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The eigensolutions of wave propagation for repetitive structures

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Abstract. The eigen-equation of a wave travelling over repetitive structure is derived directly from the stiffness matrix formulation, in a form which can be used for the case of the cross stiffness submatrix $\boldsymbol{K}_{ab}$ being singular. The weighted adjoint symplectic orthonormality relation is proved first. Then the general method of solution is derived, which can be used either to find all the eigensolutions, or to find the main eigensolutions for large scale problems.

Key words: repetitive structure; eigenvalue problem; symplectic matrix; wave propagation.

1. Introduction

Wave propagation along repetitive structure has attracted much attention in recent years (Mead 1973, 1975, Yong and Lin 1989, 1990, Von Flotow 1986, Signorelli and Von Flotow 1988, Miller and Von Flotow 1989, Miller et al. 1990). Based on the analogy theory between structural mechanics and optimal control (Zhong and Zhong 1990, 1992a, 1993, Zhong et al. 1992), it can be anticipated that the Hamiltonian system theory and the corresponding symplectic mathematics can play an important role for wave propagation problems (Zhong and Williams 1991, 1992, Zhong and Yang 1992). However, the corresponding matrix expressions and their numerical compuctation requires very efficient formulations and methods. The problem of wave propagation along repetitive structure belongs to structural dynamics, for which the displacement formulation is natural and leads to the dynamic stiffness matrix. Therefore, the eigensolution, the adjoint symplectic orthogonality, the adjoint symplectic subspace iteration, and the eigenvector expansion method of symplectic mathematics are better formulated in the dynamic stiffness matrix representation directly, rather than in the transfer matrix or other representations. This dynamic stiffness representation avoids the unneces-

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sary numerical problems which otherwise arise from the inversion of the cross substiffness matrix K_{ab} needed to obtain the transfer matrix for the repetitive structure. The formulation and algorithm are now given for general repetitive structures, rather than just for some simplified models.

2. The fundamental equations

A typical multi-connected substructural chain is shown schematically in Fig. 1. Let the internal and left-hand and right-hand external displacements of a typical substructure be denoted by u_i, u_a and u_b respectively. The u_a will match the u_b when the substructure is moved one bay to the right, so that u_a and u_b share the same dimension, n say. The internal vector u_i can be eliminated beforehand (Zhong and Yang 1992), giving the external dynamic stiffness matrix of the typical substructure as

$$K(\omega) = \begin{bmatrix} K_{aa} & K_{ab} \\ K_{ba} & K_{bb} \end{bmatrix}, \quad \text{with} \quad u = \begin{bmatrix} u_a \\ u_b \end{bmatrix} \tag{1}$$

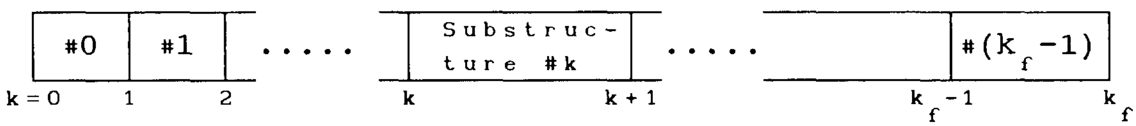


Fig. 1 The substructural chain

The cost of this simplification is that the submatrices K_{aa}, K_{ab} and K_{bb} are now transcendental functions of the circular frequency ω, but $K(\omega)$ is still symmetric for a nondissipative system, i.e. $K_{aa}^T = K_{aa}$, $K_{bb}^T = K_{bb}$ and $K_{ab}^T = K_{ba}$, The stiffness matrix corresponds to using the displacement method in structural analysis, which in turn corresponds to the Lagrangian formulation for a dynamical system. The dynamic potential energy for the typical substructure can be introduced (Zhong and Williams 1991) as

$$U = \frac{1}{2} \begin{Bmatrix} u_a \\ u_b \end{Bmatrix}^T K \begin{Bmatrix} u_a \\ u_b \end{Bmatrix} \tag{2}$$

The inter-substructural force vectors n_a and n_b can be introduced as follows. The principle of virtual work gives $n = \partial U / \partial u$. Therefore n_a and n_b follow from Eq.(2) as

$$\begin{aligned} n_a &= \frac{\partial U}{\partial u_a} = K_{aa} u_a + K_{ab} u_b \\ n_b &= -\frac{\partial U}{\partial u_b} = -K_{ba} u_a - K_{bb} u_b \end{aligned} \tag{3}$$

where the minus signs follow from the action and reaction rule. The whole state vector v is defined as the combination of displacement and internal force vectors, such that

$$v = \left\{ \begin{matrix} u \\ n \end{matrix} \right\} \tag{4}$$

Thus Eq. (3) can be expressed in transfer matrix form, which relates the whole state vector at boundaries a and b by

$$v_b = S\, v_a, \quad \text{or} \quad \left\{ \begin{matrix} u_b \\ n_b \end{matrix} \right\} = S \left\{ \begin{matrix} u_a \\ n_a \end{matrix} \right\}, \quad (a{=}k,\ b{=}k+1 \text{ junctions}) \tag{5}$$

where S is the transfer matrix. It can be shown (Zhong and Williams 1991) that

$$S(\omega) = \begin{bmatrix} S_{aa} & S_{ab} \\ S_{ba} & S_{bb} \end{bmatrix}, \tag{6}$$

where

$$\left. \begin{matrix} S_{aa} = -K_{ab}^{-1} K_{aa}, & S_{ab} = K_{ab}^{-1} \\ S_{ba} = -K_{ba} + K_{bb} K_{ab}^{-1} K_{aa}, & S_{bb} = -K_{bb} K_{ab}^{-1}, \end{matrix} \right\} \tag{7}$$

It has also been verified (Yong and Lin 1990) that S is a symplectic matrix, i. e.

$$S^{-T} = J\, S\, J^{-1}, \quad \text{or} \quad S^T J\, S = J \tag{8}$$

where

$$J = \begin{bmatrix} O & I_n \\ -I_n & O \end{bmatrix}, \quad J^T = J^{-1} = -J, \tag{9}$$

and I_n is an n-dimensional unit matrix. For a symplectic matrix, it is known that: its determinant is 1; if μ is an eigenvalue then so is $1/\mu$; the symplectic orthonormality relationship exists between its eigenvectors; and an arbitrary whole state vector can be expanded in terms of the eigenvectors (Yong and Lin 1990, Zhong 1992)

Direct derivation of the symplectic matrix S requires the inversion of the matrix K_{ab}, which is not always possible, e.g. it is not possible for the periodical structure shown in Fig. 2. However, even if the matrix K_{ab} is invertible the computation of the matrix S is likely to involve numerical ill-conditioning problems.

To avoid such problems, representation by the displacement vector directly is preferable. It is easily seen from Eqs. (3) and (4) that

$$v_a = \left\{ \begin{matrix} u_a \\ n_a \end{matrix} \right\} = L \times \left\{ \begin{matrix} u_a \\ u_b \end{matrix} \right\}, \quad v_b = N \times \left\{ \begin{matrix} u_a \\ u_b \end{matrix} \right\}, \quad L = \begin{bmatrix} I_n & O \\ K_{aa} & K_{ab} \end{bmatrix}, \quad N = \begin{bmatrix} O & I_n \\ -K_{ba} & -K_{bb} \end{bmatrix} \tag{10}$$

where L and N are invertible if K_{ab} is invertible, and it can be verified by using Eqs. (6) and (7) that $S = NL^{-1}$. Because u_a for substructure $k+1$ of Fig. 1 is equal to u_b for substructure k, it is possible to write the transfer equation in terms of displacement vectors as

$$L \times w_{k+1} = N \times w_k, \quad \text{where} \quad w_k = \{u_a^T,\ u_b^T\}_k^T \tag{11}$$

where the subscripts k and $k+1$ represent the substructure number, and the subscripts a and b refer to the left-hand and right-hand boundaries respectively. Obviously, Eq. (11) does not require K_{ab} to be invertible. Although Eq. (11) is expressed in terms of displacements, it is not the displacement method, because the displacement continuity condition $\{\boldsymbol{u}_a\}_{k+1}=\{\boldsymbol{u}_b\}_k$ is not satisfied beforehand but is instead treated as an equation in the formulation.

Instead of solving the eigenequation with the symplectic matrix S directly, the generalized eigenequation

$$\mu\times\boldsymbol{L}\times\boldsymbol{w} = \boldsymbol{N}\times\boldsymbol{w} \tag{12}$$

can be solved, where μ is the eigenvalue and $\boldsymbol{w}$ is the eigenvector of the typical substructure. Now the identity

$$\boldsymbol{L}^T\boldsymbol{J}\boldsymbol{L} = \boldsymbol{N}^T\boldsymbol{J}\boldsymbol{N} = \begin{bmatrix} \boldsymbol{O} & \boldsymbol{K}_{ab} \\ -\boldsymbol{K}_{ba} & \boldsymbol{O} \end{bmatrix} \tag{13}$$

is easily verified by substituting from Eqs. (9) and (10) and using the facts that $\boldsymbol{K}_{aa}^T=\boldsymbol{K}_{aa}$, $\boldsymbol{K}_{bb}^T=\boldsymbol{K}_{bb}$ and $\boldsymbol{K}_{ab}^T=\boldsymbol{K}_{ba}$. Hence left multiplying Eq. (12) by $\boldsymbol{N}^T\boldsymbol{J}$ and using Eq. (13) gives

$$\mu\boldsymbol{N}^T(\boldsymbol{J}\boldsymbol{L}\boldsymbol{w}) = \boldsymbol{L}^T(\boldsymbol{J}\boldsymbol{L}\boldsymbol{w}) \tag{14}$$

Transposing Eq. (14) gives $(\boldsymbol{J}\boldsymbol{L}\boldsymbol{w})^T(\boldsymbol{N}-\mu^{-1}\boldsymbol{L})=0$, which means that $(\boldsymbol{J}\boldsymbol{L}\boldsymbol{w})$ is the left eigenvector, so that μ^{-1} is also an eigenvalue. Therefore, the $2n$ eigenvalues of Eq. (12), when ordered appropriately, can be subdivided into the two groups of

$$\text{(a)}\quad \mu_i \quad (i=1, 2, \cdots, n), \quad \text{with} \quad |\mu_i| \leq 1 \tag{15}$$

which corresponds to those waves that are travelling to the right, and

$$\text{(b)}\quad \mu_{n+i} = \mu_i^{-1} \quad (i=1, 2, \cdots, n), \quad |\mu_{n+i}| \geq 1 \tag{16}$$

which corresponds to the waves travelling to the left. For most problems the eigenvalues are single roots, and this is assumed in the following derivation. Therefore for each eigenvalue there will be a single eigenvector.

3. Weighted adjoint symplectic orthogonality between the eigenvectors

When all the eigensolutions of Eq. (12) are found, a very important application is the eigenvector expansion of an arbitrary vector. Noting the adjoint symplectic orthogonality relation between the eigenvectors of the symplectic matrix S (Zhong 1992), there must be a weighted adjoint symplectic orthonormality relation between the eigenvectors of Eq. (12). Provided that there are two eigensolutions i and j of Eq. (12)

$$\boldsymbol{N}\times\boldsymbol{w}_i = \mu_i\times\boldsymbol{L}\times\boldsymbol{w}_i, \quad \boldsymbol{N}\times\boldsymbol{w}_j = \mu_j\times\boldsymbol{L}\times\boldsymbol{w}_j, \quad i, j \leq 2n \tag{17a, b}$$

Transforming Eq. (17a) to the form of Eq. (14), left multiplying by $\boldsymbol{w}_j^T$ and then taking the transpose gives $\boldsymbol{w}_i^T\boldsymbol{L}^T\boldsymbol{J}\boldsymbol{N}\boldsymbol{w}_j=\mu_i^{-1}(\boldsymbol{w}_i^T\boldsymbol{L}^T\boldsymbol{J}\boldsymbol{L}\boldsymbol{w}_j)$. However, left multiplying Eq. (17b) by $\boldsymbol{w}_i^T\boldsymbol{L}^T\boldsymbol{J}$, gives $\boldsymbol{w}_i^T\boldsymbol{L}^T\boldsymbol{J}\boldsymbol{N}\boldsymbol{w}_j=\mu_j\times\boldsymbol{w}_i^T\boldsymbol{L}^T\boldsymbol{J}\boldsymbol{L}\boldsymbol{w}_j$. Subtracting these two equations gives

$$(\mu_j - \mu_i^{-1})\times(\boldsymbol{w}_i^T\boldsymbol{L}^T\boldsymbol{J}\boldsymbol{L}\boldsymbol{w}_j)=0 \tag{18}$$

which leads to the weighted adjoint symplectic orthogonality as

$$w_i^T(L^TJL)w_j = 0, \quad \text{when } \mu_j - 1/\mu_i \neq 0, \quad [\text{ i. e. } j \neq \mathrm{mod}_{2n}(n+i)] \tag{19}$$

where L^TJL is the weighting matrix, see Eq. (13). Therefore an eigenvector is symplectic orthogonal to all other eigenvectors, including itself but excluding its adjoint. The i-th eigenvector has only one adjoint eigenvector, the $(n+i)$-th, which is given by Eqs. (15) and (16).

An adjoint symplectic normalization relationship can be introduced for the adjoint eigenvectors, i.e.

$$w_i^T L^TJLw_{i+n}=1, \quad \text{or} \quad w_{i+n}^T L^TJLw_i=-1, \quad (i \leq n) \tag{20}$$

Because the adjoint eigenvectors have two arbitrary constant factors and Eq. (20) supplies only one condition, another condition can be supplied, which can be taken as

$$w_i^T w_i = w_{i+n}^T w_{i+n} \tag{21}$$

Let $y_i = Lw_i (i \leq 2n)$. Then it can be seen from Eqs. (5) and (10) that the y_i are the eigenvectors of the symplectic matrix S. Eqs. (19) and (20) then become the usual symplectic orthonormality relationships. The derivation above has bypassed the problem of K_{ab} possibly not being invertible. However, in such cases the weighting matrix L^TJL is not of full rank. Hence further investigation follows, to ensure the completeness of the eigenvector set used for the expansion solution.

4. The case of K_{ab} not being of full rank

In the case of K_{ab} being singular, the matrices L and N of Eq. (10) are also singular, because of the way that K_{ab} appears in them. The singular eigenvectors w_∞ corresponding to the infinite eigenvalue can be found, by using Eqs. (10) and (12), as the solution of

$$Lw_\infty = 0, \quad w_\infty^T = \{ 0, u_{b\infty}^T \}^T, \quad \text{where} \quad K_{ab} u_{b\infty} = 0 \tag{22}$$

Because K_{ab} is singular, there exist n_∞ linearly independent vectors $u_{b\infty}$ and hence n_∞ linearly independent w_∞. On the other hand, corresponding to the eigenvalue $\mu=0$, the singular eigenvectors w_0 are given as the solutions of

$$Nw_0 = 0, \quad w_0^T = \{ u_{a0}^T, 0 \}^T, \quad \text{where} \quad K_{ab}^T u_{a0} = 0 \tag{23}$$

The number of linearly independent solutions of Eq. (23) is also n_∞. Obviously these $2n_\infty$ solutions for w_∞ and w_0 are linearly independent and the subspace which they span is orthogonal to the weighting matrix operator $L^T JL$. According to Eq. (13), the rank of the weighting matrix is $2(n-n_\infty)$, which means that the effective subspace of the weighting matrix L^TJL complements the subspace spanned by the $2n_\infty$ vectors w_∞ and w_0, such that their direct sum composes the complete space.

5. Expansion in terms of the eigenvectors

Any $2n$ dimensional external displacement vector of the typical substructure can be expressed as a linear combination of the eigenvectors. However, the eigenvectors in section 3 must be supplemented by the $\boldsymbol{w}_\infty$ and $\boldsymbol{w}_0$, so that the adjoint symplectic orthonormality relationship should also be given for these special eigenvectors. For these eigenvectors, the eigenequation (12) can be considered as a set of linear simultaneous equations with the coefficient matrix $\boldsymbol{N}$ or $\boldsymbol{L}$ see Eqs. (23) and (22). The solvability condition for a singular matrix should tend to look like the solution of homogeneous linear equations with its transpose matrix. By using Eqs. (10), (22) and (23), it is easy to verify that

$$\boldsymbol{N}^T \{(\boldsymbol{K}_{bb}\boldsymbol{u}_{b\infty})^T, \boldsymbol{u}_{b\infty}^T\}^T = 0, \quad \text{and} \quad \boldsymbol{L}^T \{(-\boldsymbol{K}_{aa}\boldsymbol{u}_{a0})^T, \boldsymbol{u}_{a0}^T\}^T = 0 \tag{24}$$

Now the solvability condition for the eigenequation (12) becomes
$\{(\boldsymbol{K}_{bb}\boldsymbol{u}_{b\infty})^T,\ \boldsymbol{u}_{b\infty}^T\}\boldsymbol{L}\boldsymbol{w}_i = 0$, i.e.

$$\boldsymbol{u}_{b\infty}^T \left[(\boldsymbol{K}_{aa}+\boldsymbol{K}_{bb}),\ \boldsymbol{K}_{ab} \right] \boldsymbol{w}_i = 0, \ (i=1,\ 2,\ \cdots,\ n-n_\infty;\ n+1,\ \cdots,\ 2n-n_\infty) \tag{25a}$$

Similarly

$$\boldsymbol{u}_{a0}^T \left[\boldsymbol{K}_{ab}^T,\ (\boldsymbol{K}_{aa}+\boldsymbol{K}_{bb}) \right] \boldsymbol{w}_i = 0, \ (i=1,\ 2,\ \cdots,\ n-n_\infty;\ n+1,\ \cdots,\ 2n-n_\infty) \tag{25b}$$

The above equations can be regarded as the orthogonal relationships between the normal and singular eigenvectors. Combining gives

$$(\boldsymbol{w}_0 \text{ or } \boldsymbol{w}_\infty)^T \boldsymbol{J}\boldsymbol{K}_*\boldsymbol{w}_i = 0, \quad \text{where} \quad \boldsymbol{K}_* = \begin{bmatrix} \boldsymbol{K}_{aa}+\boldsymbol{K}_{bb} & \boldsymbol{K}_{ab} \\ \boldsymbol{K}_{ba} & \boldsymbol{K}_{aa}+\boldsymbol{K}_{bb} \end{bmatrix} \tag{25}$$

The adjoint orthogonal relationship between the singular eigenvectors can be derived as follows. Obviously, by using Eqs. (22), (23) and (25),

$$\boldsymbol{w}_\infty^T \boldsymbol{J}\boldsymbol{K}_*\boldsymbol{w}_\infty = 0, \quad \boldsymbol{w}_0^T \boldsymbol{J}\boldsymbol{K}_*\boldsymbol{w}_0 = 0, \tag{26}$$

and an appropriate linear combination gives

$$\boldsymbol{w}_{0i}^T \boldsymbol{J}\boldsymbol{K}_*\boldsymbol{w}_{\infty j} = \boldsymbol{u}_{a0i}^T(\boldsymbol{K}_{aa}+\boldsymbol{K}_{bb})\boldsymbol{u}_{b\infty j} = \delta_{ij} \quad (i,\ j \le n_\infty) \tag{27}$$

Eqs. (25)-(27) supply the required adjoint symplectic orthogonality relationships. However, the special case that

$$(\boldsymbol{K}_{aa}+\boldsymbol{K}_{bb})\ (\boldsymbol{u}_{a0} \text{ or } \boldsymbol{u}_{b\infty}) = 0 \tag{28}$$

for some $\boldsymbol{u}_{a0}$ or $\boldsymbol{u}_{b\infty}$ should be mentioned. In such a case, Eq. (27) does not hold and the Jordan canonical form for the singular eigenvalue (0 or ∞) will happen, so that special treatment is required.

Excluding such special cases and assuming that Eq. (27) holds, then the eigenvector expansion for an arbitrary external vector $\boldsymbol{w}$ holds, giving

$$w = \sum_{i=1}^{n-n_\infty} (a_i w_i + b_i w_{n+i}) + \sum_{i=1}^{n_\infty} (a_{n+i-n_\infty} w_{0i} + b_{n+i-n_\infty} w_{\infty i}) \tag{29}$$

Based on the orthogonal and normalized relationships of Eqs. (19) and (20), the singular solutions of Eqs. (22) and (23) and the weighting matrix Eq. (13), one can derive

$$a_i = -w_{n+i}^T L^T J L w, \quad b_i = -w_i^T L^T J L w \quad (i \leq n - n_\infty) \tag{30}$$

Based on the orthogonality relationships of Eqs. (25)-(27), the other coefficients can be

$$a_{n+i-n_\infty} = -w_{\infty i}^T J K_* w, \quad b_{n+i-n_\infty} = w_{0i}^T J L_* w \quad (i \leq n_\infty) \tag{31}$$

6. Solving the eigenproblem

According to the description above, the eigensolutions can be classified as singular (coming from Eqs. (22) and (23)) and normal solutions. It is easy to find the singular eigensolutions from the matrix K_{ab}, by using the Gauss elimination process.

For the non-singular eigensolutions, left multiplying Eq. (12) by $L^T J$ and by $N^T J$ in turn gives, after using Eq. (13)

$$(L^T J N) w_i = \mu_i L^T J L w_i, \quad \text{and} \quad (N^T J L) w_i = \mu_i^{-1} L^T J L w_i \tag{32}$$

Adding these together gives

$$(L^T J N + N^T J L) w_i = (\mu_i + \mu_i^{-1})(L^T J L) w_i \tag{33}$$

Because the weighting matrix $L^T J L$ has rank $2(n - n_\infty)$, only the eigensolutions within the subspace spanned by the weighting matrix $L^T J L$ are considered. Among these eigensolutions the adjoint pair i and $(n+i)$, $(i \leq n - n_\infty)$ corresponds to a single eigenvalue of Eq. (33), i. e.

$$\lambda_i = \mu_i + \mu_i^{-1} = \mu_i + \mu_{n+i} \tag{34}$$

Hence, every eigenvalue of Eq. (33) is duplicated, and so has two corresponding linearly independent eigenvectors w_i and w_{n+i}. It should be noted that the eigenvectors obtained by solving Eq. (33) can be linear combinations of the eigenvectors w_i and w_{n+i} of Eq. (12), and so are not necessarily identically the eigenvectors of Eq. (12). However, the eigenvectors w_i and w_{n+i} can be found via the linear combinations of the eigenvectors of Eq. (33), and the corresponding eigenvalues found via Eq. (34). Therefore the solution of Eq. (33) is a critical step. Substituting the L and N of Eq. (10) into Eq. (33) gives

$$\begin{bmatrix} K_{ab} - K_{ab}^T & -(K_{aa} + K_{bb}) \\ (K_{aa} + K_{bb}) & K_{ab} - K_{ab}^T \end{bmatrix} w_i = \lambda_i \begin{bmatrix} O & K_{ab} \\ -K_{ba} & O \end{bmatrix} w_i \tag{35}$$

Here both matrices are anti-symmetric, with the left-hand one depending on $(K_{aa} + K_{bb})$, but not on K_{aa} or K_{bb} individually. Hence for the eigenvalue problem the two diagonal stiffness submatrices of $K(\omega)$ can be mutually shifted, such as both K_{aa} and K_{bb} can be substituted by $(K_{aa} + K_{bb})/2$, as is done when dealing with singular control problems (Zhong and Cheng

1991)

Because $\boldsymbol{K}_{ab}$ is not of full rank, there are $2n_\infty$ singular eigenvectors corresponding to the eigenvalue $\lambda=\infty$. Therefore, Eq. (35) should be reduced to the order of $2m=2(n-n_\infty)$. Based on the adjoint symplectic subspace iteration method for Hamiltonian or symplectic matrices (Zhong 1992, Zhong and Zhong 1991), combined with the orthonormality relationships of Eqs. (19) and (20), one can use the weighted adjoint symplectic orthonormality basis with the weighting matrix $\boldsymbol{L}^T\boldsymbol{J}\boldsymbol{L}$. The $2m$ basis vectors are not necessarily the eigenvectors of Eq. (35), but one can generate such basis vectors from $2m$ roughly selected vectors by the adjoint symplectic weighted orthonormality algorithm given in the next section.

7. The adjoint symplectic weighted orthonormality algorithm(ASWONA)

It is well known that from n arbitrarily selected linearly independent vectors, the Gram-Schmidt orthonormality algorithm can be used to generate a set of n orthonormalized basis vectors. Similarly, from $2m$ ($m=n-n_\infty$) linearly independent under the weighting matrix $\boldsymbol{L}^T\boldsymbol{J}\boldsymbol{L}$ vectors, a set of $2m$ adjoint symplectic weighted orthonormalized basis vectors $\boldsymbol{\psi}_i$ ($i\leq 2m$) can be generated, such that

$$\boldsymbol{\psi}_i^T\boldsymbol{L}^T\boldsymbol{J}\boldsymbol{L}\boldsymbol{\psi}_j = 0 \quad \text{when } i\neq \mathrm{mod}_{2m}(m+j) \tag{36}$$

$$\boldsymbol{\psi}_i^T\boldsymbol{L}^T\boldsymbol{J}\boldsymbol{L}\boldsymbol{\psi}_{i+m} = 1, \quad \text{or} \quad \boldsymbol{\psi}_{i+m}^T\boldsymbol{L}^T\boldsymbol{J}\boldsymbol{L}\boldsymbol{\psi}_i = -1, \quad (i\leq m) \tag{37}$$

Note that the vectors $\boldsymbol{\psi}_i$ in this section are not necessarily eigenvectors. Similarly to the adjoint symplectic orthonormality algorithm given in (Zhong 1992, Zhong and Zhong 1991), an ASWONA can be proposed here. The weighted symplectic orthogonalization equation for a given vector $\boldsymbol{\psi}_i$ to a pair of adjoint vectors $\boldsymbol{\psi}_j$ and $\boldsymbol{\psi}_{j+m}$ is given first as

$$\boldsymbol{\psi}_i := \boldsymbol{\psi}_i-(\boldsymbol{\psi}_{(j')}^T\boldsymbol{L}^T\boldsymbol{J}\boldsymbol{L}\boldsymbol{\psi}_i/\boldsymbol{\psi}_{(j')}^T\boldsymbol{L}^T\boldsymbol{J}\boldsymbol{L}\boldsymbol{\psi}_j)\boldsymbol{\psi}_j, \quad j'=\mathrm{mod}_{2m}(m+j), \quad (i, j\leq 2m) \tag{38}$$

and the normalization equation is similar to Eq. (20). Then the ASWONA can be described as

For $i:=1$ **to** m **do begin** comment: to orthonormalize $\boldsymbol{\psi}_i$ and $\boldsymbol{\psi}_{i+m}$
 {read in $\boldsymbol{\psi}_i$ and $\boldsymbol{\psi}_{i+m}$ from the database}
 For $j:=1$ **to** $i-1$ **do begin** comment the j-th had been orthonormalized
  {read in $\boldsymbol{\psi}_j$ and $\boldsymbol{\psi}_{j+m}$ from the database}
  {orthogonalize $\boldsymbol{\psi}_i$ and $\boldsymbol{\psi}_{i+m}$ with the $\boldsymbol{\psi}_j$ and $\boldsymbol{\psi}_{j+m}$ by Eq. (38)}
 End;
 {symplectic normalization Eq. (20); then write to the database}
End; (39)

Before the execution of the above ASWONA, the initial $\boldsymbol{\psi}_i$ and $\boldsymbol{\psi}_{i+m}$ are considered to be available, and should be linearly independent with respect to the weighting matrix $\boldsymbol{L}^T\boldsymbol{J}\boldsymbol{L}$. After the execution of the ASWONA, all the adjoint vector pairs $\boldsymbol{\psi}_i$ and $\boldsymbol{\psi}_{i+m}$ ($i=1, 2, \cdots, m$) compose a $2m$ dimensional adjoint symplectic orthonormalized complete basis under the weighting matrix $\boldsymbol{L}^T\boldsymbol{J}\boldsymbol{L}$. When the n_∞ pairs of singular vectors $\boldsymbol{w}_o$ and $\boldsymbol{w}_\infty$ are also included, they compose the complete set of basis vectors for the $2n$ dimensional space. However, $\boldsymbol{w}_o$ and $\boldsymbol{w}_\infty$ are easily found, so that the $m=n-n_\infty$ pairs of eigensolutions are of concern, which gives the eigenproblem in the adjoint symplectic weighted subspace.

8. The eigenproblem in the adjoint symplectic subspace

The right-hand side matrix of Eq. (35) is singular, so that the $2m$ dimensional subspace is extracted for the eigensolution. The adjoint symplectic weighted basis vectors $\boldsymbol{\psi}_i$ and $\boldsymbol{\psi}_{i+m}$ ($i = 1, 2, \cdots, m$) have been obtained via the ASWONA, Eq. (39). Now the $2n\times 2m$ matrix $\boldsymbol{\Psi}$ can be composed from these basis vectors, giving

$$\boldsymbol{\Psi} = [\,\boldsymbol{\psi}_1, \cdots, \boldsymbol{\psi}_m;\ \boldsymbol{\psi}_{m+1}, \cdots, \boldsymbol{\psi}_{2m}] \tag{40}$$

The eigensolution of Eqs. (33) or (35) can be expanded in terms of these basis vectors as

$$\boldsymbol{w} = \sum_{i=1}^{m} (a_i\boldsymbol{\psi}_i + a_{i+m}\boldsymbol{\psi}_{i+m}) = \boldsymbol{\Psi}\times\boldsymbol{a} \tag{41}$$

Substituting Eq. (41) into Eq. (33), then left multiplying by $\boldsymbol{\Psi}^T$ and using the weighted adjoint symplectic orthonormality relationships of Eqs. (36) and (37) gives

$$\boldsymbol{A}\times\boldsymbol{a} = \lambda\boldsymbol{J}_m\times\boldsymbol{a}, \quad \text{or} \quad -\boldsymbol{J}_m\boldsymbol{A}\times\boldsymbol{a} = \lambda\boldsymbol{a}, \quad \boldsymbol{J}_m = \begin{bmatrix} \boldsymbol{O} & \boldsymbol{I}_m \\ -\boldsymbol{I}_m & \boldsymbol{O} \end{bmatrix} \tag{42}$$

Eq. (42) has been reduced to a $2m$ dimensional subspace, and

$$\boldsymbol{A} = \boldsymbol{\Psi}^T\times(\boldsymbol{L}^T\boldsymbol{J}\boldsymbol{N} + \boldsymbol{N}^T\boldsymbol{J}\boldsymbol{L})\times\boldsymbol{\Psi} \tag{43}$$

By using Eq. (9), it is easily verified that $\boldsymbol{A}$ is an anti-symmetric matrix. Hence the matrix

$$\boldsymbol{M} = -\boldsymbol{J}_m\boldsymbol{A} = \begin{bmatrix} \boldsymbol{M}_{aa} & \boldsymbol{M}_{ab} \\ \boldsymbol{M}_{ba} & \boldsymbol{M}_{bb} \end{bmatrix} \tag{44}$$

which enables Eq. (42) to be written as

$$\boldsymbol{M}\times\boldsymbol{a} = \lambda\boldsymbol{a}, \tag{45}$$

has the characteristics of a square Hamiltonian matrix (Van Loan 1984, Lin 1987), i.e.

$$\boldsymbol{M}_{bb} = \boldsymbol{M}_{aa}^T, \quad \boldsymbol{M}_{ab} = -\boldsymbol{M}_{ab}^T, \quad \boldsymbol{M}_{ba} = -\boldsymbol{M}_{ba}^T \tag{46}$$

There are efficient methods for the eigensolution of Eq. (45) (Van Loan 1984, Lin 1987). When the dimension m is large, the main eigensolutions can be found via the adjoint symplectic subspace iteration method (Zhong 1992, Zhong and Zhong 1991). However, the eigenproblem of Eq. (42), with the anti-symmetric matrix $\boldsymbol{A}$ of Eq (43), can also be solved directly (Zhong and Zhong 1992b).

Combining all the above factors gives a good numerical method for solving the eigenequation of the wave propagation problem for substructural chain type structures.

9. Numerical examples

For the simple substructural chain type structure of Fig. 2(a), whose typical substructure is given in Fig. 2(b), it is required to solve the static problem, i.e. $\omega=0$, as a demonstration. The number of external displacements for each end is $n=3$, and

$$\boldsymbol{K}_{aa}=\begin{bmatrix}1&0&0\\0&1&-1\\0&-1&2\end{bmatrix},\quad \boldsymbol{K}_{bb}=\begin{bmatrix}2&-1&0\\-1&1&0\\0&0&2\end{bmatrix},\quad \boldsymbol{K}_{ab}=\begin{bmatrix}-1&0&0\\0&0&0\\1&-1&0\end{bmatrix},$$

Obviously, the rank of $\boldsymbol{K}_{ab}$ is $m=2$ and $n_\infty=1$. It is easy to find from Eqs. (22) and (23) that

$$\boldsymbol{w}_0=\{0\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\}^T,\quad \boldsymbol{w}_\infty=\{0\ \ 0\ \ 0\ \ 0\ \ 0\ \ -1\}^T,$$

and then Eqs. (25) and (13) give

$$\boldsymbol{JK}_*=\begin{bmatrix}-1&0&1&3&-1&0\\0&0&-1&-1&2&-1\\0&0&0&0&-1&4\\-3&1&0&1&0&0\\1&-2&1&0&0&0\\0&1&-4&-1&1&0\end{bmatrix},\quad \boldsymbol{L}^T\boldsymbol{JL}=\begin{bmatrix}0&0&0&-1&0&0\\0&0&0&0&0&0\\0&0&0&1&-1&0\\1&0&-1&0&0&0\\0&0&1&0&0&0\\0&0&0&0&0&0\end{bmatrix}$$

After the ASWONA in the subspace of $\boldsymbol{L}^T\boldsymbol{JL}$, the matrix composed of the the basis vectors is

$$\boldsymbol{\Psi}=\begin{bmatrix}-1&1&0&0\\0&4&1&1\\0&1&0&0\\0&0&1&0\\0&0&0&-1\\0&-1&-1&-2\end{bmatrix};\quad \text{and from Eq. (43),}\quad \boldsymbol{A}=\begin{bmatrix}0&1&3&1\\-1&0&0&5\\-3&0&0&1\\-1&-5&-1&0\end{bmatrix}$$

Then the eigenproblem of Eq. (45) must be solved. The eigenvalue must be duplicated. However the Jordan form will not happen for twofold eigenvalues. The present example has $m=2$ so the eigenvalue of Eq. (45) can be solved by expanding the determinant equation (Zhong and Zhong 1991). The matrix $\boldsymbol{M}$, obtained from Eq. (44) and its corresponding eigenvalues and eigenvector matrix $\boldsymbol{A}_e$ are

$$\boldsymbol{M}=\begin{bmatrix}3&0&0&-1\\1&5&1&0\\0&1&3&1\\-1&0&0&5\end{bmatrix},\quad \text{and}\quad \begin{array}{c}\lambda=\begin{array}{cccc}4-\sqrt{2}&4+\sqrt{2}&4-\sqrt{2}&4+\sqrt{2}\end{array}\\ \boldsymbol{A}_e=\begin{bmatrix}\sqrt{2}+1&1-\sqrt{2}&0&0\\-1&-1&\sqrt{2}-1&\sqrt{2}+1\\0&0&-1&1\\1&1&0&0\end{bmatrix}\end{array}$$

where the columns of $\boldsymbol{A}_e$ are eigenvectors. Now substituting $\boldsymbol{A}_e$ and $\boldsymbol{\Psi}$ into Eq. (41) gives the matrix composed of the four column eigenvectors of Eq. (35) as

$$\begin{array}{ccccc}\lambda\ = & 4-\sqrt{2} & 4+\sqrt{2} & 4-\sqrt{2} & 4+\sqrt{2}\end{array}$$

$$\begin{bmatrix}-(2+\sqrt{2})&-(2+\sqrt{2})&\sqrt{2}-1&\sqrt{2}+1\\-3&-3&4\sqrt{2}-5&4\sqrt{2}+5\\-1&-1&\sqrt{2}-1&\sqrt{2}+1\\0&0&-1&1\\-1&-1&0&0\\-1&-1&2-\sqrt{2}&-(2+\sqrt{2})\end{bmatrix}$$

The column vectors are not necessarily the eigenvectors of Eq. (12). However, the eigenvectors of Eq. (12) can be linearly composed from the two corresponding column vectors, and their eigenvalues can be solved from

$$\mu+\mu^{-1} = \lambda, \qquad \text{i.e.} \quad \mu = [\,\lambda \pm \sqrt{\lambda^2-4}\,]\,/\,2$$

so that the eigensolutions of Eq. (12) are

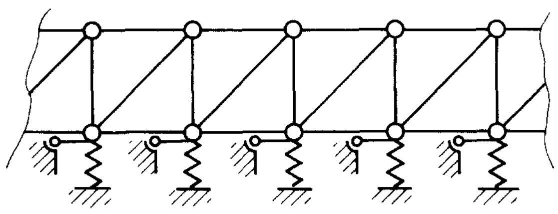
$$\begin{array}{cccc} \mu = 0.4734 & 0.1915 & 2.1124 & 5.2227 \end{array}$$

$$\begin{bmatrix} 2.1124 & 5.2227 & 0.4734 & 0.1915 \\ 1.5631 & 25.040 & 0.1231 & -0.7265 \\ 0.3258 & 7.2087 & -0.1542 & -1.3802 \\ 1 & 1 & 1 & 1 \\ 0.7400 & 4.7945 & 0.2600 & -3.7945 \\ 0.1542 & 1.3803 & 0.3258 & -7.2087 \end{bmatrix}$$

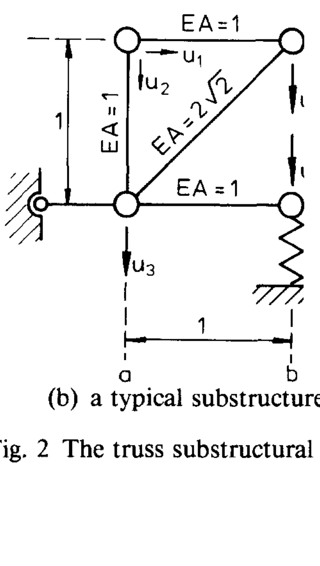
In the above equation there are only the eigensolutions with finite non-zero eigenvalues. If the two singular eigensolutions $\boldsymbol{w}_0$ and $\boldsymbol{w}_\infty$ are included, the six eigenvectors compose the complete set.

Table 1 Eigenvalues for second example.

	1	2	3	4	5	6
real	−724448.	−1/724448.	1/800.202	800.202	0.613	0.613
imag.	0.000	0.000	0.000	0.000	0.790	−0.790
	7	8	9	10	11	12
real	1.218	0.783	1.218	0.783	1.00	1.00
imag.	0.271	−0.174	−0.271	0.174	0.030	−0.030



(a) the substructural chain



(b) a typical substructure

Fig. 2 The truss substructural chain.

Hence, the complete eigensolution for the substructural chain of Fig. 2(a) has been found.

The above example is solved by hand, to give more insight into the problem. However, it is very small and also artificial. As a second more realistic example the repetitive structure shown in Fig. 3 is analysed. The frequency is selected arbitrarily as $\omega^2 = 1.0/sec/sec$ and for the beam element stiffness matrix use is made of the exact equations of reference (Howson, Banerjee and Williams 1983). The dynamic stiffness matrix is given as

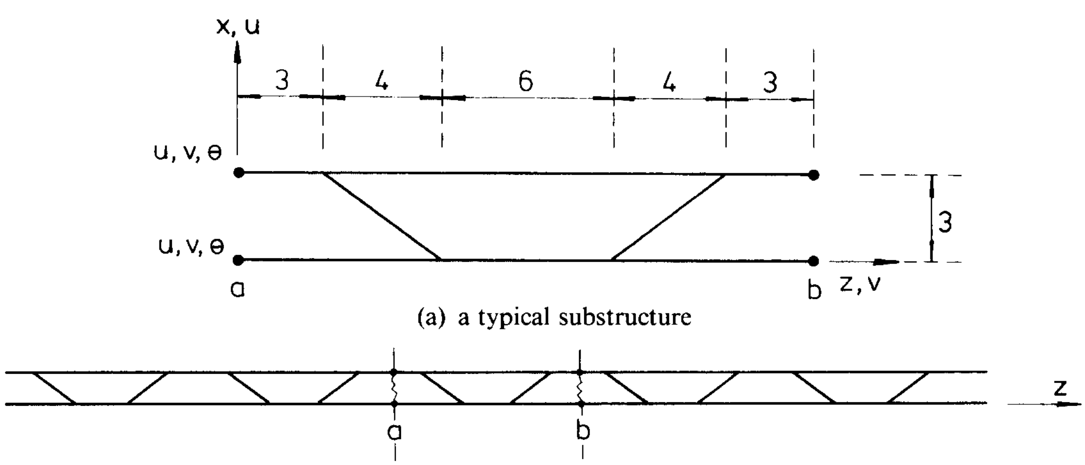
$(n=6)$

$$K_{aa} = \begin{bmatrix} -9.6237 & & & & \text{symmetric} & \\ 9.6114 & 1030.0170 & & & & \\ 6.7767 & -9.5992 & 52.9108 & & & \\ -10.0698 & -6.6561 & 18.9328 & -5.0979 & & \\ -5.1480 & -29.7285 & 0.5394 & 7.8188 & 1025.7190 & \\ 29.6183 & 22.9306 & -54.4557 & -4.2189 & -24.3200 & 70.4958 \end{bmatrix}$$

$$K_{bb} = \begin{bmatrix} -9.6231 & & & & \text{symmetric} & \\ -9.6121 & 1030.0180 & & & & \\ -6.7755 & -9.6002 & 52.9128 & & & \\ -10.0696 & 6.6559 & -18.9325 & -5.0978 & & \\ 5.1484 & -29.7287 & 0.5400 & -7.8187 & 1025.7190 & \\ -29.6178 & 22.9304 & -54.4549 & 4.2190 & -24.3198 & 70.4960 \end{bmatrix}$$

$$K_{ab} = \begin{bmatrix} 5.2847 & -11.1838 & -18.9036 & 6.5770 & 6.7616 & -16.7936 \\ 11.1839 & -967.5479 & -11.6419 & -4.6429 & -30.9382 & 11.2974 \\ 18.9041 & -11.6415 & -53.4631 & 14.3637 & 2.6443 & -37.6745 \\ 6.5771 & 4.6430 & -14.3639 & 2.4014 & -6.8260 & -6.7314 \\ -6.7613 & -30.9381 & 2.6440 & 6.8260 & -973.2586 & -18.9830 \\ 16.7941 & 11.2977 & -37.6750 & 6.7314 & -18.9830 & -18.6778 \end{bmatrix}$$

The wave number eigenvalues μ in Eq. (12) are given by Table 1 in which the eigenvectors are omitted to save space. Although the eigenvalues differ by several orders of magnitude, the eigen-solutions match Eq. (12) very precisely, being almost free from any ill-conditioning.



(a) a typical substructure

(b) the periodic structure

Fig. 3 Second example of a periodic structure. All members have flexural rigidity (EI), extensional rigidity (EA) and shear rigidity (GA) given by $EI = 200N\ m^2$, $EA = 2 \times 10^4 N$ and $GA = 5000N$.

10. Concluding remarks

Finding all of the eigensolutions is very important for wave propagation problems involving repetitive structures. The present paper derives the eigenequation directly from the stiffness matrix of the typical substructure, so that the method can be applied to cases for which the cross stiffness submatrix K_{ab} is singular. By filtering out the singular eigensolutions first, the adjoint symplectic orthonormalized basis vectors for the remaining space are generated. Then the eigenproblem can be made into a square Hamiltonian matrix type eigenproblem. Therefore, for a large scale matrix (i.e. n is large), the adjoint symplectic subspace iteration and origin shifting methods can be introduced to find the main eigensolutions. Combining all of the above techniques composes an effective algorithm for the eigensolution of wave propagation for repetitive structures.

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