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Buckling and vibration of symmetric laminated composite plates with edges elastically restrained

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Abstract. The finite strip transition matrix technique, a semi analytical method, is employed to obtain the buckling loads and the natural frequencies of symmetric cross-ply laminated composite plates with edges elastically restrained against both translation and rotation. To illustrate the accuracy and the validation of the method several example of cross play laminated composite plates were analyzed. The buckling loads and the frequency parameters are presented and compared with available results in the literature. The convergence study and the excellent agreement with known results show the reliability of the purposed technique.

Key words: buckling; vibration; finite strip; elastic restraint; laminated plates.

1. Introduction

In recent years, laminated plates have been used in many industries e.g. in the civil, automotive, aerospace, and marine industries. Determination of the critical loads and the vibration frequencies of such plates are very important for the designer. Many researchers have considered the buckling and vibration of such plates subjected to classical boundary conditions. In practical applications, many boundary conditions in the design of such plates allow certain degree of rotation or translation or both. It is well known that the classical boundary conditions represent an idealization of the elastic restraint boundary conditions. Although there are a quite few publications regarding plates with edge restrained in the literature, most of the plates considered are isotropic plates (Lee and Lin 1992, Zhou 1995, Gorman 2000), Mindlin plates (Xiang *et al*. 1997, Saha *et al.* 1996) and most of the boundary conditions considered are elastically restrained against rotation only (Kobayashi and Soda 1991, Grossi and Bahat 1995). The main aim of this paper is to provide some solutions of such plates with edges elastically restrained against both rotation and translation.

For isotropic plates, Paik and Thayamballi (2000) investigated the buckling strength characteristic of steel plating elastically restrained in two edges and simply supported at the other two edges by analytical methods. Gorman (2000) used the superposition method to obtain the buckling loads and the free vibration frequencies of rectangular elastic restrained isotropic plates subjected to uniform in plane loading. Zhou (1995) used a set of static beam function in the Rayleigh Ritz method to determine the natural frequencies of elastically rectangular plates. Xiang *et al*. (1997) used Mindline plates theory in conjuction with Ritz method to study the free vibration of thick rectangular plates with edges restrained

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against both rotation and translation.

For composite plates, Rohani and Marcellier (1999) extended the small-deflection theorem for orthotropic plates to investigate the buckling and vibration analysis of anisotropic rectangular sandwich plates with edges elastically restrained against rotation. Cheung and Zhou (2000) used a set o static beam function to analyze the vibration of orthotropic rectangular plate with elastic intermediate line support. Bank and Yin (1996) discussed uniaxial buckling of an orthotropic plate, simply supported on its loaded edges and free and rotationally restrained on its unloaded edges. They presented a parametric study and buckling curves for typical composite materials for this special case of boundary conditions.

In this paper, we analyzed the buckling and vibration of elastically restrained symmetrically laminated plates. The elastically restrained boundary conditions in this paper are satisfied exactly. The finite strip transition matrix technique (Ashour 2001, Farag and Ashour 2000 and Farag 1994) is employed to investigate the buckling and the elastically restrained boundary conditions. The convergence and the accuracy of the method are investigated and compared with known results. The applied method has been validated through the excellent agreement with other results in the literature.

2. The governing equation and the boundary conditions

2.1. The governing equation

Under the assumption of the classical deformation theory, the partial differential equation governing the vibration of rectangular plates of mass density ρ and thickness h (multiple unidirectional, symmetric cross ply composite or single orthotropic ply) under in-plane forces, as shown in Fig. 1 is given in terms of the plate deflection W: by

$$D_{11}\frac{\partial^4 W}{\partial x^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 W}{\partial x^2 y^2} + D_{22}\frac{\partial^4 W}{\partial y^4} - \overline{N}_x\frac{\partial^2 W}{\partial x^2} - \overline{N}_y\frac{\partial^2 W}{\partial y^2} - 2\overline{N}_{xy}\frac{\partial^2 W}{\partial x \partial y} = -\overline{m}\frac{\partial^2 W}{\partial t^2}. \qquad (1)$$

where $\overline{m} = \rho h_0$ is the mass per unit area, h_0 is the thickness of the plate, $\overline{N}_x$, $\overline{N}_y$, $\overline{N}_{xy}$ are the in-plane direct and shear forces per unit length and D_{11}, D_{22}, D_{12} and D_{66} are the flexural rigidities of the plate given by

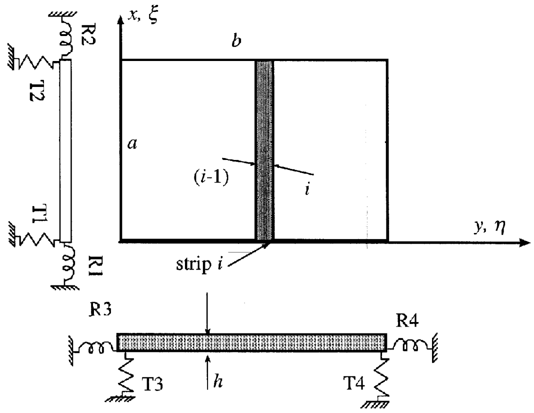

Fig. 1 Composite rectangular plate subjected to in-plane forces

$$D_{ij} = \int_{-h/2}^{h/2} \overline{Q}_{ij}^{k} z^2 dz \quad i, j = 1, 2, 6 \tag{2}$$

where z_k is the distance from the midplane of the plate to the bottom of the kth layer and $\overline{Q}_{ij}^{k}$ are the plane stress transformed reduced stiffness. The plane stress reduced stiffness of the lamina Q_{ij}^{k} are given by

$$Q_{11} = \frac{E_{11}}{(1-\nu_{12}\nu_{21})}, \quad Q_{22} = \frac{E_{22}}{(1-\nu_{21}\nu_{12})}, \quad Q_{12} = \frac{\nu_{21}E_{11}}{(1-\nu_{12}\nu_{21})}, \quad Q_{21} = Q_{12}, \quad Q_{66} = G_{12} \tag{3}$$

where E_{11}, E_{22} are the longitudinal and transverse Young's moduli parallel and perpendicular to the fibers, respectively and G_{12} is the in plane shear modulus of elasticity, ν_{12} and ν_{21} are the Poisson's ratios.

2.2. The boundary conditions

In this paper, the elastic restraint boundary conditions at edges $x = 0$ and at $x = a$ can be written as.

$$R_1\frac{\partial W}{\partial x} - \left\{D_{11}\frac{\partial^2 W}{\partial x^2} + \nu_{12}D_{11}\frac{\partial^2 W}{\partial y^2}\right\} = 0, \quad T_1 W + D_{11}\frac{\partial^3 W}{\partial x^3} + (4D_{66} + \nu_{12}D_{22})\frac{\partial^3 W}{\partial y^2 x} = 0 \tag{4}$$

at $x = 0$,

$$R_2\frac{\partial W}{\partial x} + \left\{D_{11}\frac{\partial^2 W}{\partial x^2} + \nu_{12}D_{11}\frac{\partial^2 W}{\partial y^2}\right\} = 0, \quad T_2 W - D_{11}\frac{\partial^3 W}{\partial x^3} + (4D_{66} + \nu_{12}D_{22})\frac{\partial^3 W}{\partial y^2 x} = 0 \tag{5}$$

at $x = a$, where T_1 and T_2 are the translational stiffness per unit length of the edges $x = 0$ and $x = a$ respectively; R_1 and R_2 are the rotational stiffness per unit length at the edges $x = 0$ and $x = a$ respectively and the boundary conditions considered along the y-direction are any combinations of the elastic restrained against both rotation and translation. At $y = 0$, the boundary conditions can be described as:

$$R_3\frac{\partial W}{\partial y} - D_{22}\left\{\frac{\partial^2 W}{\partial y^2} + \nu_{12}\frac{\partial^2 W}{\partial x^2}\right\} = 0, \quad T_3 W + D_{22}\frac{\partial^3 W}{\partial y^3} + (4D_{66} + \nu_{12}D_{22})\frac{\partial^3 W}{\partial x^2 y} = 0 \tag{6}$$

and at $y = b$

$$R_4\frac{\partial W}{\partial y} + D_{22}\left\{\frac{\partial^2 W}{\partial y^2} + \nu_{12}\frac{\partial^2 W}{\partial x^2}\right\} = 0, \quad T_4 W - D_{22}\frac{\partial^3 W}{\partial y^3} + (4D_{66} + \nu_{12}D_{22})\frac{\partial^3 W}{\partial x^2 y} = 0 \tag{7}$$

where T_3 and T_4 are the translational stiffness per unit length of the edges $y = 0$ and $y = b$ respectively; R_3 and R_4 are the rotational stiffness per unit length at the edges $y = 0$ and $y = b$ respectively.

3. Method of solution

For a striped plate in the x-direction as shown in Fig. 1, the shape function $W(\xi, \eta)$ may be assumed in the form

$$W(\xi,\eta)=\sum_{m=1}^{M}X_m(\xi)Y_m(\eta) \tag{8}$$

where, $\xi = x/a$, $\xi = y/b$, $Y_m(\eta)$ is unknown function to be determined and $X_m(\xi)$ is chosen a priori, the basic function in ξ- direction. The most commonly used is the eigenfunction obtained from the solution of the differential equation of a beam vibration under the prescribed conditions of the stripe at $\xi = 0$ and $\xi = 1$. The beam function in the x-direction is considered as a strip element of the plate and the flexural rigidity EI of the beam can replaced by $(1-\nu^2)D_{11}$ and for $\nu = 0.3$, it can be just approximated by $EI \approx D_{11}$. The free vibration of a beam of length a can be described by the non-dimensional differential equation.

$$\frac{d^4X}{d\xi^4} = \frac{\rho A}{D_{11}}a^4\omega^2X, \tag{9}$$

and the boundary conditions for a beam elastically restrained against both rotation and translation are:

$$F_{R1}\frac{dX(\xi)}{d\xi}-\left\{\frac{d^2X(\xi)}{d\xi^2}\right\} = 0, \quad F_{T1}X(\xi)+\frac{d^3X(\xi)}{d\xi^3} = 0 \tag{10}$$

at $\xi = 0$, where $F_{R1} = \dfrac{R_1a}{D_{11}}$, $F_{T1} = \dfrac{T_1a^3}{D_{11}}$ and

$$F_{R2}\frac{dX(\xi)}{d\xi}+\left\{\frac{d^2X(\xi)}{d\xi^2}\right\} = 0, \quad F_{T2}X(\xi)-\frac{d^3X(\xi)}{d\xi^3} = 0 \tag{11}$$

at $\xi = 1$, where $F_{R2} = \dfrac{R_2a}{D_{11}}$, $F_{T2} = \dfrac{T_2a^3}{D_{11}}$.

The solution of the beam equation can be written as

$$X_i(\xi) = A\sin(\mu_i\xi)+B\cos(\mu_i\xi)+C\sinh(\mu_i\xi)+D\cosh(\mu_i\xi) \tag{12}$$

where $\mu_i^4 = \dfrac{\rho A}{D_{11}}\omega_i a^4$. One can obtain a system of homogenous linear equations by stratifying the boundary conditions Eqs. (10), (11) at $\xi = 0$ and 1, the roots of the characteristic equation of this system represent the eigenvalues and the corresponding eigenvectors can be obtained from the system of equations. Thus, the shape function in the x-direction $X_i(\xi)$ can be obtained. Substituting the strip function in Eq. (1), multiplying both sides by $X_n(\xi)$, and integrating from 0 to 1, one obtains

$$\beta^4\frac{d^4Y_m}{d\eta^4}+\beta^2\sum_{m=1}^{M}\left\{2\frac{D_{12}+2D_{66}}{D_{22}}\frac{d^2Y_m}{d\eta^2}\frac{C_{nm}}{A_{mm}}\right\}+\left[\frac{D_{11}}{D_{22}}\frac{E_{mm}}{A_{mm}}\right]Y_m-\sum_{m=1}^{M}\frac{\bar{N}_xa^2}{D_{22}}\frac{C_{nm}}{A_{mm}}Y_m$$

$$-2\beta\frac{\bar{N}_{xy}a^2}{D_{22}}\sum_{m=1}^{M}\frac{B_{nm}}{A_{mm}}\frac{dY_m}{d\eta}-\beta^2\frac{\bar{N}_ya^2}{D_{22}}\frac{d^2Y_m}{d\eta^2}-\bar{m}a^4\omega^2/D_{22}Y_m = 0 \qquad n = 1, 2, 3,..., M \tag{13}$$

where ω is the angular frequency and $\beta = a/b$ is the aspect ratio, $A_{nm}=\int_0^1 X_n(\xi)X_m(\xi)d\xi$, $B_{nm}=\int_0^1 \frac{dX_n(\xi)}{d\xi}X_m(\xi)d\xi$, $C_{nm} = \int_0^1 \frac{d^2X_n(\xi)}{d\xi^2}X_m(\xi)d\xi$, and $E_{nm} = \int_0^1 \frac{d^4X_n(\xi)}{d\xi^4}X_m(\xi)d\xi$.

It is to be noted that the orthogonality conditions yield and for $A_{mn} = 0$ and $E_{mn} = 0$ for $m \neq n$. By transforming the fourth order differential Eqs. (13) into 4M-number of first order ordinary differential equations, the following relation can be obtained at any nodal line j of the divided plate:

$$\frac{d}{d\eta}\{F\}_j = [S]_j\{F\}_j; \quad j = 1, 2, 3, \ldots, N, \tag{14}$$

where, $\{F\}_j = \{\overline{Y}_1\ \overline{Y}_2 \cdots \overline{Y}_i \cdots \overline{Y}_n\}_j^T$, $\overline{Y}_i = \left\{Y_i \frac{dY_i}{d\eta} \frac{d^2Y_i}{d\eta^2} \frac{d^3Y_i}{d\eta^3}\right\}^T$ and $[S]_j$ is a *4M* by *4M* matrix, the solution of the above coupled system of equation is carried out using the transition matrix technique [Zurmuhl 1976]

$$\{F\}_i = [T]_i\{F\}_{i-1} \tag{15}$$

where $[T]_i$ is called the transition matrix of the strip i while $\{F\}_i$ and $\{F\}_{i-1}$ are the nodal vectors of the boundaries i and i-1. The solution of the 4M first order differential equations of motion (14) is carried out using 2M-number of initial vectors $\{Y\}_0$ at $\eta = 0$. Relation (15) is applied across the divided plate until the final end at $\eta = 1$ is reached. Therefore, 2M-number of solutions S_n, $n = 1,2, \ldots, 2M$ can be obtained. The true solutions S can be obtained as a linear combination of these solutions as:

$$S = \sum_{n=1}^{2M} C_n S_n \tag{16}$$

where C_n are arbitrary constants. These constants can be determined by satisfying 2M-number of boundary conditions at $\eta = 1$.

3.1. The boundary conditions at η = 0 and η = 1

Using Eq. (8), the boundary conditions Eqs. (6,7) at $\eta = 0$, after some algebraic manipulation, can be expressed as the folowing:

$$\frac{d^2Y_n}{d\eta^2} + \frac{\nu_{12}}{\beta^2}\sum_{m=1}^{M}\frac{C_{nm}}{A_{mm}}Y_m = F_{R3}\frac{dY_n}{d\eta}, \quad \frac{d^3Y_n}{d\eta^3} + \frac{1}{\beta^2}\left(\frac{4D_{66}}{D_{22}} + \nu_{12}\right)\sum_{m=1}^{M}\frac{C_{nm}}{A_{mm}}\frac{dY_m}{d\eta} = -F_{T3}Y_n, \tag{17}$$

where $F_{T3}=\frac{b^3T_3}{D_{22}}$, and $F_{R3}=\frac{bR_3}{D_{22}}$ and at $h = 1$

$$\frac{d^2Y_n}{d\eta^2} + \frac{\nu_{12}}{\beta^2}\sum_{m=1}^{M}\frac{C_{nm}}{A_{mm}}Y_m = -F_{R4}\frac{dY_n}{d\eta}, \quad \frac{d^3Y_n}{d\eta^3} + \frac{1}{\beta^2}\left(\frac{4D_{66}}{D_{22}} + \nu_{12}\right)\sum_{m=1}^{M}\frac{C_{nm}}{A_{mm}}\frac{dY_m}{d\eta} = -F_{T4}Y_n, \tag{18}$$

where $F_{T3}=\frac{b^3T_4}{D_{22}}$, and $F_{R4}=\frac{bR_4}{D_{22}}$.

3.2. The eigenvalue problems

For the buckling problem, the matrix S can be rewritten as a stander eigenvalue problem as:

$$[[\boldsymbol{K}] - P_c[\boldsymbol{K}_g]][Y] = 0.0 \tag{19}$$

where $[K]$ is the stiffness matrix, $[K_g]$ is the geometric stiffness matrix due to the in-plane forces, P_c and Y are the eigenvalue and the eigenvector respectively. For the vibration problem, the eigenvalue problem can be written as

$$[[\boldsymbol{K}] - \lambda^2[M]][Y] = 0.0$$

where λ is the natural frequency of the plate. An iteration method is used to calculate the buckling loads and the frequency parameters.

4. Numerical results comparisons, and discussion

In order to establish a validation of our results, convergence and comparison studies are carried out. In the following analysis, the designation CSCF means that the edges $x = 0$, $x = a$, $y = 0$, $y = b$ are clamped, simply supported, clamped and Free respectively. First, isotropic square plates with all edges simply supported or all edges clamped are considered. Table 1 shows the convergence and the comparison analysis of the buckling loads for all edges simply supported plates and clamped plates. The buckling loads for SSSS and CCCC plates are shown for three cases, uniaxial buckling, biaxial buckling and pure shear buckling. The corresponding normalized buckling loads are $P_{cru} = \dfrac{N_{ycr}hb^2}{\pi^2 D}$, $P_{crb} = \dfrac{N_{xcr}hb^2}{\pi^2 D}$, $P_{crs} = \dfrac{N_{xycr}hb^2}{\pi^2 D}$ respectively, where, $D = \dfrac{Eh^3}{12(1-\nu^2)}$, ν is the Poisson's ration equal to 0.3 in this case. It is very clear that the method has a stable and fast convergence only after few terms. Also, the comparison with other numerical or exact solutions shows excellent agreement.

Second, orthotropic plates are considered for all edges are simply supported and all edges are clamped. The material properties is given by $D_{11}/D_{22} = 10$, $(D_{12} + D_{66})/D_{22} = 1.67$, and $\nu_{12} = 0.333$. The results are compared with those of Whitney (1987) for SSSS and CCCC plates and they are presented

Table 1 Buckling loads for isotropic simply supported (SSSS) and clamped (CCCC) plates

M	SSSS			CCCC		
	Uniaxial	Biaxial	Pure Shear	Uniaxial	Biaxial	Pure Shear
1	4.0000	1.9999	10.4029	10.7043	5.3395	16.5411
3	4.0000	2.0002	9.3939	10.0929	5.3079	14.7780
4	4.0000	2.0002	9.3813	10.093	5.3079	14.7696
5	4.0000	2.0002	9.3494	10.0769	5.3047	14.7225
6	4.0000	2.0002	9.3470	10.0769	5.3047	14.7199
7	4.0000	2.0002	9.3426	10.0748	5.3041	14.6904
(Reddy 1995)	4.0000	2.0000		10.08		
(Wang 1997a,b)	4.0000		9.3245	10.0740	5.3036	14.6421

Table 2 Convergence of the shear buckling load for orthotropic plates $P_{crs} = \dfrac{X_{xycr}a^2}{D_{22}}$

M	CCCC	(Whitney 1987)	SSSS	(Whitney 1987)
2	550.6842	768.0	332.152	392.9
3	504.5323	544.0	309.621	313.7
4	504.3081		309.157	310.2
5	503.474	505.2	308.592	309.0
6	503.4306		308.538	
7	503.3612	503.6	308.471	

Table 3 Convergence study of the natural frequencies of symmetric cross-play laminated (0/90/0) plates $\lambda = \dfrac{\omega a^2}{h}\sqrt{\dfrac{\rho}{E_{11}}}$

E_{22}/E_{11}=20	F_R	F_T		λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
20	20	20	2	25.9072	33.3291	69.4007	74.0579		
20	20	20	3	25.9029	33.3291	49.5528	74.0579	69.3837	
20	20	20	4	25.9029	33.3254	49.5528	74.0384	69.3837	74.5694
20	20	20	5	25.9025	33.3254	49.5506	74.0384	69.3815	74.5694
20	20	20	6	25.9025	33.3247	49.5506	74.5682	69.3815	74.0349
20	20	20		25.91*					

*Rais-Rohani and Marcellier (1999)

in Table 2. The convergence is much faster than that obtained by Whitney (1987) for such plates. Another convergence analysis for the natural frequencies of symmetric cross play laminated plates with edges equally elastically restrained against rotation (F_{R1}=F_{R2}=F_{R3}=F_{R4}=F_R) is considered in Table 3 and compared of those by Rohani and Marcellier (1999). The material properties are $E_{22}/E_{11} = 20$, $G_{12}/E_{11} = 0.5$, $\nu_{12} = .25$. The convergence in this case is also fast and stable. In all cases, excellent agreements are achieved.

In all calculation we used $M = 6$ and N =20. The biaxial buckling loads are presented in Table 4 for different combinations of classical boundary conditions and compared with some of the exact solutions (Lévy type solutions) obtained by Khedir (1988). The corresponding normalized buckling loads are given by $P_{crs} = \dfrac{N_{xycr}a^2}{\pi^2 h^3 E_1}$. The natural frequencies of elastically restrained against rotation of symmetric

Table 4 Biaxial buckling loads for cross play laminated plates (0/90/0) E_{22}/E_{11}=20 $\bar{N}_x = \bar{N}_y$ and $P_c = \dfrac{\bar{N}_x a^2}{E_1 h^3}$

B.C.	P_{crb}	Exact (Khdeir 1988)	*B.C.*	P_{crb}	*B.C.*	P_{crb}
SSSS	9.591274	9.591	SCFS	3.433958	CCSS	11.33679
SSCS	14.02611	14.026	SCSC	15.07619	CCSC	17.90293
SSFS	1.981392	1.978	SCCC	21.92654	CCCC	24.4725
SSCC	21.69436	21.709	SCFC	5.696648	CCFC	8.507665
SSFC	4.688254	4.683	SCSF	3.43304	CCFF	5.678682
SSFF	1.41948	1.420	SCCF	5.694448	SCSS	9.749077
			SCFF	2.68544	SCCS	15.07619

Table 5 Comparison of the natural frequency of symmetric cross-play laminated plate with edges elastically restrained against rotation $\lambda = \frac{\omega a^2}{h}\sqrt{\frac{\rho}{E_{11}}}$

F_{R1}=F_{R2}	F_{R3}=F_{R4}	E_{22}/E_{11}	λ_1	Ref*	λ_2	λ_3	λ_4	λ_5
0.25	0.25	10	11.059	11.06	18.963	34.503	37.431	43.016
0.25	0.25	20	14.542	14.54	22.234	38.992	51.827	56.399
0.25	0.25	30	17.34	17.34	25.087	43.032	63.017	98.527
0.25	0.25	40	19.745		27.65	46.727	72.5	
20	20	10	19.133	19.14	26.987	42.556	49.868	55.468
20	20	20	25.902	25.91	33.325	69.382	49.551	74.035
20	20	30	31.239	31.24	38.64	55.684	82.784	84.503
20	20	40	35.788		43.309	61.208	97.302	90.103

*Rais-Rohani and Marcellier (1999)

cross-play plate (F_{R1}=F_{R2}=F_{R3}=F_{R4}=F_R) are presented in Table 5. The results are compared with those of Rohani and Marcellier (1999).

Tables 6 shows the natural frequencies of elastically restrained symmetric Cross play plates with extreme values of the elastic restraints coefficients (classical boundary conditions) with some comparison of results of exact solutions available in the literature. From the boundary condition Eqs. (10), (11), (17), (18), it can be seen as $F_R \to \infty$ and $F_T \to \infty$ we obtain clamped edges. When, $F_R \to 0$ and $F_T \to \infty$, the boundary condition approaches the simply supported case and when, $F_R \to 0$ and $F_T \to 0$, the boundary condition approaches the free case. In all cases, the results agree very well with the corresponding cases. The material properties are G_{12}/E_{11} = 0.5, ν_{12} = .25 and for different E_{22}/E_{11}. Table 7 shows the natural frequency of all edges equally restrained against translation and rotation. The material properties are the same as in Table 6. In order to investigate the effect of the elastic restrained against translation coefficient F_T on the buckling loads of symmetric cross play laminated plates under different combination of in-plane loads, parametric studies are carried out and the results are presented in Fig. 2 for biaxial loads for different ratio of $r_x = N_x / N_y$ and in Fig. 3 for $n_x = n_y$ and different r_{xy}.

It can be concluded that the elastic restrained against translation coefficient F_T has a significant effect only in the range $0.1 < F_R < 100$ and for $F_R < 0.1$ the buckling loads are mainly affected by r_x. The effect of r_x on the buckling are more significant at higher values of F_R ($F_R > 100$). Also, It can be concluded the same for the other case of r_{xy}.

Finally, we investigated the effect of rotational and transversal restraints coefficients on the natural frequencies of plates. Since there are numerous amounts of data for this case, we limited our self in this paper for two cases, namely simply supported in the *x*-direction while the other two directions are elastically restrained against both rotation and translation (S-S-ER-ER) and clamped-clamped in the *x*-direction while the other two edges are elastically restrained in both rotation and translation (C-C-ER-ER) with equally elastic restraints at y = 0, and y = b. In Figs. 4(a, b), the frequency parameters surface contours of the fundamental mode of S-S-ER-ER and C-C-ER-ER plates (F_{R3}=F_{R4}=F_R, F_{T3}=F_{T4}=F_T) are plotted vs. the elastic restraint coefficients against translation as *x*-axis and the elastic restraint coefficients against rotation as *y*-axis. We can observe that the frequency parameter λ_1 is almost constant for small value of F_T ($F_T < 1$ for S-S-ER-ER and $F_T < 10$ for C-C-ER-ER), for moderate value of F_T ($1\text{-}10 < F_T < 1000$ and) the frequency parameter is mainly controlled by F_T, and for higher values

Table 6 The natural frequency of elastically restrained cross-play laminated plates

Classical BC	F_{R1}	F_{T1}	F_{R2}	F_{T2}	F_{R3}	F_{T3}	F_{R4}	F_{T4}	E_{22}/E_{11}	λ_1	(Khdeir 1988)	λ_2	λ_3	λ_4
SSCC	0	∞	0	∞	∞	∞	∞	∞	10	21.117		26.393	39.344	56.599
	0	∞	0	∞	∞	∞	∞	∞	20	29.164	29.166	33.856	46.826	69.425
	0	∞	0	∞	∞	∞	∞	∞	30	35.428	35.431	39.949	53.27	97.114
	0	∞	0	∞	∞	∞	∞	∞	40	40.741	40.743	45.229	59.015	84.622
Paik and Thayamballi (2000)										40.743		45.233	59.023	
SSFF	0	∞	0	∞	0	0	0	0	10	3.294		5.884	22.795	29.668
	0	∞	0	∞	0	0	0	0	20	3.722	3.721	14.888	30.405	37.947
	0	∞	0	∞	0	0	0	0	30	4.106	4.106	16.422	36.459	43.479
	0	∞	0	∞	0	0	0	0	40	4.457	4.457	17.826	42.68	48.38
Paik and Thayamballi (2000)												17.827	40.113	
SSFS	0	∞	0	∞	0	0	0	∞	10	4.093		14.037	24.429	30.531
	0	∞	0	∞	0	0	0	∞	20	4.447	4.443	15.655	28.941	34.271
	0	∞	0	∞	0	0	0	∞	30	4.774	4.770	17.124	37.653	49.596
	0	∞	0	∞	0	0	0	∞	40	5.08		18.476	40.758	53.812
Paik and Thayamballi (2000)										5.076		18.473	40.761	
SSCF	0	∞	0	∞	∞	∞	0	0	10	5.425		14.684	21.767	28.982
	0	∞	0	∞	∞	∞	0	0	20	6.52	6.524	16.578	29.437	35.786
	0	∞	0	∞	∞	∞	0	0	30	7.45	7.443	18.261	35.485	41.465
	0	∞	0	∞	∞	∞	0	0	40	8.274	8.269	19.794	40.643	46.45
Paik and Thayamballi (2000)										8.269		19.789	41.505	
SSSF	0	∞	0	∞	0	∞	0	0	10	3.296		7.859	17.291	29.681
	0	∞	0	∞	0	∞	0	0	20	3.722		14.892	33.507	37.371
	0	∞	0	∞	0	∞	0	0	30	4.106		16.424	36.954	40.827
	0	∞	0	∞	0	∞	0	0	40	4.457		17.827	40.109	55.342
SSSS	0	∞	0	∞	0	∞	0	∞	10	10.60	10.650	18.64	34.247	36.99
	0	∞	0	∞	0	∞	0	∞	20	13.948	13.948	21.754	38.631	51.20
	0	∞	0	∞	0	∞	0	∞	30	16.604	16.605	24.482	42.582	24.48
	0	∞	0	∞	0	∞	0	∞	40	18.891		26.936	46.2	
SSCC	0	∞	0	∞	∞	∞	∞	∞	10	21.116	21.118	26.393	39.344	56.598
	0	∞	0	∞	∞	∞	∞	∞	20	29.163	29.17	33.856	46.826	69.424
	0	∞	0	∞	∞	∞	∞	∞	30	35.428	35.43	39.949	53.269	97.201
	0	∞	0	∞	∞	∞	∞	∞	40	40.74		45.228	59.014	84.622
CCCC	∞	∞	∞	∞	∞	∞	∞	∞	10	22.308		31.077	48.243	63.327
	∞	∞	∞	∞	∞	∞	∞	∞	20	30.244		38.528	56.375	79.612
	∞	∞	∞	∞	∞	∞	∞	∞	30	36.494		44.758	63.482	92.843
	∞	∞	∞	∞	∞	∞	∞	∞	40	41.82		50.222	69.872	101.251

Table 7 The natural frequency of symmetric cross-play laminated plate with edges equally elastic restrained against rotation and translation (M=4) $\lambda = \frac{\omega a^2}{h}\sqrt{\frac{\rho}{E_{11}}}$

F_R=F_T	F_{22}=F_{11}	λ_1	λ_2	λ_3	λ_4	λ_5
0.25	20	0.935	1.349	3.411	3.727	8.847
0.25	30	1.136	1.561	4.168	4.454	9.769
0.25	40	1.306	1.749	4.807	5.077	10.612
1	20	1.863	2.596	6.373	6.952	9.839
1	30	2.262	3.015	7.788	8.311	10.892
1	40	2.601	3.383	8.982	9.477	11.853
10	20	5.778	6.863	13.707	14.300	14.755
10	30	7.017	8.117	16.039	16.743	17.695
10	40	8.067	9.202	17.609	19.308	20.211
100	20	16.218	17.645	22.983	28.060	29.250
100	30	19.692	21.120	26.623	34.235	35.357
100	40	22.638	24.099	29.821	39.455	40.555
1000	20	27.438	32.308	39.875	51.656	61.603
1000	30	33.207	38.101	45.918	58.349	75.159
1000	40	38.112	43.121	51.253	64.341	86.616
1000000	20	30.245	38.529	56.380	79.615	83.601
1000000	30	36.495	44.759	63.486	92.846	96.986
1000000	40	41.821	50.223	69.875	101.253	111.687

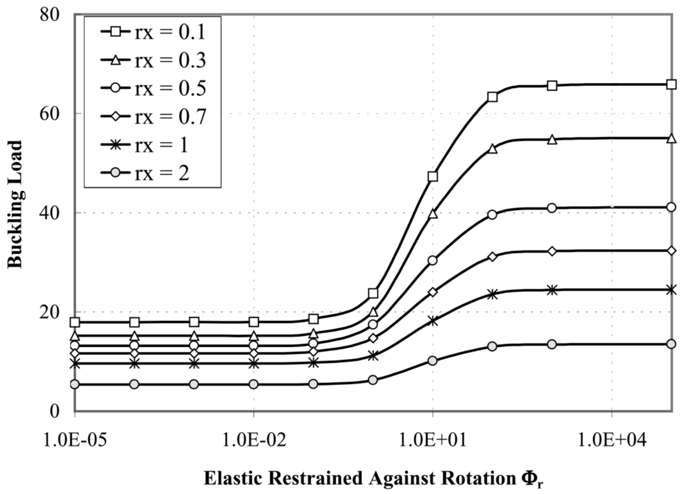

Fig. 2 The buckling loads of symmetric cross play laminated plates under biaxial loads ($r_x = N_x / N_y$)

of F_T the frequency parameter is mainly controlled by F_R. Figs. 5(a, b) show the surface contours of the second mode for the same two cases. It is observed that for this case for small values or higher values of both F_R and F_T, the second mode frequency parameter is almost constant while in the medium range both F_T and F_R affect the frequency parmater.

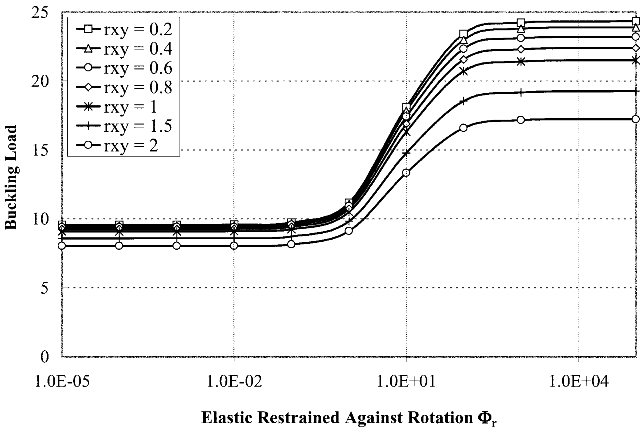

Fig. 3 The buckling loads of symmetric cross play laminated plates under different combination of in-plane loads ($N_x = N_y$, $rxy = Nxy / N_x$)

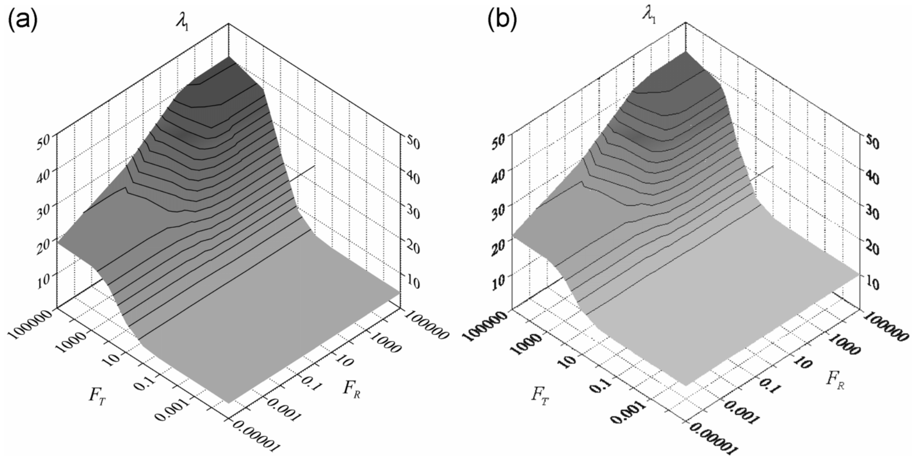

Fig. 4 The frequency parameters λ_1 for laminated plate with edges elastically restrained against both rotation and translation (a) S-S-ER-ER (b) C-C-ER-ER

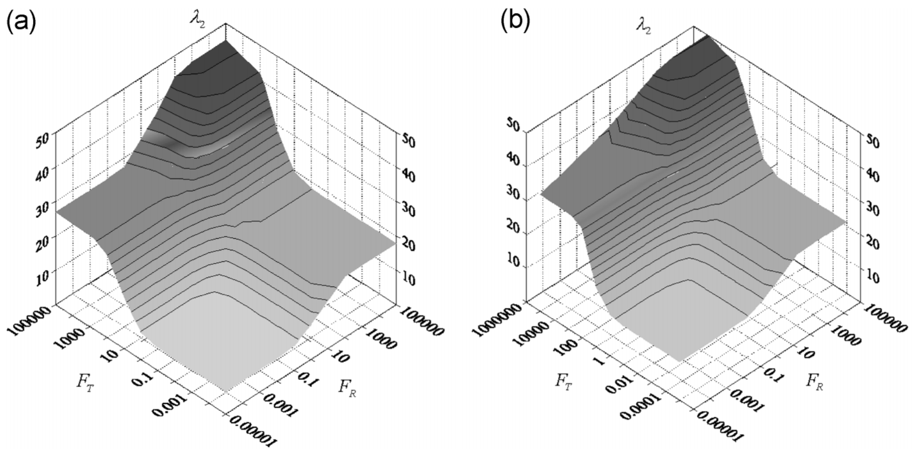

Fig. 5 The second mode frequency parameters λ_2 for laminated plate with edges elastically restrained against both rotation and translation (a) S-S-ER-ER (b) C-C-ER-ER

5. Conclusions

The vibration and buckling of cross-ply symmetrically laminated plates with edges restrained against rotation and translation is investigated using the finite strip transition matrix technique. The numerical results for isotropic plates and laminated plates with single orthotropic layer are presented and compared with some available results. In all cases considered in this paper, fast and stable convergences have been achieved after few terms of the series solution and the results agree very well with other methods. Also, the effect of the edge restrained coefficients on the buckling loads and frequency parameters for three layer cross-ply symmetrically laminated plates has been investigated.

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