac{4}{(p_B - p_A)^2}(p - p_A)(p - p_B) \tag{23}$$

to trigger a jump, when the load parameter p changes from p_A to p_B (see Fig. 5). Here, $q_{\max}$ is the maximum magnitude of q to be specified at the beginning of jump. If $q_{\max}$ is too small, it may not jump beyond the barrier of potential and may stay in the prebuckling state. If $q_{\max}$ is too large, the load q becomes too strong and a postbuckling solution may not be found for the comparatively small p. The parabolic constraint seems to be less realistic in engineering but may be computationally useful in terms of the globally convergent nonlinear solution procedure. The procedure to trace the path defined by Eqs. (19) and (20) is executed in the usual predictor-corrector steps in R^{N+2}. When $q{=}0$, a possible equilibrium solution, $\boldsymbol{u}$ and p, for Eqs. (1) and (2) is attained. The proposed static path jumping is a generalization of the global branch-switching procedure in R^{N+1} described by Eq. (15).

The proposed static path jumping requires only an application of popular path-tracing procedures, which are available in most existing finite element codes. The computational features of dynamic and static path jumping are clear when compared to each other. The significant asset of static jump is that both stable and unstable equilibrium solutions are attainable and the return (jump back) to the initial starting point from the attained solution by tracing back the defined path (reversible two-way computation) is possible. The dynamic mode jump, on the contrary, proceeds only in one direction to a stable solution and there can be no return to the starting point (irreversible one-way computation).

Prior to static or dynamic jumping, there is generally no information on the attainable solution, which depends upon the initial conditions or static load vector ***f***. Regarding the static jump, however, the mode vector ***f*** may often serve to predict the attainable stable solution, when the eigenvector at the lower limit point in the postbuckling region is substituted into ***f***. This is illustrated in the following numerical example.

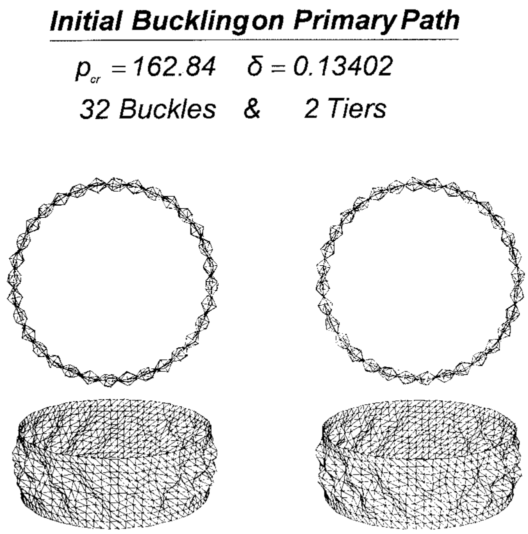

Fig. 7 Initial buckling modes

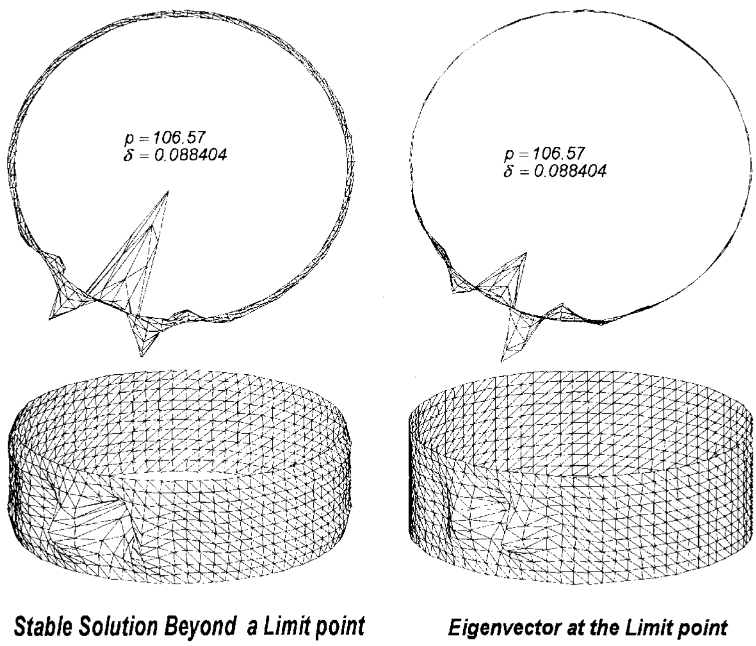

Fig. 8 One possible stable postbuckling solution and eigenvector

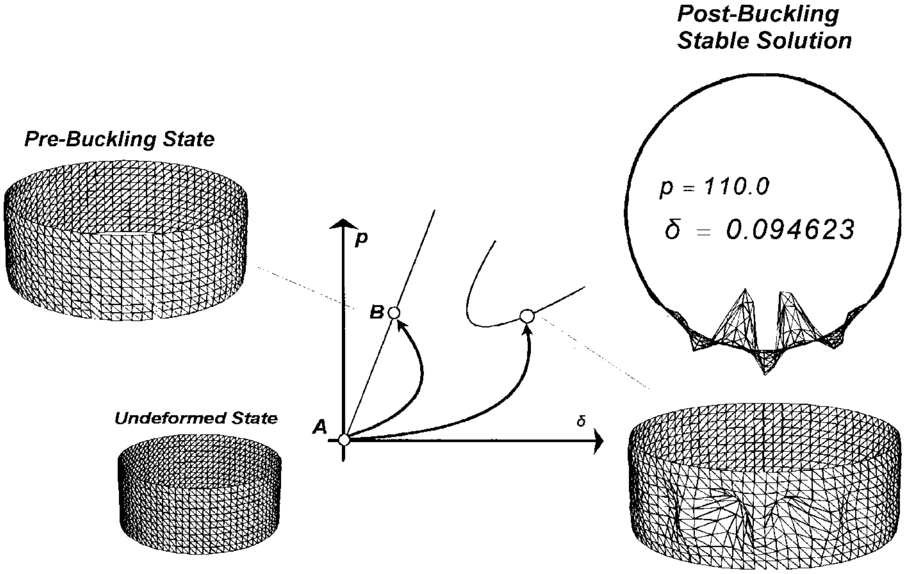

Fig. 9 Jump attempts from the undeformed state

5. Numerical example

A clamped-clamped circular cylindrical shell under axial compression, as illustrated in Fig. 6, was chosen for the numerical example. Because a portion of the cylindrical panel with symmetric constraints would not provide the target solution with all symmetry broken (Fujii *et al*. 2001), the total shell surface was idealized in the present study with a uniform end-shortening d for the compressed edge. There were 80×10 shell elements (Noguchi and Hisada 1993) and 880 nodes in the discretization with 3601 degrees-of-freedom. The total compressive load was assumed to be $p\times 1.0$ for the full cross-section, and the end-shortening δ corresponded to the load parameter p.

The first bifurcation point with two critical eigenvectors was computed and the multiple buckling modes had 32 buckles in the circumferential direction and 2 tiers in the longitudinal direction (see Fig. 7). After this initial buckling (P_{cr}=162.84 and δ_{cr}=0.13402), careful tracing of the descending branches in the p-δ plot, pinpointing bifurcation points and branch-switching enabled the calculations to proceed into the postbuckling region and to find several stable and non-symmetric solutions. Among these solutions, one example beyond a lower limit point is illustrated on the left-hand side in Fig. 8. It is hard to realize the non-symmetric configuration of the solution with the strongly localized deformation. However, the critical eigenvector, which is identical to the tangent vector of the equilibrium branch at the limit point, is non-symmetric, as shown on the right-hand side in Fig. 8, so that the illustrated equilibrium point is certainly one target solution.

What needs to be specified to trigger a static jump, are $\boldsymbol{f}$ in Eq. (18) and $q_{\max}$, p_A, p_B in Eq. (23). For the path jump from the undeformed and unloaded state, the eigenvector in Fig. 8, for example, was substituted into the vector $\boldsymbol{f}$, and the different magnitudes for $q_{\max}$, p_A, and p_B were tested in some trials to attain postbuckling equilibria including stable and unstable solutions (see Fig. 9). If the perturbation is too small (that is, $q_{\max}$ is too small), the prebuckling state is recovered, as illustrated in Fig. 9 on the left. For an appropriate choice of the parameters, the jump into postbuckling region is made successfully and the equilibrium solutions found. In Fig. 9 on the right, one such stable solution with all symmetry broken is shown. This is obviously different from the solution shown in Fig. 8. For the perfect shell model, there may exist more solutions at the lower load level, so that all the postbuckling equilibria computed in the present paper provide only an upper bound for the stable and non-symmetric solution existing at the lowest load level.

6. Conclusions

To attain possible stable equilibria existing at the deep bottom of the postbuckling region of a perfect cylindrical shell under axial compression, procedures in computational bifurcation theory were first applied to trace carefully postbuckling branches with the load decreasing to the target solutions. For bifurcation points, pinpointing and branch-switching were repeated until the deformed configuration of the perfect shell lost all symmetry and the stiffness matrix became positive-definite. The iterative procedure to pinpoint bifurcation points worked well when the critical eigenpair was identified during the iteration. However, the close proximity of branches and bifurcation points often manifested itself in unstable equilibrium iteration or in convergence to unexpected branches during path-tracing and branch-switching. A more sophisticated Newton-Raphson method is yet to be established to control the equilibrium iteration.

The static path jump into the postbuckling region was introduced and worked well in the numerical example. Stable postbuckling solutions were actually computed at load levels much lower than the initial buckling load. The direct jump from the prebuckling state into the postbuckling region is particularly significant in shell stability problems. The critical eigenvector at a limit point with all symmetry lost in the configuration is one of the most promising choices of the load disturbance $\boldsymbol{f}$. The alternative to this choice may be such that the vector $\boldsymbol{f}$ "excites" the shape of the predicted deformation at the target.

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