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The torsional stiffness of bars with L, [, +, I, and Z cross-section

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Abstract. In literature for thin-walled sections with L, [, +, I, and Z- shapes the approximate torsion equations for stiffness are used which were proposed by Bach (Hsu 1984), p.30. New formulae for torsional stiffness of bars with L, [, +, I, and Z cross section valid not only for thin-walled sections are presented in this paper. These formulae are obtained by appropriate polynomial approximation of stiffness results obtained by means of method of fundamental solutions. On the base of obtained results the validity of Bach's formulae are verified when cross section is not thin-walled.

Keywords: elastic torsion; torsional stiffness formulae; method of fundamental solution; polynomial approximation; L, [, +, I, and Z cross sections.

1. Introduction

The study of the torsion of the rods is important and basic in design of steel structural elements and it is a classical problem in theory of elasticity (Timoshenko and Goodier 1970, and Sokolnikoff 1956). The problem can be formulated as a boundary value problem of the Laplace equation for the warping function or a boundary value problem of Poisson equation for Prandtl's stress function. The exact solutions of torsion problem have been found for some simple cross-sectional shapes such as circle, annulus, ellipse, rectangle and triangle. In a case of steel structures though the rods very often have L, [, +, I, and Z shaped cross-sections. Despite the popularity of these cross sections, an exact method for solving their torsion problem has not been obtained yet. In Arutyunyan and Abramian's book (1963) one can find some approximate methods for the considered cross sections. In this method, the boundary problem is reduced to an infinite set of ordinary differential equations. In paper (Chen and Chen 1981)

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the authors divide the original section into several rectangular regions, and the Duhem theorem of continuation of harmonic functions is used. This leads to an infinite system of linear equations for certain unknown coefficients. A special version of the finite element method was used in paper (Chen 1982) for the considered cross sections. In literature for the thin-walled sections with L, [, +, I, and ʃ - shapes, the approximate torsion equations for stiffness proposed by Bach (Hsu 1984), p.30 were used. These equations are based on the following three assumptions:

1. The real steel cross section with L, [, +, I, and ʃ shapes can be approximated by rectangular components.
2. The width of each rectangular component is thin as compared to the overall dimension.
3. The shape of the cross section remains unchanged after twisting. In other words, the angle of twist is the same for all rectangular components

According to these assumptions, the torsional moment T is given by formula

$$T = G \cdot \omega \sum_{i=1}^{NR} \frac{1}{3} c_i^3 b_i \tag{1}$$

where G is the shear modulus of a rod material, ω is an angle of twist of rod per unit length, NR is a number of rectangular components, b_i is higher dimension of i-th rectangle, c_i is lower dimension of i-th rectangle.

Eq. (1) results from analytical solution of appropriate torsion boundary value problem for “thin rectangle” and assumption of superposition stiffness for rectangular components.

The purpose of the present paper is a proposition of new formulae for torsional stiffness of bars with L, [, +, I, and ʃ cross sections**,** appropriate not only for thin-walled sections. These formulae are obtained on the base of the polynomial approximation of stiffness calculated by means of method of fundamental solutions. Five torsion boundary value problems were considered: L - section, [-section, +-section, I-section and ʃ-section. The validity of Eq. (1) is discussed when cross section is not thin-walled.

2. Method of fundamental solutions

To solve the torsion problem for noncircular cross sections, St. Venant invented a “semi-inverse” method. In this method some features of the displacements are first assumed. The remaining features are then determined to satisfy all the equations of the theory of elasticity. The following two assumptions are made to describe the displacement components for noncircular sections:

- The shape of the cross sections remains unchanged after twisting,
- Warping of the cross section is identical throughout the length of the noncircular member.

The torsion problem of prismatic bars is often formulated in terms of the stress function which satisfies Poisson’s equation (Timoshenko and Goodier 1970):

$$\frac{\partial^2 \psi(x,y)}{\partial x^2} + \frac{\partial^2 \psi(x,y)}{\partial y^2} = -2G\omega \tag{2}$$

with the following boundary condition:

$$\psi(x,y) = 0 \quad \text{in} \quad \Gamma \tag{3}$$

where $\psi(x,y)$ is the stress function , Γ is the boundary of the cross-section.

Numerical methods are usually necessary for solution of the boundary value problem (2-3) and in this way to examine the formula (1). The most widely used universal are finite difference methods e.g. (Ely and Zienkiewicz 1960), the finite element methods, e.g. (Herrmann 1965), or the boundary element methods, e.g. (Chou and Mohr 1990). These methods belong to so-called mesh methods. In last decades the meshless method became more and more popular. One of such method is the method of fundamental solutions (MFS).

The MFS was first proposed by the Georgian researchers Kupradze and Aleksidze (1963, 1964). Its numerical implementation was carried out by Mathon and Johnston (1977). The mathematical analysis (convergence and stability) of this method was considered in papers (Bogomolny 1985, Katsurada and Okamoto 1988, Kitagawa 1988, Katsurada 1990, Kitagawa 1991, Katsurada and Okamoto 1996). Comprehensive reviews on the MFS for various applications can be found in Fairweather and Karageorghis (1998) Golberg and Chen (1998), Fairweather *et al.* (2003). This method is applicable when a fundamental solution of the differential is known. In the literature, this method is also known by other names such as the superposition method (Koopman *et al.* 1988), the desingularized method (Cao 1991), the fundamental collocation method (Burges and Mahajerin 1987), and the charge simulation method (Katsurada and Okamoto 1988).

In the MFS, the approximate solution of the problem is represented in the form of a linear superposition of source functions (fundamental solutions) with singular points that are located outside the domain of the problem. These points, called source points, are located on a "pseudo-boundary" outside the region. The pseudo-boundary has no common points with the boundary of the region. Because the fundamental solution satisfies the differential equation at any point except at the source point, it follows that this representation satisfies exactly the governing equation whereas the boundary conditions are only satisfied approximately. Therefore, the MFS belongs to the group of Trefftz methods for which it is essential that the governing equation is exactly satisfied. The weight coefficients which occur in the approximate solution are determined by the satisfaction of the boundary condition, usually on a set of boundary points (collocation points).

It is better to use dimensionless variables in calculations. The one way is to assume that a is the characteristic length (characteristic dimension of cross-section). Then, dimensionless variables are introduced:

$$X = \frac{x}{a}, \quad Y = \frac{y}{a}, \quad \Psi(X, Y) = \frac{\psi(x, y)}{a^2 G \omega} \tag{4}$$

Now the governing Eq. (2) with the boundary condition (3) is written as:

$$\frac{\partial^2 \Psi(X, Y)}{\partial X^2} + \frac{\partial^2 \Psi(X, Y)}{\partial Y^2} = -2 \tag{5}$$

$$\Psi(X, Y) = 0 \quad \text{in} \quad \tilde{\Gamma} \tag{6}$$

where $\tilde{\Gamma}$ is the boundary of the cross-section in dimensionless coordinates.

Instead of the stress function $\Psi(X, Y)$, one can introduce a harmonic function $\Phi(X, Y)$ by the substitution:

$$\Psi(X, Y) = \Phi(X, Y) - \frac{1}{2}(X^2 + Y^2) \tag{7}$$

and, by inserting this in Eq. (5), we obtain the Laplace equation:

$$\frac{\partial^2 \Phi}{\partial X^2} + \frac{\partial^2 \Phi}{\partial X^2} = 0 \quad \text{in} \quad \Omega \tag{8}$$

with the boundary condition:

$$\Phi(X, Y) = \frac{1}{2}(X^2 + Y^2) \quad \text{in} \quad \tilde{\Gamma} \tag{9}$$

Formulation of boundary values problems for five considered cross section in term of $\Phi(X,Y)$ are given in Fig. 1-5. In all problems, the thickness E is a geometrical parameter which changes in a permissible range, i.e., $0 < E < 1$ for problems I, IV, V and $0 < E < 0.5$ for problems II, III.

According to MFS the approximate solution of Eq. (8) is assumed in the form:

$$\Phi(X, Y) = \sum_{j=1}^{NC} c_j \varphi(X, Y, \xi_j, \eta_j) \tag{10}$$

where NC is the number of source points, c_j are the unknown constants, $\varphi(X, Y, \xi_j, \eta_j) = \ln[(X-\xi_j)^2 + (Y-\eta_j)^2]$are fundamental solutions of the 2-D Laplace Eq. (8), and $(\xi_j, \eta_j)$are the coordinates of the source points.

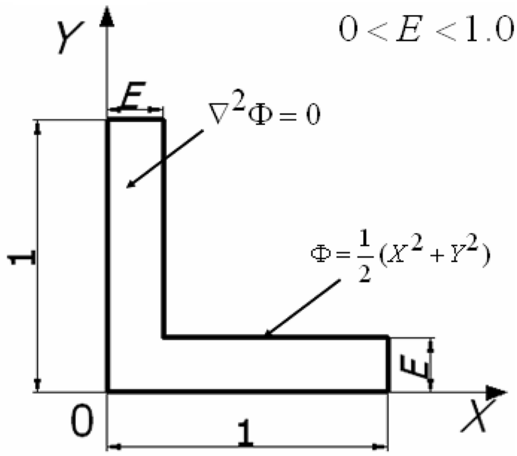

Fig. 1 Formulation of boundary value problem for L - cross section of a bar. Problem I

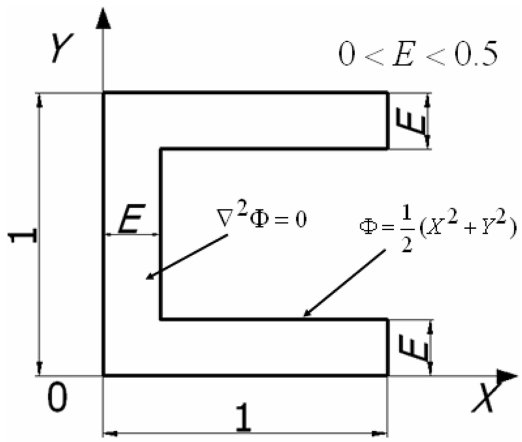

Fig. 2 Formulation of boundary value problem for ⌶ - cross section of a bar. Problem II

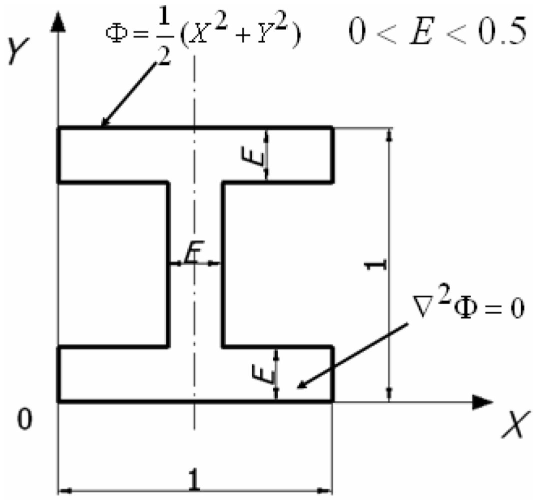

Fig. 3 Formulation of boundary value problem for [-cross section of a bar. Problem III

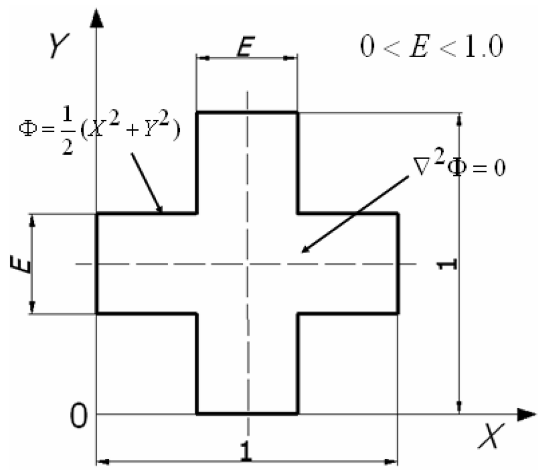

Fig. 4 Formulation of boundary value problem for +-cross section of a bar. Problem IV

Then assumed form of the solution has a singularity at the source points. Because the stress function in the cross-section of the rod does not have singular points, the source points are placed outside the considered region.

One of the fundamental problems in the application of the MFS is the determination of the positions of the source points. Generally, there have been two approaches to choosing the source points, fixed and adaptive. Most authors assume a fixed position for the sources with the designation of their coordinates as input data. If the linear boundary value problem is solved, the unknown weight coefficients of the fundamental solutions are obtained by solving linear algebraic equations.

In the second case, the coordinates of the position of source points are treated as unknowns and they are found together with the coefficients of the fundamental solutions by forcing the approximation to satisfy the boundary conditions. However, even in the case of linear boundary value problems, its numerical implementation leads to a nonlinear problem. That is why publications of the MFS with

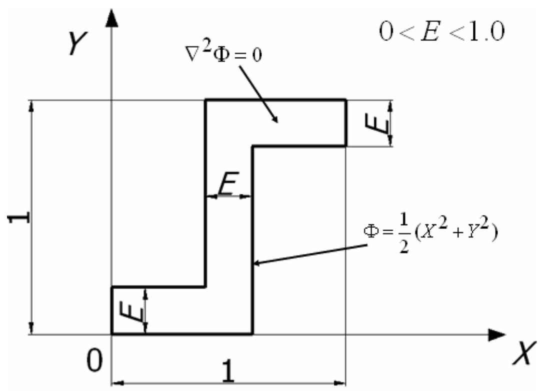

Fig. 5 Formulation of boundary value problem for ⌡-cross section of a bar. Problem V

unknown source position are rare. Two essential arbitrary ways of determination of source points are distinguished. The problem comes down to a designation of the shape of "pseudo-boundary" on which the source points are placed. First, when the "pseudo-boundary" is a circle (see Fig. 6), e.g. (Bogomolny 1985), and second when the "pseudo-boundary" is a contour geometrically similar to the boundary contour of the region under consideration as in Fig. 7, e.g. (Kołodziej and Kleiber 1989). Irrespective of the shape of "pseudo-boundary" the source points are usually uniformly distributed (see Fig. 6, 7). This problem is of particular importance in the case of value boundary problems in the concave regions as considered in this paper. Some authors, in the case of complex regions, suggest using the method of fundamental solutions in combination with the method domain of decomposition, e.g. (Partridge and Sensale 2000). The complex region is divided into subregions of a simpler shape. Then the MFS is used in every subregion subject to appropriate linking conditions.

When the region is not divided and the source points are arbitrary determined as the input parameters of the method then we have to deal with the following problem: what should the radius of the circle of the "pseudo-boundary" be or at what distance from the boundary should the "pseudo-boundary" be placed? This problem was considered in authors' previous paper (Gorzelanczyk and Kolodziej 2007). From the numerical experiments carried out in that paper it has been concluded that in all examined

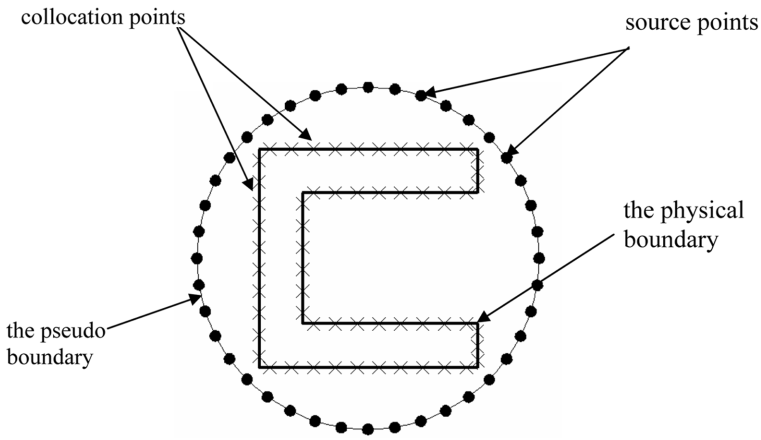

Fig. 6 Arrangement of the source points on the circular contour

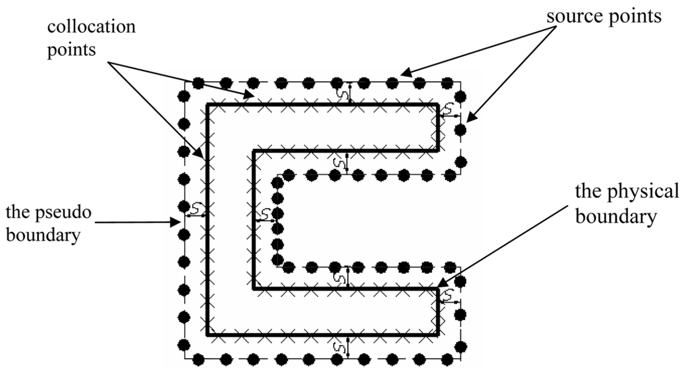

Fig. 7 Arrangement of the source points on the similar contour

cases referring to torsion of prismatic rods, the error values are smaller when the source contour is geometrically similar to the boundary of the region in comparison with the circular source contour.

The unknown constants c_j has been chosen to satisfy the boundary condition at N points on the boundary $\tilde{\Gamma}$. In this way, one obtains N linear equations for the unknowns c_j in the form:

$$\sum_{j=1}^{NC} c_j \ln[(X_i - \xi_j)^2 + (Y_i - \eta_j)^2] = \frac{1}{2}(X_i^2 + Y_i^2) \quad i = 1, 2 \ldots, N \tag{11}$$

where (X_i, Y_i), $i = 1,2,...,N$ are points on the boundary which are called collocation points (Fig. 1). Than, this method may be treated as a boundary collocation method in which fundamental solutions have been chosen, at the test functions.

The linear system of Eq. (11) may be written as:

$$\sum_{j=1}^{NC} A_{ij} c_j = b_i \qquad i = 1, 2 \ldots, N \tag{12}$$

where $A_{ij} = \ln[(X_i - \xi_j)^2 + (Y_i - \eta_j)^2]$.

The system of Eq. (12) includes NC unknowns while number of equations in the system equals N. To solve this system of equations, the condition $NC \leq N$ has to be satisfied. If the number of equations in the system equals the number of unknowns, then the system may be solved using Gaussian elimination. If the number of unknowns is less than the number of equations, the system of equations is overdefined and it may be solved using the least squares method. In this case, system (12) is written in the form:

$$\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{c} = \mathbf{A}^{\mathrm{T}}\mathbf{b} \tag{13}$$

In system (13) the number of unknowns equals the number of equations. After determining the coefficients in (10), the stress function is known at every point of the region.

The interested reader can find the details about optimal number of collocation and source points and distance between source contour and boundary contour for five considered cross sections in our previous paper (Gorzelanczyk and Kolodziej 2007).

3. Torsional stiffness

The torsional moment can be related to the integral of the shear stresses over the area. Using

$$T = \iint(\tau_{zy}x - \tau_{zx}y)\,dxdy \tag{14}$$

and substituting the relation between the stresses and stress function, and integrating over the region gives

$$T = 2\iint\psi\, dxdy \tag{15}$$

Introducing non-dimensional variables into (15) the torsional moment can be related to non-dimensional stress function

$$T = G \cdot \omega \cdot a^4 \cdot 2\iint\Psi(X, Y)\,dXdY \tag{16}$$

Then non-dimensional torsional stiffness can by defined as

$$SZ = \frac{T}{G \cdot \omega \cdot a^4} = 2\iint\Psi(X,Y)\,dXdY \tag{17}$$

Putting approximate solution given by Eqs. (7) and (10) into (17) and performing the integration analytically we obtain

$$SZ = 2\sum_{j=1}^{NC} c_j \sum_{r=1}^{NR} I_{jr} - \frac{1}{3}\sum_{r=1}^{NR}[(b_r^3 - a_r^3)(d_r - c_r) + (b_r - a_r)(d_r^3 - c_r^3)] \tag{18}$$

where NR is a number of rectangles on which considered cross section is divided, a_r, b_r, c_r, d_r are values which determine the coordinates of apexes of r-th rectangle (see Fig. 8), and

$$I_{jr} = \int_a^b\int_c^d \ln[(x-\xi_j)^2 + (y-\eta_j)^2]\,dxdy = \frac{\pi}{2}\left\{\begin{array}{l} \dfrac{c\cdot(c-\eta_j)\cdot(c-2\eta_j)}{|c-\eta_j|}\cdot\left[\dfrac{(a-\xi_j)}{|a-\xi_j|} - \dfrac{(b-\xi_j)}{|b-\xi_j|}\right] - \\ -\dfrac{d\cdot(d-\eta_j)\cdot(d-2\eta_j)}{|d-\eta_j|}\cdot\left[\dfrac{(a-\xi_j)}{|a-\xi_j|} - \dfrac{(b-\xi_j)}{|b-\xi_j|}\right] \end{array}\right\} +$$

$$+ [(a-\xi_j)^2 - (c-\eta_j)^2]\cdot\tan^{-1}\left(\frac{c-\eta_j}{a-\xi_j}\right) - [(a-\xi_j)^2 - (d-\eta_j)^2]\cdot\tan^{-1}\left(\frac{d-\eta_j}{a-\xi_j}\right) -$$

$$-[(b-\xi_j)^2 - (c-\eta_j)^2]\cdot\tan^{-1}\left(\frac{c-\eta_j}{b-\xi_j}\right) + [(b-\xi_j)^2 - (d-\eta_j)^2]\cdot\tan^{-1}\left(\frac{d-\eta_j}{b-\xi_j}\right) +$$

$$+ (a-\xi_j)\cdot(c-\eta_j)\cdot\ln[(a-\xi_j)^2 + (c-\eta_j)^2] - (a-\xi_j)\cdot(d-\eta_j)\cdot\ln[(a-\xi_j)^2 + (d-\eta_j)^2] -$$

$$-(b-\xi_j)\cdot(c-\eta_j)\cdot\ln[(b-\xi_j)^2 + (c-\eta_j)^2] + (b-\xi_j)\cdot(d-\eta_j)\cdot\ln[(b-\xi_j)^2 + (d-\eta_j)^2] +$$

$$+ 3(a-b)\cdot(d-c) \tag{19}$$

In this way formula (18) presents non-dimensional torsional stiffness for cross sections, which can divide, on finite number of rectangles. Only coefficients c_j are obtained numerically from satisfaction of boundary conditions in collocation sense.

Results of non-dimensional torsional stiffness for five cross sections considered in this paper are given in Tables 1-5. On the other hand the torsional stiffness for these cross sections if they are thin-walled (small values of E parameter) can be calculated on the base of Bach method mentioned in the

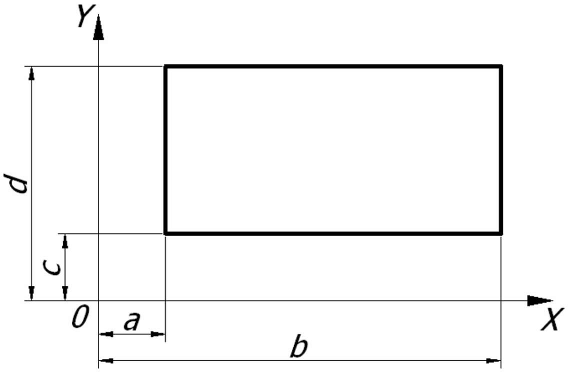

Fig. 8 Notation of integral limits for rectangle

Introduction what gives the following formula:

$$SZ_{Bach} = \frac{T}{G \cdot \omega} = \sum_{i=1}^{NR} \frac{1}{3} c_i^3 b_i \tag{20}$$

where NR is number of rectangular components on which cross section is divided, b_i is higher dimension of i-th rectangle, c_i is lower dimension of i-th rectangle.

Then from Bach theory we can calculate the following formulae for considered cross sections: for L, + and ʃ - shapes

Table 1 Comparison of results obtained on the base MFS (formula (18)) with results given by Bach's formula (21) for L - cross section .

E	SZ on the base (18)	Relative percentage difference between (18) and (21)
0.05	8.04E-5	1.09%
0.1	0.000619	2.32%
0.15	0.00201	3.49%
0.2	0.00458	4.83%
0.25	0.00857	6.33%
0.3	0.0142	7.93%
0.35	0.0215	9.69%
0.4	0.0306	11.7%
0.45	0.0413	14.0%
0.5	0.0535	16.8%
0.55	0.0668	20.4%
0.6	0.0808	24.8%
0.65	0.0947	30.5%
0.7	0.108	37.8%
0.75	0.120	47.2%
0.8	0.129	59.1%
0.85	0.135	74.0%
0.9	0.139	92.3%
0.95	0.140	114.0%

Table 2 Comparison of results obtained on the base MFS (formula (18)) with results given by Bach's formula (22) for I - cross section

E	SZ on the base (18)	Relative percentage difference between (18) and (22)
0.025	1.54E-5	0.303%
0.05	0.000122	0.726%
0.075	0.000410	1.17%
0.1	0.000949	1.63%
0.125	0.00183	2.11%
0.15	0.00312	2.58%
0.175	0.00488	3.07%
0.2	0.00719	3.54%
0.225	0.0101	4.04%
0.25	0.0138	5.91%
0.275	0.0179	4.97%
0.3	0.0228	5.40%
0.325	0.0285	5.79%
0.35	0.0350	6.15%
0.375	0.0423	6.50%
0.4	0.0594	21.0%
.0425	0.0593	7.28%
0.45	0.0692	7.80%

$$SZ_{Bach} = \frac{1}{3} \cdot E^3 \cdot 1 + \frac{1}{3} \cdot E^3 \cdot (1 - E) = \frac{1}{3} \cdot E^3 \cdot (2 - E) \tag{21}$$

and for [, and I, - shapes

$$SZ_{Bach} = \frac{1}{3} \cdot E^3 \cdot (3 - 2 \cdot E) \tag{22}$$

From the steel structure design point of view Bach type formulae are very useful. They permit to calculate the torsional stiffness in a very simple and fast way. There is some interest for similar formulae valid not only for the thin-walled sections. One can observe that formulae form Bach theory are polynomials of E parameter and that polynomials have only terms with E^3 and E^4. Moreover if $E = 0$ then the torsional stiffness must equal zero. It suggests that results obtained for torsional stiffness, on the base of MFS, which are not only for thin-walled sections, can be approximated by formula

$$SZ = \sum_{j=3}^{7} b_j E^j \tag{23}$$

where b_j are unknown coefficients which must be determine by least square method.

Using least square method on the base formula (23) and results given in Tables 1-5, the following approximate formulae for nondimensional stiffness are obtained:
for L - cross section

$$SZ = 0.748 \cdot E^3 - 1.19 \cdot E^4 + 2.29 \cdot E^5 - 3.01 \cdot E^6 + 1.31 \cdot E^7 \tag{24}$$

Table 3 Comparison of results obtained on the base MFS (formula (18)) with results given by Bach's formula (22) for [-cross section

E	SZ on the base (18)	Relative percentage difference between (18) and (22)
0.05	0.000120	0.376%
0.075	0.000399	0.556%
0.1	0.000927	0.740%
0.125	0.00177	0.923%
0.15	0.00300	1.12%
0.175	0.00467	1.33%
0.2	0.00682	1.56%
0.225	0.00951	1.79%
0.25	0.0128	2.02%
0.275	0.0166	2.27%
0.3	0.0211	2.53%
0.325	0.0262	2.80%
0.35	0.0319	3.08%
0.375	0.0383	3.35%
0.4	0.0453	3.62%
0.425	0.0530	3.85%
0.45	0.0613	4.01%
0.475	0.0705	3.94%

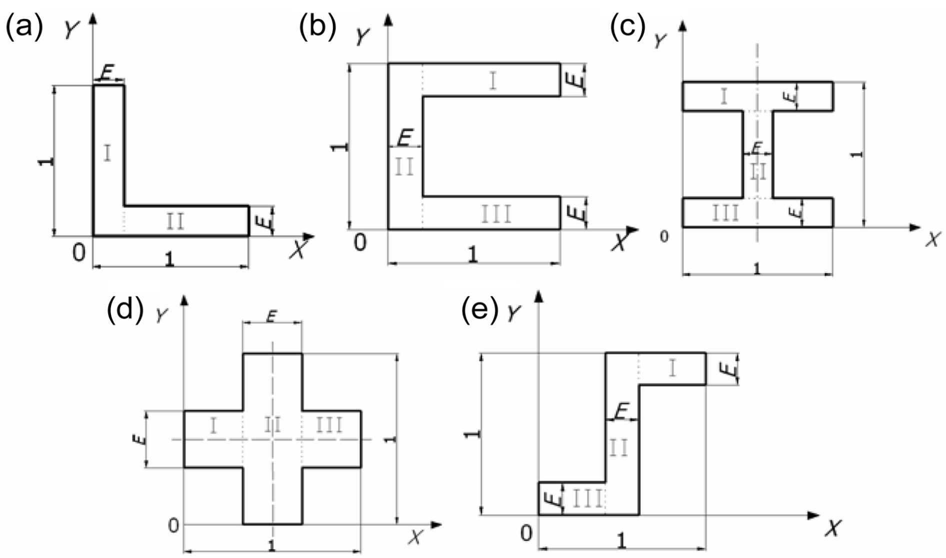

Fig. 9 Manner of division of considered cross sections on rectangles

for I - cross section

$$SZ = -0.776 \cdot E^3 + 29.3 \cdot E^4 - 175.0 \cdot E^5 + 428.0 \cdot E^6 - 373.0 \cdot E^7 \tag{25}$$

Table 4 Comparison of results obtained on the base MFS (formula (18)) with results given by Bach's formula (21) for +-cross section

E	SZ on the base (18)	Relative percentage difference between (18) and (21)
0.05	8.27E-5	1.73%
0.1	0.000657	3.62%
0.15	0.00220	5.57%
0.2	0.00519	7.48%
0.25	0.0100	9.23%
0.3	0.0171	10.5%
0.35	0.0265	11.1%
0.4	0.0382	10.7%
0.45	0.0517	9.00%
0.5	0.0665	5.97%
0.55	0.0815	1.34%
0.6	0.0959	5.07%
0.65	0.109	13.5%
0.7	0.120	24.0%
0.75	0.128	37.0%
0.8	0.134	52.5%
0.85	0.138	70.6%
0.9	0.140	91.1%
0.95	0.140	114.0%

Table 5 Comparison of results obtained on the base MFS (formula (18)) with results given by Bach's formula (21) for I, and -cross section.

E	SZ on the base (18)	Relative percentage difference between (18) and (21)
0.05	8.08E-5	0.563%
0.1	0.000627	1.10%
0.15	0.00205	1.67%
0.2	0.00469	2.280%
0.25	0.00885	2.98%
0.3	0.0147	3.81%
0.35	0.0225	4.86%
0.4	0.0322	6.17%
0.45	0.0437	7.83%
0.5	0.0568	10.1%
0.55	0.0711	13.2%
0.6	0.0857	17.6%
0.65	0.0999	23.7%
0.7	0.113	31.9%
0.75	0.123	42.5%
0.8	0.131	55.9%
0.85	0.137	72.3%
0.9	0.140	91.7 %
0.95	0.141	114.0 %

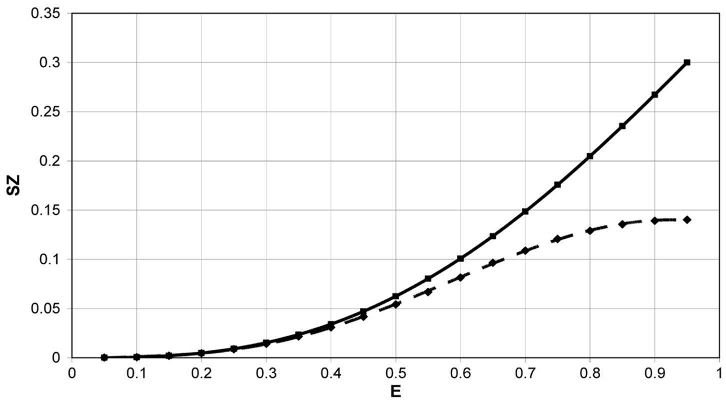

Fig. 10 Comparison of results given by proposed here formula (24)-solid line and Bach's formula (21)-broken line for L - cross section

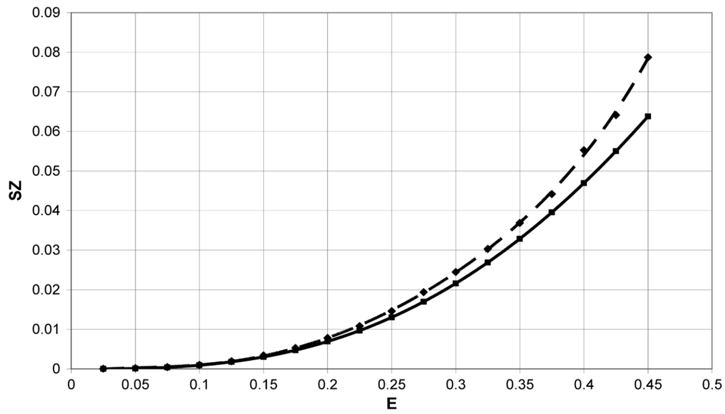

Fig. 11 Comparison of results given by proposed here formula (25)-solid line and Bach;s formula (22)- broken line for I-cross section

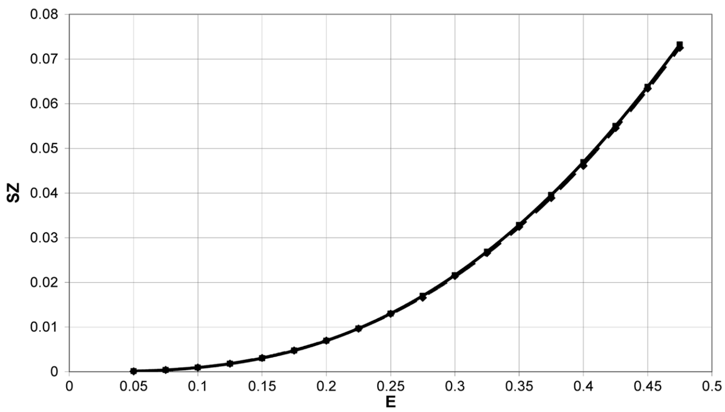

Fig. 12 Comparison of results given by proposed here formula (26)-solid line and Bach's formula (22)-broken line for [- cross section

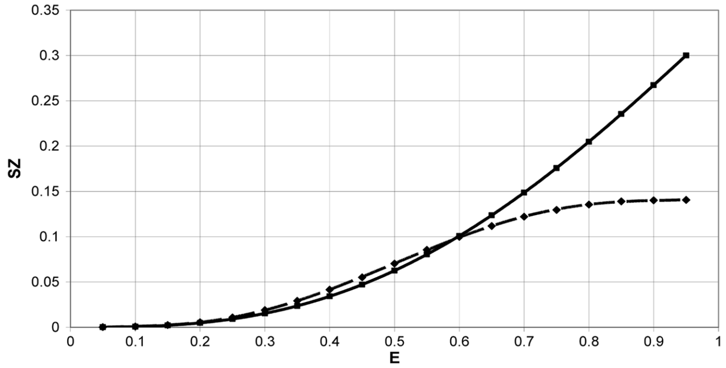

Fig. 13 Comparison of results given by proposed here formula (27)-solid line and Bach's formula (21)- broken line ┼-cross section

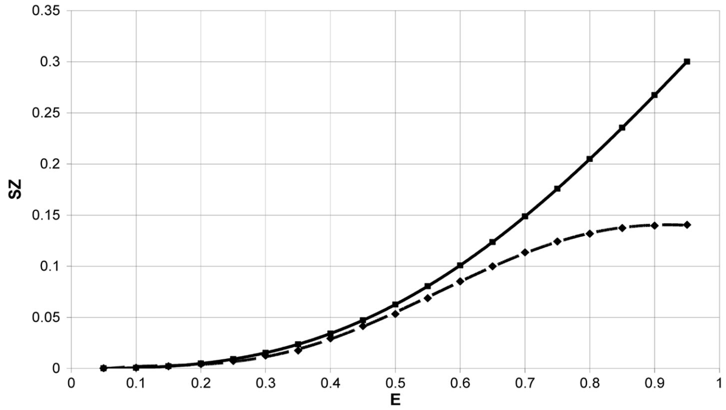

Fig. 14 Comparison of results given by proposed here formula (28)-solid line and Bach's formula (21)-broken line for ⌠-cross section

for [- cross section

$$SZ = 1.04 \cdot E^3 - 1.31 \cdot E^4 + 3.14 \cdot E^5 - 7.33 \cdot E^6 + 6.26 \cdot E^7 \qquad (26)$$

for ┼ - cross section

$$SZ = 0.302 \cdot E^3 + 2.67 \cdot E^4 - 6.56 \cdot E^5 + 4.84 \cdot E^6 - 1.11 \cdot E^7 \qquad (27)$$

for ⌠ - cross section

$$SZ = 0.683 \cdot E^3 - 0.763 \cdot E^4 + 1.77 \cdot E^5 - 3.08 \cdot E^6 + 1.53 \cdot E^7 \qquad (28)$$

Comparison of results given by proposed here formulae (24-28) and Bach's formulae are given on Figs. 10-14. Relative differences between these results are presented also in Tables 1-5.

4. Discussion results and conclusions

The method of fundamental solutions was applied with success to solution of boundary value problems related with torsion of prismatic rods with cross section shaped as L, [, +, I, and ʃ. Thanks to this method the closed formulae for stress functions in whole considered cross sections are obtained. These formulae are a linear superposition of fundamental solutions of 2-D Laplace equation plus a particular solution of nonhomogeneous equation. The coefficients in these solutions are determined from satisfaction of boundary condition in collocation sense. The form of the obtained formulae permits an analytical calculation of integrals through the considered cross section from stress function and in consequence the closed formulae for nondimnsional stiffness. In relation to the well known Bach's method for calculation of torsional stiffness for thin-walled sections, formulae obtained in this paper for torsional stiffness of L, [, +, I, and ʃ shaped cross sections are valid for any given thickness (parameter E). The obtained formulae permit to verify the region of applicability of Bach's method. After comparing the results of both methods (see Tables 1-5, and Figs. 10-14) one can state that the formulae obtained from Bach's theory are right for engineering calculations in a region that depends on the shape of a cross section.

These regions are as follow:

- for L shaped cross section until $E = 0.3$ of non-dimensional thickness,
- for I shaped cross section until $E = 0.2$ of non-dimensional thickness,
- for [shaped cross section until $E = 0.3$ of non-dimensional thickness,
- for + shaped cross section until $E = 0.6$ of non-dimensional thickness,
- for ʃ shaped cross section until $E = 0.3$ of non-dimensional thickness.

In this paper five cross sections are considered in which parameter of thickness E was changed from 0 to 1 or from 0 to 0.5 what gives in limit case unit square cross section or unit square with crack. The proposed method can be used for another cross sections composed of rectangles, which in a limit case don't have to be square, without problems.

In this paper torsion problem was considered in frame linear elasticity. It is well known that if the shearing strain is large one have elastoplastic torsion problem (Mendelson 1968). In such case governing equation is non-linear but it can be expressed as Poisson equation with non-linear right hand side in plastic region (Mendelson 1975). The MFS is easy applicable for linear problems because fundamental solutions are known for linear equations. However for non-linear torsion problem it also can be applied by using same extension of this method such analog equation method given Wang *et al.* (2005).

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