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Effect of energy dissipation on plane waves in sandwiched layered thermoelastic medium

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Abstract. In the present investigation, a plane *P* (longitudinal) wave is made incident upon a transversely isotropic magnetothermoelastic solid slab of uniform thickness, interposed between two different semi-infinite viscoelastic solids. The transversely isotropic magnetothermoelastic sandwiched layer is homogeneous with combined effects of two temperature, rotation and Hall current in the context of GN Type-II and Type-III (1993) theory of thermoelasticity. The amplitude ratios of various reflected and refracted waves are obtained by using appropriate boundary conditions. The effect of energy dissipation on various amplitude ratios of longitudinal wave with angle of incidence are depicted graphically. Some cases of interest are also deduced from the present investigation.

Keywords: interface; reflection; transmission; transversely isotropic thermoelastic; viscoelastic half space; amplitude ratios

1. Introduction

As the importance of anisotropic devices has increased in many fields of optics and microwaves, wave propagation in anisotropic media has been widely studied over in the last decades. The anisotropic nature basically stems from the polarization or magnetization that can occur in materials when external fields pass by. Mathematical modeling of plane wave propagation along with the free boundary of an elastic half-space has been subject of continued interest for many years. Keith and Crampin (1977) derived a formulation for calculating the energy division among waves generated by plane waves incident on a boundary of anisotropic media. Reflection of plane waves at the free surface of a transversely isotropic thermoelastic diffusive solid half-space has been discussed by Kumar and Mukhopadhyay (2010), Othman (2010). Wave propagation has remained the study of concern of many researchers (Sharma and Marin 2013, Kumar and Gupta 2013, Kumar 2015, Lata *et al*. 2016, Othman and Abd-Elaziz 2017).

Chen and Gurtin (1968) and Chen *et al*. (1968, 1969) have formulated a theory of heat conduction in deformable bodies which depends upon two distinct temperatures, the conductive temperature φ and the thermo dynamical temperature T. For time independent situations, the difference between these two temperatures is proportional to the heat supply, and in absence of heat supply, the two temperatures are identical. For time dependent problems, the two temperatures are different, regardless of the presence of heat supply. The two temperatures T, φ and the strain are found to have representations in the form of a travelling wave plus a response, which occurs instantaneously throughout the body (Boley and Tolins 1962). The wave propagation in two temperature theory of thermoelasticity was investigated by Warren and Chen (1973).

Green and Naghdi (1991) postulated a new concept in thermoelasticity theories and proposed three models which are subsequently referred to as GN-I, II, and III models. The linearized version of model-I corresponds to classical thermoelastic model (based on Fourier's law). The linearized version of model-II and III permit propagation of thermal waves at finite speed. Green-Naghdi's second model (GN-II), in particular exhibits a feature that is not present in other established thermoelastic models as it does not sustain dissipation of thermal energy (Green and Naghdi 1993). In this model, the constitutive equations are derived by starting with the reduced energy equation and by including the thermal displacement gradient among other constitutive variables. Green-Naghdi's third model (GN-III) admits dissipation of energy. In this model the constitutive equations are derived by starting with the reduced energy equation, where the thermal displacement gradient in addition to the temperature gradient, are among the constitutive variables. Green and Naghdi (1992) included the derivation of a complete set of governing equations of a linearized version of the theory for homogeneous and isotropic materials in terms of the displacement and temperature fields and a proof of the uniqueness of the solution for the corresponding initial boundary value problem.

A comprehensive work has been done in thermoelasticity theory with and without energy dissipation and thermoelasticity with two temperatures. Youssef (2011), constructed a new theory of generalized thermoelasticity by

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taking into account two-temperature generalized thermoelasticity theory for a homogeneous and isotropic body without energy dissipation. Several researchers studied various problems involving two temperature e.g., (Youssef 2006, Sharma and Marin 2013, Sharma and Bhargav 2014, Sharma *et al*. 2013, Sharma and Kumar 2013, Kumar *et al*. 2016a, b, c, 2017, Ezzat *et al*. 2016).

In view of the fact that most of the large bodies like the earth, the moon and other planets have an angular velocity, as well as earth itself behaves like a huge magnet, it is important to study the propagation of thermoelastic waves in a rotating medium under the influence of magnetic field. So, the attempts are being made to study the propagation of finite thermoelastic waves in an infinite elastic medium rotating with angular velocity. Several authors (Das and Kanoria 2014, Atwa and Jahangir 2014, Ezzat and AI-Bary 2016, 2017) have studied various problems in generalized thermoelasticity to study the effect of rotation.

Sandwich structures are widely used in diverse applications such as spacecraft, aircraft, automobiles, boats and ships due to their substantial bending strength and impact resistance at a light weight. The dynamic applications have motivated various studies of wave propagation and dynamic flexural deformation of multilayer beams and plates. Elphinstone and Lakhtakia (1994) have investigated the response of a plane wave incident on a chiral solid slab sandwiched between two elastic half spaces. Khurana and Tomar (2009) discussed longitudinal wave response of chiral slab interposed between micropolar elastic solid half spaces. Wave propagation in sandwich layer has been investigated by many researchers (Chaudhary *et al*. 2010, Deshpande and Fleck 2005, Liu and Bhattacharya 2009, Vlase *et al*. 2017).

Different authors discussed different types of problems in viscoelasticity. Freudenthal (1954) pointed out that most solids when subjected to dynamic loading exhibit viscous effects. The Kelvin -Voigt model is one of the macroscopic mechanical models often used to describe the viscoelastic behaviour of a material. This model represents the delayed elastic response subjected to stress where the deformation is time dependent. Iesan and Scalia (1989) studied some theorems in the theory of thermoviscoelasticity. Borrelli and Patria (1991) investigated the discontinuity of waves through a linear thermoviscoelastic solid of integral type. Pal (2000) studied the problem of torsional body forces in viscoelastic half-space. Corr *et al*. (2001) investigated the non linear generalized Maxwell fluid model for viscoelastic materials. Ezzat *et al*. (2010) presented research on thermo-electric-visco elastic materials. Effect of viscosity on wave propagation in anisotropic thermoelastic medium with three-phase-lag model was discussed by Kumar *et al*. (2012). Sharma *et al*. (2013) analysed effect of viscousity on wave propagation in anisotropic thermoelastic with Green-Naghdi theory Type-II and Type-III. AI-Basyouni *et al*. (2014) discussed effect of rotation, magnetic field and a periodic loading on radial vibrations thermo-viscoelastic non-homogeneous media. Thermo-viscoelastic materials with fractional relaxation operators were discussed by Ezzat *et al*. (2015). Yadav *et al*. (2015) investigated a state problem of Two-Temperature generalized thermovisco-elasticity with fractional order strain subjected to moving heat source.

Here in this paper, we consider transversely isotropic magnetothermoelastic solid slab of uniform thickness, interposed between two different semi-infinite homogeneous isotropic elastic solids. A plane longitudinal or transverse wave propagating through one of the viscoelastic solid half spaces, is made incident upon transversely isotropic magnetothermoelastic solid. We have presented the reflection and transmission coefficients obtained separately, corresponding to the appropriate set of boundary conditions. The effect of energy dissipation on variations in the modulus of the amplitude ratios with the angle of incidence are depicted graphically.

2. Basic equations

The constitutive relations for a transversely isotropic thermoelastic medium are given by

$$t_{ij} = C_{ijkl}\, e_{kl} - \beta_{ij} T \tag{1}$$

Equation of motion for a transversely isotropic thermoelastic medium rotating uniformly with an angular velocity $\mathbf{\Omega} = \Omega n$, where n is a unit vector representing the direction of axis of rotation and taking into account Lorentz force

$$t_{ij,j} + F_i = \rho\left\{\ddot{u}_i + \left(\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times u)\right)_i + (2\boldsymbol{\Omega}\times\dot{u})_i\right\} \tag{2}$$

Following Chandrasekharaiah (1998) and Youssef (2011), the heat conduction equation with two temperature and with and without energy dissipation is given by

$$K_{ij}\,\varphi_{,ij} + K^{*}_{IJ}\,\dot{\varphi}_{,ij} = \beta_{ij} T_0 \ddot{e_{ij}} + \rho C_E \ddot{T} \tag{3}$$

The above equations are supplemented by generalized Ohm's law for media with finite conductivity and including the Hall current effect

$$\boldsymbol{J} = \frac{\sigma_0}{1+m^2}\left(\boldsymbol{E} + \mu_0\left(\dot{\boldsymbol{u}}\times\boldsymbol{H} - \frac{1}{en_e}\boldsymbol{J}\times\boldsymbol{H}_0\right)\right) \tag{4}$$

and the strain displacement relations are

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \qquad i,j = 1,2,3 \tag{5}$$

Here, $F_i = \mu_0(\boldsymbol{J}\times\boldsymbol{H}_0)_i$ are the components of Lorentz force.

$\beta_{ij} = C_{ijkl}\,\alpha_{ij}$ and $T = \varphi - a_{ij}\,\varphi_{,ij}$

$\beta_{ij} = \beta_i\delta_{ij}, \; K_{ij} = K_i\delta_{ij}, \; K_{ij}{}^{*} = K_i{}^{*}\delta_{ij},$

i is not summed

Following Achenbach (1973), the constitutive relations for the viscoelastic half space are

$$t_{ij,j}{}^{l} = 2\mu^l u^l_{i,j} + \lambda^l u^l_{k,k}\delta_{ij}$$
$$(i,j,k = 1,2,3 \quad \text{and} \quad l = 1,2), \tag{6}$$

and, equations of motion are

$$\mu^l u^l_{i,jj} + (\lambda^l + \mu^l)u^l_{i,ij} = \rho^e \frac{\partial^2 \boldsymbol{u}_i{}^l}{\partial t^2}$$
$$(i,j = 1,2,3 \quad \text{and} \quad l = 1,2) \tag{7}$$

$C_{ijkl}\left(C_{ijkl} = C_{klij} = C_{jikl} = C_{ijlk}\right)$ are elastic parameters, β_{ij} is the thermal tensor, T is the temperature, T_0 is the reference temperature, t_{ij} are the components of stress tensor, e_{kl} are the components of strain tensor, u_i are the displacement components, ρ is the density, C_E is the specific heat, K_{ij} is the thermal conductivity, K^*_{IJ} is the materialistic constant, a_{ij} are the two temperature parameters, α_{ij} is the coefficient of linear thermal expansion, Ω is the angular velocity of the solid, H is the magnetic strength, $\dot{\boldsymbol{u}}$ is the velocity vector, $\boldsymbol{E}$ is the intensity vector of the electric field, $\boldsymbol{J}$ is the current density vector, $m\left(= \omega_e t_e = \frac{\sigma_0\mu_0 H_0}{e n_e}\right)$ is the Hall parameter, t_e is the electron collision time, $\omega_e = \frac{e\mu_0 H_0}{m_e}$ is the electronic frequency, e is the charge of an electron, m_e is the mass of the electron, $\sigma_0 = \frac{e^2 t_e n_e}{m_e}$, is the electrical conductivity and n_e is the number of density of electrons. λ^l, μ^l, ρ^l are the Lame's constants and density in the elastic half space. $\boldsymbol{u}_i{}^l$ $(i = 1,2,3$ and $l = 1,2)$ are the components of displacement vector, $t_{ij,j}{}^l$ are the components of stress in elastic half space.

3. Formulation of the problem

Consider a layered model consisting of a transversely isotropic magnetothermoelastic slab of finite thickness H, which is rotating uniformly with an angular velocity $\boldsymbol{\Omega}$ initially at uniform temperature T_0 with Hall current effect, is interposed between two distinct elastic half spaces. Introducing the Cartesian co-ordinate system (x_1, x_2, x_3) such that $x_1 -$ and $x_3 -$ axis are on horizontal plane and $x_3 -$ axis is pointing vertically downwards. Let the intermediate layer occupying the region $M\,[0 \le x_3 \le H]$ be delineated by the planes $x_3 = 0$ and $x_3 = H$ as shown in Fig. 1 and the two elastic half spaces be occupying the regions $M^{(1)}: [x_3 < 0]$ and $M^{(2)}: [x_3 > H]$. For two dimensional problem, the displacement vectors $\boldsymbol{u}$, $\boldsymbol{u}^l$ $(l = 1,2)$ in transversely isotropic magnetothermoelastic and in elastic half space are taken as

$$\boldsymbol{u} = (u_1, 0, u_3) \quad \text{and} \quad \boldsymbol{u}^l = (u_1^l, 0, u_3^l),$$
$$(l = 1,2) \tag{8}$$

We also assume that

$$E = 0, \qquad \boldsymbol{\Omega} = (0,\ \Omega,\ 0) \tag{9}$$

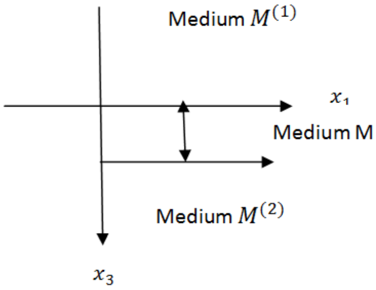

Fig. 1 Sandwiched layered medium

The generalized Ohm's law gives

$$J_2 = 0 \tag{10}$$

the current density components J_1 and J_3 using Eq. (10) are given as

$$J_1 = \frac{\sigma_0\mu_0 H_0}{1+m^2}\left(m\frac{\partial u_1}{\partial t} - \frac{\partial u_3}{\partial t}\right) \tag{11}$$

$$J_3 = \frac{\sigma_0\mu_0 H_0}{1+m^2}\left(\frac{\partial u_1}{\partial t} + m\frac{\partial u_3}{\partial t}\right) \tag{12}$$

Following Slaughter (2002), using appropriate transformations, on the set of Eqs. (2) and (3) and with the aid of Eqs. (8)-(12), the field equations for transversely isotropic magnetothermoelastic medium are

$$\begin{aligned} &c_{11}\frac{\partial^2 u_1}{\partial x^2} + c_{13}\frac{\partial^2 u_3}{\partial x_1 \partial x_3} + c_{44}\left(\frac{\partial^2 u_1}{\partial {x_3}^2} + \frac{\partial^2 u_3}{\partial x_1 \partial x_3}\right) \\ &-\beta_1\frac{\partial}{\partial x_1}\left\{\varphi - \left(a_1\frac{\partial^2\varphi}{\partial {x_1}^2} + a_3\frac{\partial^2\varphi}{\partial {x_3}^2}\right)\right\} - \mu_0 J_3 H_0 \\ &= \rho\left(\frac{\partial^2 u_1}{\partial t^2} - \Omega^2 u_1 + 2\Omega\frac{\partial u_3}{\partial t}\right) \end{aligned} \tag{13}$$

$$\begin{aligned} &(c_{13}+c_{44})\frac{\partial^2 u_1}{\partial x_1 \partial x_3} + c_{44}\frac{\partial^2 u_3}{\partial {x_1}^2} + c_{33}\frac{\partial^2 u_3}{\partial {x_3}^2} \\ &-\beta_3\frac{\partial}{\partial x_3}\left\{\varphi - \left(a_1\frac{\partial^2\varphi}{\partial {x_1}^2} + a_3\frac{\partial^2\varphi}{\partial {x_3}^2}\right)\right\} + \mu_0 J_1 H_0 \\ &= \rho\left(\frac{\partial^2 u_3}{\partial t^2} - \Omega^2 u_3 - 2\Omega\frac{\partial u_1}{\partial t}\right) \end{aligned} \tag{14}$$

$$\begin{aligned} &\left(k_1 + {k_1}^*\frac{\partial}{\partial t}\right)\frac{\partial^2\varphi}{\partial {x_1}^2} + \left(k_3 + {k_3}^*\frac{\partial}{\partial t}\right)\frac{\partial^2\varphi}{\partial {x_3}^2} \\ &= T_0\frac{\partial^2}{\partial t^2}\left\{\beta_1\frac{\partial u_1}{\partial x_1} + \beta_3\frac{\partial u_3}{\partial x_3}\right\} + \rho C_E \ddot{T} \end{aligned} \tag{15}$$

and the stress components are

$$t_{33} = c_{13}e_{11} + c_{33}e_{33} - \beta_3 T \tag{16}$$

$$t_{13} = 2c_{44}e_{13} \tag{17}$$

where

$$T = \varphi - \left(a_1\frac{\partial^2\varphi}{\partial x_1^2} + a_3\frac{\partial^2\varphi}{\partial x_3^2}\right)$$

$$\beta_1 = (c_{11}+c_{12})\alpha_1 + c_{13}\alpha_3,$$
$$\beta_3 = 2c_{13}\alpha_1 + c_{33}\alpha +_3$$

In the above equations we use the contracting subscript notations $(11 \to 1, 22 \to 2, 33 \to 3, 23 \to 4, 31 \to 5, 12 \to 6)$ to relate c_{ijkl} to c_{mn}

The field equations for the elastic half spaces are

$$\mu^l\left(\frac{\partial^2 u_1^l}{\partial {x_1}^2} + \frac{\partial^2 u_1^l}{\partial {x_3}^2}\right) + (\lambda^l + \mu^l)\frac{\partial^2 u_3^l}{\partial x_1 \partial x_3} = \rho^l \frac{\partial^2 {u_1}^l}{\partial t^2}, \quad (l = 1, 2) \tag{18}$$

$$\mu^l\left(\frac{\partial^2 u_3^l}{\partial {x_1}^2} + \frac{\partial^2 u_3^l}{\partial {x_3}^2}\right) + (\lambda^l + \mu^l)\frac{\partial^2 u_1^l}{\partial x_1 \partial x_3} = \rho^l \frac{\partial^2 {u_3}^l}{\partial t^2}, \quad (l = 1, 2) \tag{19}$$

In order to account for the material damping behaviour the material coefficients λ and μ are assumed to be a function of time operator $D = \frac{\partial}{\partial t}$, i.e.

$$\lambda^l = \lambda^{l^*} \quad \text{where} \quad \lambda^{l^*} = \lambda^l(D) \qquad (l = 1, 2)$$

$$\mu^l = \mu^{l^*} \quad \text{where} \quad {\mu^l}^* = \mu^l(D) \qquad (l = 1, 2)$$

Assuming that the viscoelastic nature of the material is described by the Voigt model of linear viscoelasticity (Kaliski 1963), we write

$$\lambda^l = \lambda^{l^*}\left(1 + Q_1 \frac{\partial}{\partial t}\right)$$

$$\mu^l = \mu^{l^*}\left(1 + Q_2 \frac{\partial}{\partial t}\right),$$

The stress components for elastic half spaces in the $x_1 - x_3$ plane are

$$t_{13}^l = \mu^l\left(\frac{\partial {u_1}^l}{\partial x_3} + \frac{\partial {u_3}^l}{\partial x_1}\right), \quad (l = 1, 2) \tag{20}$$

$$t_{33}^l = (\lambda^l)\frac{\partial {u_1}^l}{\partial x_1} + (\lambda^l + 2\mu^l)\frac{\partial {u_3}^l}{\partial x_3}, \quad (l = 1, 2) \tag{21}$$

To facilitate the solution, we introduce the dimensionless quantities

$$x_1' = \frac{x_1}{L}, \quad x_3' = \frac{x_3}{L},$$
$$({u_1}', u_3', {u_1}'^l, {u_3}'^l) = \frac{\rho c_1^2}{L\beta_1 T_0}(u_1, u_3, {u_1}^l, {u_3}^l),$$
$$T' = \frac{T}{T_0}, \quad t' = \frac{c_1}{L}t, \quad t_{11}' = \frac{t_{11}}{\beta_1 T_0},$$
$$J' = \frac{\rho c_1^2}{\beta_1 T_0}J(t_{33}', t_{31}', t_{13}'^l, t_{33}'^l) = \left(\frac{t_{33}}{\beta_1 T_0}, \frac{t_{31}}{\beta_1 T_0}, \frac{t_{13}^l}{\beta_1 T_0}, \frac{t_{33}^l}{\beta_1 T_0}\right),$$
$$\varphi' = \frac{\varphi}{T_0}, \quad a_1' = \frac{a_1}{L}, \quad a_3' = \frac{a_3}{L}, \quad h' = \frac{h}{H_0},$$
$$M = \frac{\sigma_0 \mu_0 H_0}{\rho c_1 L}, \quad \Omega' = \frac{L}{c_1}\Omega \quad \text{where} \quad c_{11} = \rho c_1^2 \tag{22}$$

Using dimensionless quantities defined by Eq. (22) in the Eqs. (13)-(15) and Eqs. (18)-(19), and suppressing the primes, the resulting equations yield

$$\begin{aligned}
&\frac{\partial^2 u_1}{\partial {x_1}^2} + \delta_4 \frac{\partial^2 u_3}{\partial x_1 \partial x_3} + \delta_2\left(\frac{\partial^2 u_1}{\partial {x_3}^2} + \frac{\partial^2 u_3}{\partial x_1 \partial x_3}\right) \\
&- \frac{\partial}{\partial x_1}\left\{\varphi - \left(\frac{a_1}{L}\frac{\partial^2 \varphi}{\partial {x_1}^2} + \frac{a_3}{L}\frac{\partial^2 \varphi}{\partial {x_3}^2}\right)\right\} \\
&- \frac{M}{1 + m^2}\mu_0 H_0\left(\frac{\partial u_1}{\partial t} + m\frac{\partial u_3}{\partial t}\right) \\
&= \frac{\partial^2 u_1}{\partial t^2} - \Omega^2 u_1 + 2\Omega\frac{\partial u_3}{\partial t}
\end{aligned} \tag{23}$$

$$\begin{aligned}
&\delta_1 \frac{\partial^2 u_1}{\partial x_1 \partial x_3} + \delta_2 \frac{\partial^2 u_3}{\partial {x_1}^2} + \delta_3 \frac{\partial^2 u_3}{\partial {x_3}^2} \\
&- \frac{\beta_3}{\beta_1}\frac{\partial}{\partial x_3}\left\{\varphi - \left(\frac{a_1}{L}\frac{\partial^2 \varphi}{\partial {x_1}^2} + \frac{a_3}{L}\frac{\partial^2 \varphi}{\partial {x_3}^2}\right)\right\} \\
&+ \frac{M}{1 + m^2}\mu_0 H_0\left(m\frac{\partial u_1}{\partial t} - \frac{\partial u_3}{\partial t}\right) \\
&= \frac{\partial^2 u_3}{\partial t^2} - \Omega^2 u_3 - 2\Omega\frac{\partial u_1}{\partial t}
\end{aligned} \tag{24}$$

$$\begin{aligned}
&\varepsilon_1\left(1 + \frac{\varepsilon_3}{\varepsilon_1}\frac{\partial}{\partial t}\right)\frac{\partial^2 \varphi}{\partial {x_1}^2} + \varepsilon_2\left(1 + \frac{\varepsilon_4}{\varepsilon_2}\frac{\partial}{\partial t}\right)\frac{\partial^2 \varphi}{\partial {x_3}^2} \\
&= {\varepsilon_5}'{\beta_1}^2\frac{\partial^2}{\partial t^2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\beta_3}{\beta_1}\frac{\partial u_3}{\partial x_3}\right) \\
&+ \frac{\partial^2}{\partial t^2}\left(\left\{\varphi - \frac{a_1}{L}\frac{\partial^2 \varphi}{\partial {x_1}^2} + \frac{a_3}{L}\frac{\partial^2 \varphi}{\partial {x_3}^2}\right\}\right)
\end{aligned} \tag{25}$$

$$\delta_1 = \frac{(c_{13} + c_{44})}{c_{11}}, \quad \delta_2 = \frac{c_{44}}{c_{11}}, \quad \delta_3 = \frac{c_{33}}{c_{11}}, \quad \delta_4 = \frac{c_{13}}{c_{11}},$$
$$\varepsilon_1 = \frac{k_1}{\rho C_E c_1^2}, \quad \varepsilon_2 = \frac{k_3}{\rho C_E c_1^2}, \quad \varepsilon_3 = \frac{{k_1}^*}{L\rho C_E c_1},$$
$$\varepsilon_4 = \frac{{k_3}^*}{L\rho C_E c_1}, \quad {\varepsilon_5}' = \frac{T_0}{\rho^2 C_E c_1^2}$$

For the mediums $M^{(1)}$ and $M^{(2)}$, we have

$$\nabla^2 \phi^l = \frac{1}{(\alpha'^{(l)})^2}\left(\frac{\partial^2 \varphi^l}{\partial t^2}\right) \tag{26}$$

$$\nabla^2 \psi^l = \frac{1}{\beta'^{(l)2}}\left(\frac{\partial^2 \psi^l}{\partial t^2}\right) \tag{27}$$

where

$\alpha'^{(l)} = \frac{\alpha^{(l)}}{c_1}$, $\beta'^{(l)} = \frac{\beta^{(l)}}{c_1}$, $\alpha^{(l)} = \sqrt{\frac{(\lambda^l + 2\mu^l)}{\rho^l}}$ and $\beta^{(l)} = \sqrt{\frac{\mu^l}{\rho^l}}$ are velocities of longitudinal and transverse waves respectively for the mediums $M^{(1)}$ and $M^{(2)}$ for $(l = 1, 2)$

and ϕ^l and ψ^l are the scalar potentials defined by

$$u_1^l = \frac{\partial \phi^l}{\partial x_1} - \frac{\partial \psi^l}{\partial x_3}, \qquad u_3^l = \frac{\partial \phi^l}{\partial x_3} + \frac{\partial \psi^l}{\partial x_1} \tag{28}$$

We seek a wave solution of the form for transversely isotropic magnetothermoelastic solid as

$$\begin{pmatrix} u_1 \\ u_3 \\ \varphi \end{pmatrix} = \begin{pmatrix} U_1 \\ U_3 \\ \varphi^* \end{pmatrix} \exp\{ik(x_1 \sin\theta + x_3 \cos\theta) - i\omega t)\} \tag{29}$$

where $(\sin\theta, \cos\theta)$ denotes the projection of the wave normal onto the $x_1 - x_3$ plane, k and ω are respectively the wave number and angular frequency of plane waves propagating in $x_1 - x_3$ plane.

Upon using Eq. (29) in Eqs. (23)-(25) and then eliminating U_1, U_3 and φ^* from the resulting equations yields the following characteristic equation

$$Ak^6 + Bk^4 + Ck^2 + D = 0 \tag{30}$$

where A, B, C, D are given in Appendix A.

The roots of Eq. (30) gives six values of k, in which we are interested to those roots whose imaginary parts are positive. Corresponding to these roots, there exists three waves corresponding to decreasing orders of their velocities, namely quasi-longitudinal, quasi-transverse and quasi-thermal waves. The phase velocity is given by

$$V_j = \frac{\omega}{|Re(k_j)|}, \qquad j = 1, 2, 3$$

where V_j, j = 1, 2, 3 are the phase velocities of QL, QTS and QT waves respectively.

4. Wave solution

Let a plane P or SV wave travelling through the elastic half space $M^{(1)}$ be incident at the interface $x_3 = 0$ and makes an angle $\theta_0^{(1)}$ with the x_3 −axis. A part of this incident energy will be reflected back into the medium $M^{(1)}$ and rest will be transmitted into the medium M. Now the wave associated with transmitted energy will proceed through the medium M to interact with the boundary $x_3 = H$, where again some part of this energy will be reflected and rest will be transmitted into the medium $M^{(2)}$. The reflected energy further proceeds back to interact with the boundary $x_3 = 0$, and the process will repeat. To satisfy the boundary conditions at both the interfaces, i.e., $x_3 = 0$ and $x_3 = H$, we shall take the following reflected and refracted waves into consideration.

A plane longitudinal or transverse wave, making an angle θ_0 with the x_3 −axis is incident at the interface through the elastic half space $M^{(1)}$. This wave results in

Reflected waves

(i) One reflected longitudinal wave travelling with speed $\alpha^{(1)}$ and making an angle $\theta_1^{(1)}$ with the x_3 −axis and one transverse wave propagating with speed $\beta^{(1)}$ and making an angle $\theta_2^{(1)}$ with the x_3 −axis in the medium $M^{(1)}$.

(ii) A reflected longitudinal wave, transverse wave and a thermal wave travelling with speeds v_1, v_2 and v_3 and making angles θ_1, θ_2 and θ_3 with x_3 −axis in the medium M.

Refracted waves

(i) A set consisting of longitudinal wave, transverse wave and a thermal wave travelling with speeds

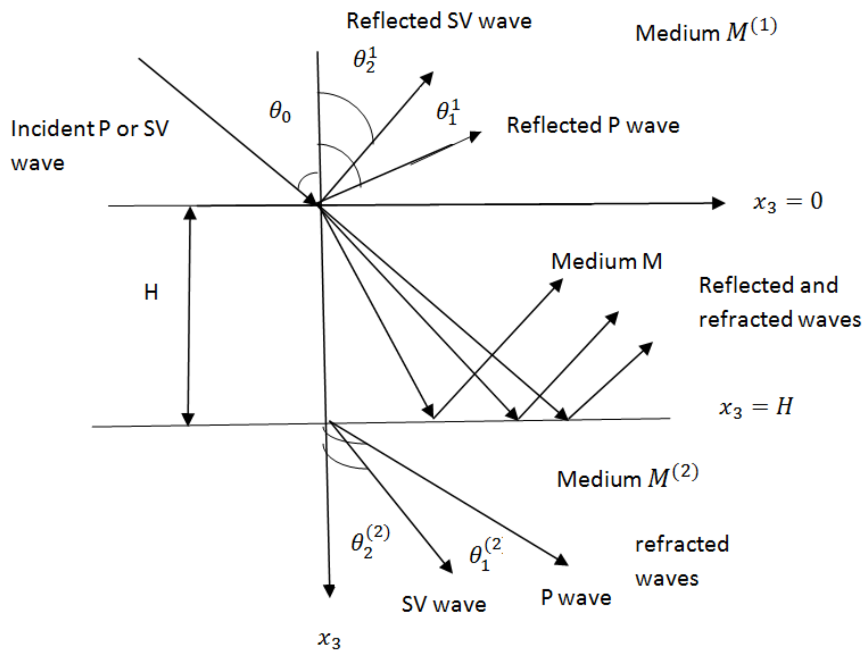

Fig. 2 Geometry of the problem

v_1, v_2 and v_3 and making angles θ_1, θ_2 and θ_3 with x_3 −axis. In the medium M.

(ii) A longitudinal wave travelling with speed $\alpha^{(2)}$ and making an angle $\theta_1^{(2)}$ with the x_3 −axis and one transverse wave propagating with speed $\beta^{(2)}$ and making an angle $\theta_2^{(2)}$ with the x_3 −axis in the medium $M^{(2)}$.

We assume the wave solution for the mediums M, $M^{(1)}$ and $M^{(2)}$ as

$$\text{Medium } M: \quad u_1 = \sum_{j=1}^{3}(A_j P_j^{+}) + \sum_{j=4}^{6}(A_j P_{j-3}^{-}) \tag{31}$$

$$u_3 = \sum_{j=1}^{3}(d_j A_j P_j^{+}) + \sum_{j=4}^{6}(d_j A_j P_{j-3}^{-}) \tag{32}$$

$$\varphi = \sum_{j=1}^{3}(l_j A_j P_j^{+}) + \sum_{j=4}^{6}(l_j A_j P_{j-3}^{-}) \tag{33}$$

where the coupling constants d_j and l_j are given in Appendix B.

$$\text{Medium } M^{(1)}: \quad \phi^{(1)} = A_0^{(1)} P_0^{+(1)} + A_1^{(1)} P_0^{-(1)} \tag{34}$$

$$\psi^1 = B_0^{(1)} Q_1^{+(1)} + B_1^{(1)} Q_1^{-(1)} \tag{35}$$

$$\text{Medium } M^{(2)}: \quad \phi^{(2)} = A_0^{(2)} P_0^{+(2)} \tag{36}$$

$$\psi^{(2)} = B_0^{(2)} Q_1^{+(2)} \tag{37}$$

Where

$$P_0^{+(l)} = \exp[i\omega\{(\sin\theta_0^{(l)} x_1 + cos\theta_0^{(l)} x_3)/\alpha'^{(l)} - t\}], \quad l = 1,2$$

$$P_0^{-(1)} = \exp[i\omega\{(\sin\theta_0^{(1)} x_1 - cos\theta_0^{(1)} x_3)/\alpha'^{(1)} - t\}],$$

$$Q_1^{+(i)} = \exp[i\omega\{(\sin\theta_1^{(l)} x_1 + cos\theta_1^{(l)} x_3)/\beta'^{(l)} - t\}], \quad l = 1,2$$

$$Q_1^{-(1)} = \exp[i\omega\{(\sin\theta_1^{(1)} x_1 - cos\theta_1^{(1)} x_3)/\beta'^{(1)} - t\}]$$

$$P_j^{+} = \exp\{ik_j(\sin\theta_j x_1 + cos\theta_j x_3) - i\omega t\}$$

$$P_j^{-} = \exp\{ik_j(\sin\theta_j x_1 - cos\theta_j x_3) - i\omega t\}, \quad j = 1,2,3$$

where

$$k_j = \frac{\omega}{v_j}.$$

5. Boundary conditions

B.(1) The boundary conditions to be satisfied at the interface $x_3 = 0$ are

(i) continuity of the stress component $t_{33} = t_{33}^{(1)}$ (38)

(ii) $t_{31} = t_{31}^{(1)}$ (39)

(iii) continuity of displacement components $u_1 = u_1^{(1)}$ (40)

(iv) $u_3 = u_3^{(1)}$ (41)

(v) thermally insulated boundary $\dfrac{\partial\varphi}{\partial x_3} = 0$ (42)

B.(2) The boundary conditions to be satisfied at the interface $x_3 = H$ are

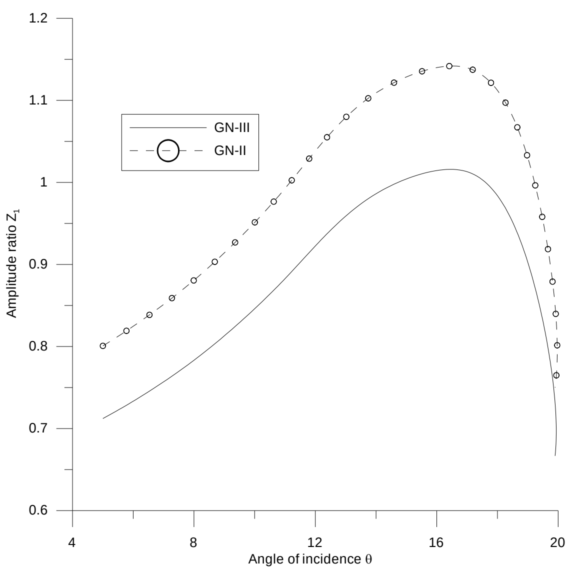

Fig. 3 Variations of amplitude ratio Z_1 with angle of incidence θ

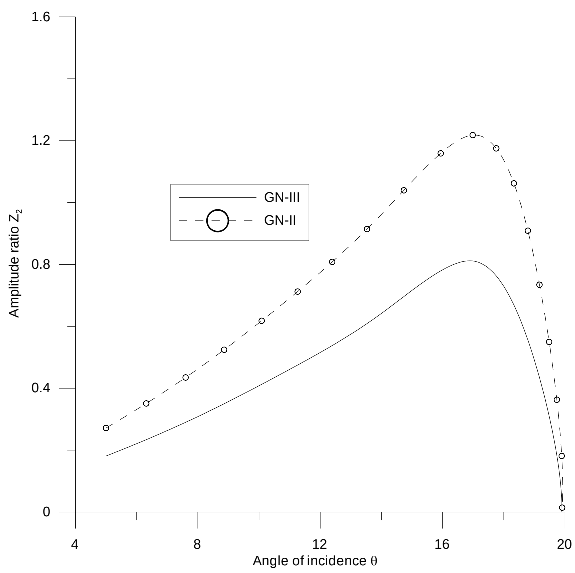

Fig. 4 Variations of amplitude ratio Z_2 with angle of incidence θ

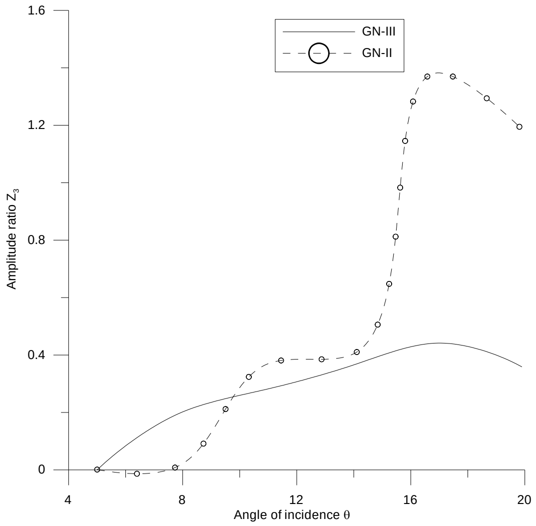

Fig. 5 Variations of amplitude ratio Z_3 with angle of incidence θ

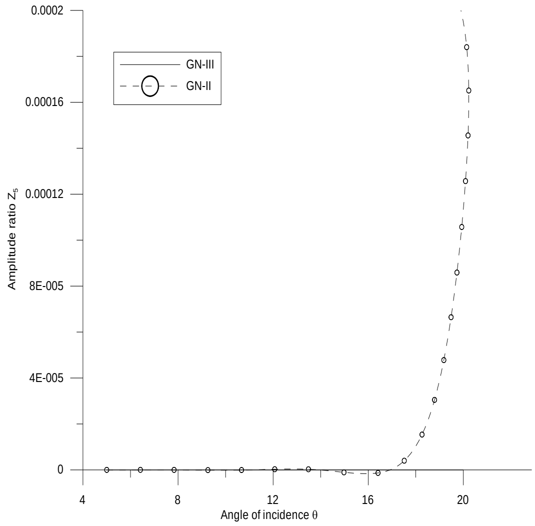

Fig. 6 Variations of amplitude ratio Z_4 with angle of incidence θ

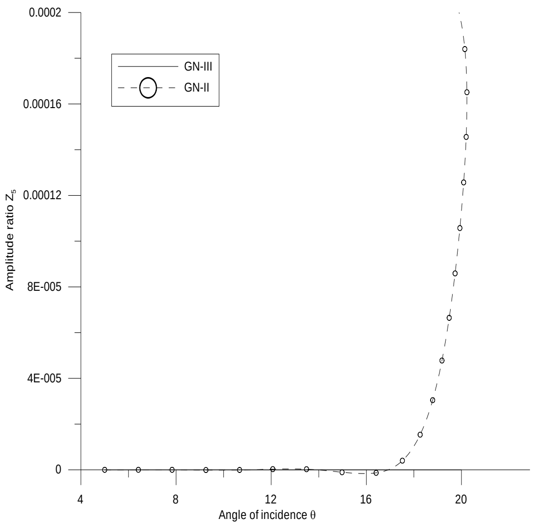

Fig. 7 Variations of amplitude ratio Z_5 with angle of incidence θ

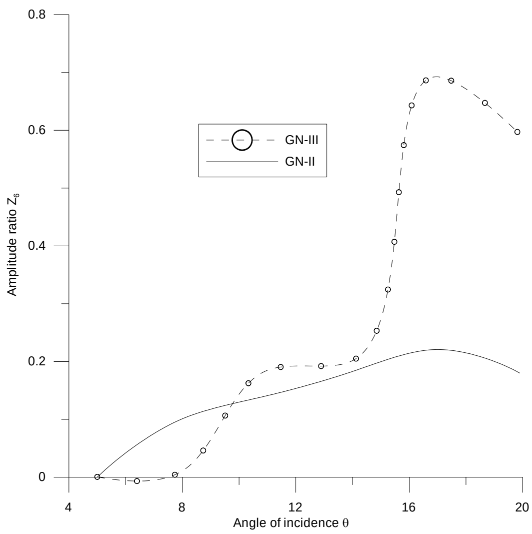

Fig. 8 Variations of amplitude ratio Z_6 with angle of incidence θ

(i) continuity of the stress component $t_{33} = t_{33}^{(2)}$ (43)

(ii) $t_{31} = t_{31}^{(2)}$ (44)

(iii) continuity of displacement components $u_1 = u_1^{(2)}$ (45)

(iv) $u_3 = u_3^{(2)}$ (46)

(v) thermally insulated boundary $\dfrac{\partial \varphi}{\partial x_3} = 0$ (47)

Amplitude ratios

Incident P wave

Using Eqs. (31)-(37) in the Eqs. (38)-(47) with the aid of Eqs. (23)-(27), we obtain a non homogeneous system of equations

$$AX = B \tag{48}$$

where $A = \left[a_{ij}\right]_{10\times10}$, $X = [z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9, z_{10}]^t$ where 't' in the superscript represents the transpose of the matrix, $z_1 = \frac{A_1^{(1)}}{A_0^{(1)}}$, $z_2 = \frac{B_1^{(1)}}{A_0^{(1)}}$ are the reflection coefficients in the medium $M^{(1)}$, $z_i = \frac{A_i}{A_0^{(1)}}, i = 3, 4, 5$ are the transmission coefficients in the medium M, $z_i = \frac{A_i}{A_0^{(1)}}$, $i = 6, 7, 8$ are the reflection coefficients in the medium M, $z_9 = \frac{A_0^{(2)}}{A_0^{(1)}}$, $z_{10} = \frac{B_0^{(2)}}{A_0^{(1)}}$ are the reflection coefficients in the medium $M^{(2)}$. Using Cramer's rule, the system of equations given in Eq. (48) enables us to amplitude ratios of various reflected and transmitted waves. The values of a_{ij}

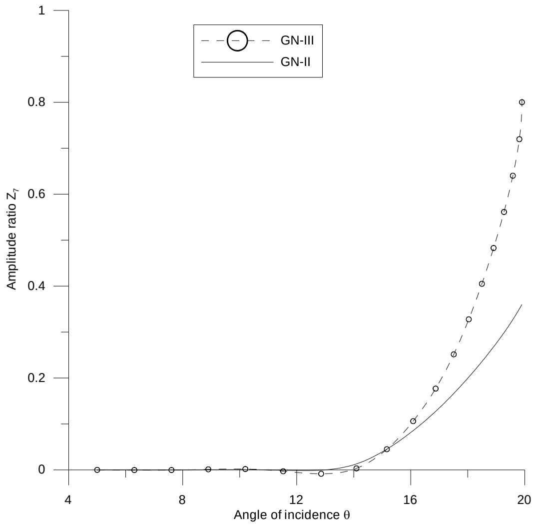

Fig. 9 Variations of amplitude ratio Z_7 with angle of incidence θ

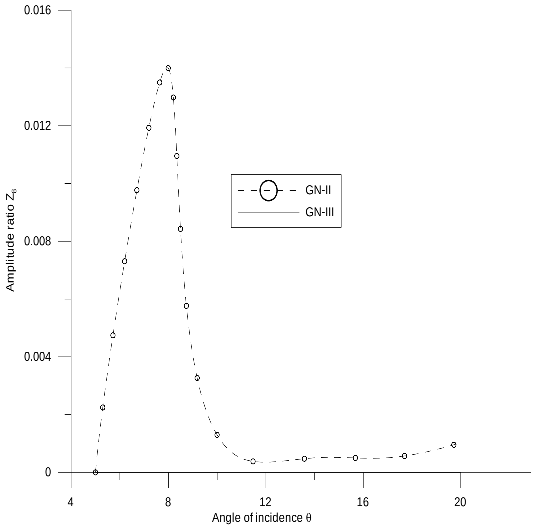

Fig. 10 Variations of amplitude ratio Z_8 with angle of incidence θ

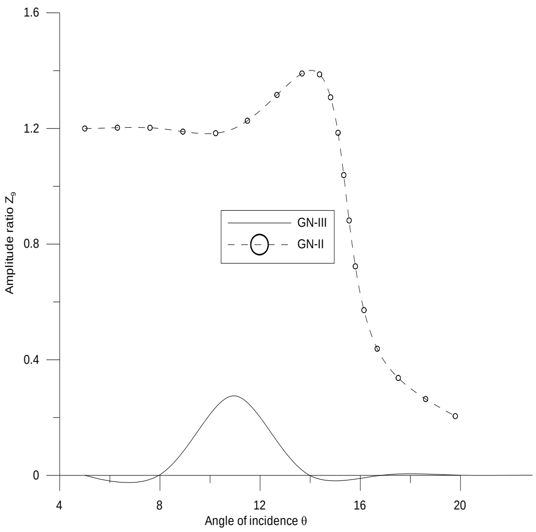

Fig. 11 Variations of amplitude ratio Z_9 with angle of incidence θ

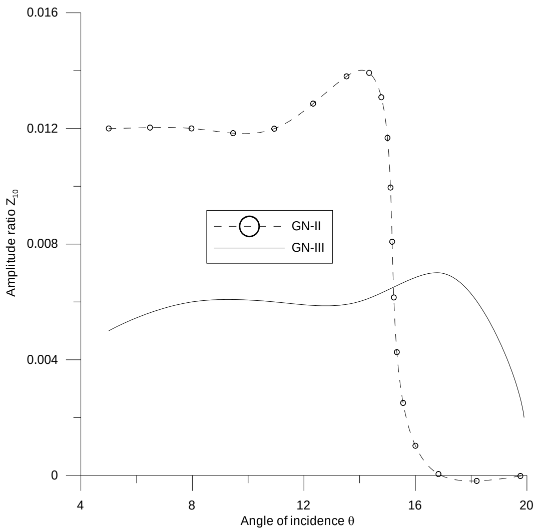

Fig. 12 Variations of amplitude ratio Z_{10} with angle of incidence θ

and B are given in Appendix C.

Incident SV wave

In system of equations Eq. (48), if we replace $A_0^{(1)}$ by $B_0^{(1)}$ and equate $A_0^{(1)} = 0$, we obtain the amplitude ratios corresponding to incident SV wave.

6. Particular cases

(i) If $k_1{}^* = k_3{}^* = 0$, then from Appendix C, we obtain the corresponding expressions for transversely isotropic magnetothermoelastic solid slab of uniform thickness, interposed between two different semi-infinite viscoelastic solids without energy dissipation and with two temperature with Hall current effect and rotation.

(ii) If $a_1 = a_3 = 0$, then we obtain the expressions for transversely isotropic magnetothermoelastic solid slab of uniform thickness, interposed between two different semi-infinite viscoelastic solids with and without energy dissipation along with Hall current effect and rotation.

(iii) If we take $c_{11} = \lambda + 2\mu = c_{33}$, $c_{12} = c_{13} = \lambda$, $c_{44} = \mu$, $\beta_1 = \beta_3 = \beta$, $\alpha_1 = \alpha_3 = \alpha$, $K_1 = K_3 = K$ and $a_1 = a_3 = a$, we obtain the corresponding expressions in isotropic magnetothermoelastic solid slab of uniform thickness, interposed between two different semi-infinite viscoelastic solids with two temperature and with and without energy dissipation along with combined effects of Hall current and rotation.

(iv) If $m = 0$, we obtain the expressions for transversely isotropic magnetothermoelastic solid slab of uniform thickness, interposed between two different semi-infinite viscoelastic solids and with and without energy dissipation and with two temperature along with rotation.

7. Numerical results and discussion

For the purpose of numerical evaluation,

(i) Cobalt material has been chosen for transversely isotropic magnetothermoelastic solid (medium M), following Dhaliwal and Singh (1980), as
$c_{11} = 3.071 \times 10^{11} Nm^{-2}$,
$c_{12} = 1.650 \times 10^{11} Nm^{-2}$,
$c_{33} = 3.581 \times 10^{11} Nm^{-2}$,
$c_{13} = 1.027 \times 10^{11} Nm^{-2}$,
$c_{44} = 1.510 \times 10^{11} Nm^{-2}$,
$\rho = 8.836 \times 10^{3} \mathrm{Kgm}^{-3}$,
$T_0 = 298°K,\ C_E = 4.27 \times 10^{2} JKg^{-1} deg^{-1}$,
$K_1 = .690 \times 10^{2}\ wm^{-1} deg^{-1}$,
$K_3 = .690 \times 10^{2} wm^{-1} deg^{-1}$,
$\beta_1 = 7.04 \times 10^{6} Nm^{-2} deg^{-1}$,
$\beta_3 = 6.90 \times 10^{6} Nm^{-2} deg^{-1}$,
$K_1^* = 0.02 \times 10^{2} Nsec^{-2} deg^{-1}$,
$K_3^* = 0.04 \times 10^{2} Nsec^{-2} deg^{-1}$,
$\mu_0 = 1.2571 \times 10^{-6} Hm^{-1}$,
$H_0 = 1 Jm^{-1} nb^{-1},\ \varepsilon_0 = 8.838 \times 10^{-12} Fm^{-1}$
with non-dimensional parameter L = 1 and
$\sigma_0 = 9.36 \times 10^{5} \mathrm{col}^2/\mathrm{Cal.\,cm.sec},\ \Omega = 3$,
$t_0 = 0.02, M = 3$ and two temperature parameters is taken as $a_1 = 0.03$ and $a_3 v = 0.06$.

(ii) Copper material has been chosen for elastic solid (medium $M^{(1)}$), following Youssef (2006)s as
$\lambda^{1*} = 7.76 \times 10^{10}\ \mathrm{Kgm}^{-1}\mathrm{s}^{-2}$,
$\mu^{1*} = 3.278 \times 10^{10}\ Kgm^{-1} s^{-2}$,
$C_E = 0.6331 \times 10^{3} JKg^{-1} K^{-1}$,
$\rho = 8.954 \times 10^{3} Kgm^{-3},\ Q_1 = 1$ and $Q_2 = 1$.

(iii) Following Dhaliwal and Singh (1980), Magnesium material has been taken for the medium $M^{(2)}$ as
$\lambda^{2*} = 2.17 \times 10^{10} Nm^{2}$,
$\mu^{2*} = 3.278 \times 10^{10} Nm^{2}$,
$\omega_1 = 3.58 \times 10^{11} S^{-1},\ \rho = 1.74 \times 10^{3} Kgm^{-3}$,
$T_0 =$ 298 K, $C_E = 1.04 \times 10^{3} Jkg^{-1} deg^{-1}$

Matlab software 8.4.0. has been used for numerical computation of the resulting quantities. The values of Amplitude ratios of various reflected and refracted waves, when P wave is incident, with respect to angle of incidence θ have been computed and are depicted graphically in Figs. 3-12. A comparison has been made to show the effect of two theories of GN Type-II and GN Type-III.

In the Figs. 3-12

(1) Solid line corresponds to GN theory of Type-III
(2) The small dashed line with centre symbol circle corresponds to GN theory of Type-II.

Incident P wave

Fig. 3 shows the variations of amplitude ratio z_1 with angle of incidence θ. Here, we notice that for GN Theory of Type -II and Type-III variations are in ascending order upto the range $0^0 \leq \theta \leq 16^0$ and descend afterwards. However the variations corresponding to GN-II are higher than GN-III.

Fig. 4, exhibits the variations of amplitude ratio z_2 with angle of incidence θ. Here, we notice that variations increase continuously in the range $0^0 \leq \theta \leq 18^0$ and decrease sharply afterwards corresponding to GN-II and GN-III.

Fig. 5 shows the trends of variations of amplitude ratio Z_3 with respect to angle of incidence θ. Here, we notice that corresponding to GN theory of Type-III, the variations increase slowly whereas increase sharply and in oscillatory form corresponding to GN-II.

Fig. 6 exhibits the trends of variations of amplitude ratio Z_4 with respect to angle of incidence θ. We notice that for GN Type-II, initially the values of Z_4 lie on the boundary surface upto $\theta = 18^0$ but immediately after this range, a sudden jump in the variations is noticed whereas no variations are noticed in this range corresponding to GN-III.

Fig. 7 shows the trends of variations of amplitude ratio Z_5 with respect to angle of incidence θ. We notice that initially the values are steady state for both the cases but as θ approaches 18^0, the values of amplitude ratio start varying and sharply increase corresponding to GN-II whereas corresponding to GN-III, the variations are not visible in this range.

Fig. 8 displays the variations in amplitude ratio Z_6 with respect to angle of incidence θ. Here corresponding to GN theory of Type-II, the values of amplitude ratio move away from the boundary surface in form of ascending pattern of waves as θ increases whereas the trends are different corresponding to GN Type -III, as here the variations approach boundary surface away from the initial range with small variations.

Fig. 9 displays the variations in amplitude ratio Z_7 with respect to angle of incidence θ. It is noticed that corresponding to GN Type-II and Type-III, upto $\theta = 14^0$ the variations are negligible but arise suddenly immediately afterwards and keep on increasing then in the rest.

Fig. 10 displays the variations in amplitude ratio Z_8 with respect to angle of incidence θ. Here, we notice that corresponding to GN-II, in the range $0^0 \leq \theta \leq 8^0$, the variations increase sharply and decrease sharply in the range $8^0 \leq \theta \leq 12^0$ and remain near the boundary surface afterwards whereas no variations are noticed in this range corresponding to GN-III.

Fig. 11 exhibits the variations in amplitude ratio Z_9 with respect to angle of incidence θ. Here corresponding to both the theories oscillatory pattern is observed and variations decrease with sharp blunt corresponding to GN-II whereas oscillate smoothly for GN-III. Fig. 12 displays the variations in amplitude ratio Z_{10} with respect to angle of incidence θ. Here also an oscillatory pattern is seen corresponding to both the cases and as we move away, the variations are near the boundary surface.

8. Conclusions

From the above graphs, we conclude that

- More variations are noticed corresponding to GN-II than GN-III in the amplitude ratios.
- In some of the amplitude ratios, in the considered range, variations are negligible corresponding to GN-III.
- Variations corresponding to GN-II move away from the boundary surface whereas near the boundary surface corresponding to GN-III.
- Variations corresponding to GN theory of Type-III move towards the boundary surface as θ increases whereas the trends are not same for GN theory of Type-II
- So to detect the variations in the earth surface corresponding to this problem, model GN-II is better as this model gives us more variations and more better results. However the limitation is that our range is small. In the higher range, the results may be different. The present theoretical results may provide interesting information for experimental scientists/ researchers/ seismologists working on this subject. The used methods in the present article is applicable to a wide range of problems in thermodynamics and thermoelasticity.

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CC

Appendix A

Where

$$A = \zeta_4\zeta_5\zeta_6 - cos^2\theta\ \zeta_2\zeta_7p_1 - \delta_1\zeta_8^2\zeta_1\zeta_4 + \zeta_2\zeta_7\zeta_8 + \zeta_2\zeta_7\zeta_8^2\zeta_1,$$

$$B = -\zeta_1\zeta_4\zeta_6 - \zeta_1 cos^2\theta\ \zeta_2\zeta_7p_1 + \omega^2\zeta_6\zeta_5 - \zeta_4\zeta_5\zeta_1 + \zeta_5 cos^2\theta\ \zeta_7p_1 - \delta_1\zeta_4\zeta_3\zeta_8 + \zeta_2\zeta_7\zeta_8 - \delta_1\omega^2\zeta_1\zeta_8^2 + \zeta_3\zeta_8\zeta_1\zeta_4 - \zeta 1_8^2\zeta_1\zeta_7 - \zeta_2\zeta_8\zeta_3\zeta_7 + \delta_1\zeta_8\zeta_7 + p_1\zeta_7\zeta_6 sin^2\theta - sin^2\theta\ \zeta_2\zeta_7p_1$$

$$C = -\zeta_5\zeta_1\omega^2 - \zeta_1\zeta_6\omega^2 + \zeta_1^2\zeta_4 - \zeta_1 cos^2\theta\ \zeta_7p_1 - \zeta_3\delta_1\omega^2\zeta_8 + \zeta_3^2\zeta_4 + \zeta_8\zeta_3\zeta_1\omega^2 - \zeta_1\ \varepsilon_5{}'\beta_1^2\omega^2 sin^2\theta$$

$$D = \omega^2(\zeta_1^2 - \zeta_3^2), \quad \zeta_1 = \left(\frac{\varepsilon_0\mu_0^2\ H_0^2}{\rho} + 1\right)\omega^2 + \Omega^2,$$

$$\zeta_2 = \frac{a_1}{L} sin^2\theta + \frac{a_3}{L} cos^2\theta, \quad \zeta_3 = -2i\omega\Omega$$

$$\zeta_4 = \zeta_2\omega^2 - sin^2\theta(\varepsilon_1 + i\varepsilon_3) - cos^2\theta\ (\varepsilon_2 + i\varepsilon_4),$$

$$\zeta_5 = \sin^2\theta + \delta_2\cos^2\theta, \qquad \zeta_6 = \delta_2\sin^2\theta + \delta_3\cos^2\theta$$

$$\zeta_7 = \varepsilon_5{}'\omega^2\beta_1\beta_3, \qquad \zeta_8 = -\sin\theta\cos\theta, \qquad p_5 = \frac{\beta_3}{\beta_1}$$

$$D = k_j^4\big(\varepsilon_{13}\delta_2\sin^4\theta_j + \varepsilon_{13}\delta_3\eta_j^2 + \delta_2\varepsilon_{24}\eta_j^2 + \varepsilon_{24}\delta_3\cos^4\theta_j - \delta_2\omega^2\sin^2\theta_j\zeta_j - \omega^2\delta_3\cos^2\theta_j\eta_j - \beta_3^2\varepsilon_5'\omega^2\cos^2\theta_j\eta_j\big) + k_j^2\Big\{\big(-\sin^2\theta_j\delta_2 - \delta_3\cos^2\theta_j\big)\omega^2 + \left(\frac{M_0}{m}i\omega + \omega^2 + \Omega^2\right)\big(-\varepsilon_{13}\sin^2\theta_j - \varepsilon_{24}\cos^2\theta_j + \omega^2\zeta_j\big) - \beta_3^2\varepsilon_5'\omega^2\cos^2\theta_j\Big\} + \left(\frac{M_0}{m}i\omega + \omega^2 + \Omega^2\right)\omega^2,$$

$$j = 1,2,3$$

$$d_j = [k_j^4\{\eta_j\big(\delta_1\sin^2\theta_j\varepsilon_{13} + \delta_1\cos^2\theta_j\varepsilon_{24} + \delta_1\zeta_j - \varepsilon_5'\beta_1\beta_3\omega^2\zeta_j\}+ k_j^2\{-\delta_1\omega^2\eta_j + (M_0 + 2\Omega)i\omega(\varepsilon_{13}\sin^2\theta_j + \cos^2\theta_j\varepsilon_{24}\} - \omega^2(M_0 + 2\Omega)i\omega\ ]/D,$$

$$j = 4, 5, 6$$

$$l_j = [k_j^3\{\delta_1\eta_j\varepsilon_5'\beta_1\beta_3\omega^2 i\cos\theta_j + (\delta_2 i\sin^3\theta_j + \delta_3 i\cos^2\theta_j\sin\theta_j)\varepsilon_5'\beta_1^2\omega^2\} + ik_j\{(M_0 i\omega + 2i\omega\Omega)\big(\varepsilon_5'\beta_1\beta_3\omega^2\cos\theta_j\big) + \left(\frac{M_0}{m}i\omega + \omega^2 + \Omega^2\right)(\varepsilon_5'\sin\theta_j\beta_1^2\omega^2)\}] /D$$

$$j = 4, 5, 6$$

$$\eta_j = \sin\theta_j\cos\theta_j, \quad \varepsilon_{13} = \varepsilon_1 - i\varepsilon_3\omega,$$

$$\varepsilon_{24} = \varepsilon_2 - i\varepsilon_4\omega, \quad \zeta_j = \frac{a_1}{L}\sin^2\theta_j + \frac{a_3}{L}\cos^2\theta_j,$$

$$M_0 = \frac{M}{1 + m^2}\ \mu_0 H_0 m$$

Appendix B

$$d_j = [k_j^4\{\eta_j\big(\delta_1 sin^2\theta_j\varepsilon_{13} + \delta_1 cos^2\theta_j\varepsilon_{24} - \delta_1\zeta_j + \varepsilon_5'\beta_1\beta_3\omega^2\zeta_j\} + k_j^2\{-\delta_1\omega^2\eta_j + (M_0 + 2\Omega)i\omega(\varepsilon_{13} sin^2\theta_j + cos^2\theta_j\varepsilon_{24}\} - \omega^2(M_0 + 2\Omega)i\omega\ ]/D,$$

$$j = 1, 2, 3$$

$$l_j = [k_j^3\{-\delta_1\eta_j\varepsilon_5'\beta_1\beta_3\omega^2 icos\theta_j + (\delta_2 isin^3\theta_j + \delta_3 icos^2\theta_j sin\theta_j)\varepsilon_5'\beta_1^2\omega^2\} + ik_j\{(M_0 i\omega + 2i\omega\Omega)\big(\varepsilon_5'\beta_1\beta_3\omega^2 cos\theta_j\big) + \left(\frac{M_0}{m}i\omega + \omega^2 + \Omega^2\right)(\varepsilon_5' sin\theta_j\beta_1^2\omega^2)\}] /D,$$

$$j = 1,2,3$$

Appendix C

$$a_{11} = i\sin\theta, \quad a_{12} = i/\beta'^{(1)}\sqrt{\omega^2 - \beta'^{(1)^2} sin^2\theta},$$

$$a_{13} = -1, \quad a_{14} = -1, \quad a_{15} = -1, \quad a_{16} = -1,$$

$$a_{17} = -1, \quad a_{18} = -1, \quad a_{19} = 0, \quad a_{1,10} = 0,$$

$$a_{21} = -i/\alpha'^{(1)}\sqrt{\omega^2 - \alpha'^{(1)^2}(\sin^2\theta)}, \quad a_{22} = i\sin\theta,$$

$$a_{21} = -i/\alpha'^{(1)}\sqrt{\omega^2 - \alpha'^{(1)^2}(\sin^2\theta)}, \quad a_{22} = i\sin\theta,$$

$$a_{a_{23}=-d_1, a_{24}=-d_2, a_{25}=-d_3, a_{26}=-d_4, 27} = -d_5,$$

$$a_{28} = -d_6, \qquad a_{29} = 0, \qquad a_{2,10} = 0,$$

$$a_{31} = (\omega^2 - \alpha'^{(1)^2})(\sin\theta)^2 - \frac{\alpha'^{(1)^2}\gamma'^{(1)^2}}{\omega^2}(\sin\theta)^2,$$

$$a_{32} = \frac{(\alpha'^{(1)^2} - \gamma'^{(1)^2})}{\beta'^{(1)}}\sin\theta\sqrt{\omega^2 - \beta'^{(1)^2}\sin^2\theta}$$

$$a_{3j} = -\Delta_j, \qquad j = 3, 4, 5$$

where

$$\Delta_j = \frac{c_{11}}{\rho c_1^2} i \sin\theta + \frac{c_{13}}{\rho c_1^2} i \frac{d_j}{v_j}\sqrt{\omega^2 - v_j{}^2(\sin\theta)^2} - \frac{\beta_3}{\beta_1} l_j .$$