Steel and Composite Structures, *Vol. 25, No. 4 (2017) 497-503*
DOI: https://doi.org/10.12989/scs.2017.25.4.497

Nonlinear vibration of multi-body systems with linear and nonlinear springs

Mahmoud Bayat*[1], Iman Pakar[2] and Mahdi Bayat[3]

[1]*Young Researchers and Elite Club, Roudehen Branch, Islamic Azad University, Roudehen, Iran*
[2]*Young Researchers and Elite club, Mashhad Branch, Islamic Azad University, Mashhad, Iran*
[3]*Department of Civil Engineering, Roudehen Branch, Islamic Azad University, Roudehen, Iran*

(*Received December 19, 2016, Revised June 11, 2017, Accepted September 20, 2017*)

Abstract. In this paper, nonlinear vibration of multi-degree of freedom systems are studied. It has been tried to develop the mathematical model of systems by second-order nonlinear partial differential equations. The masses are connected with linear and nonlinear springs in series. A great effort has been done to solve the nonlinear governing equations analytically. A new analytical method called Variational Iteration Method (VIM) is proposed and successfully applied to the problem. The linear and nonlinear frequencies are obtained and the results are compared with numerical solutions. The first order of Variational Iteration Method (VIM) leads us to high accurate solution.

Keywords: nonlinear vibration; two-degree-of-freedom; Variational Iteration Method (VIM)

1. Introduction

Some dynamical systems that require two independent coordinates, or degrees of freedom, to describe their motions, are called two degree of freedom systems (TDOF). There has been an increasing attention in recent years towards the motion of the nonlinear TDOF oscillation systems (Chen 1987, Cveticanin 2001, 2002, Masri 1972, Vakakis *et al*. 2004).Coupled vibrating systems has been widely investigate and used in many practical engineering components. The governing equation of motion of multi-degree of freedom systems are consist of two second-order differential equations with cubic nonlinearities.

Generally, the fact of preparing a solution for the equations of motion for a mechanical system associated with linear and nonlinear properties was attempted through the transformation into a set of differential algebraic equations using intermediate variables; here is the equations of motion for a TDOF system are transformed into the Duffing equations (Lai *et al*. 2007). To prepare an analytical solution for nonlinear differential equations are extremely difficult.

It is very interesting for the scientific to applied mathematics to solve dynamic problems analytically. Recently, particular attention much attention has been devoted to the developments of the new approximate analytical methods to prepare analytical solutions for nonlinear differential equations such as (Baki *et al*. 2011, He 2002, 2010, Akgoz *et al*. 2011, Bayat 2015a,b, 2012, Lau *et al*. 1983, MehdipourI *et al*. 2010, Pakar *et al*. 2015, Sedighi *et al*. 2016, 2015, Shen *et al*. 2009, Wu 2011, Öziş *et al*. 2017, Hashemietal 2013, Khavaji *et al*. 2012, He 1999). In this paper, we have applied Vairational Iteration Method (VIM) for nonlinear vibration of two degree of freedom systems. The methodfirst was proposed by He (1999). It has been shown that the first iteration of the Vairational Iteration Method (VIM) leads us to a high accuracy of the solution and has an excellent agreement with the exact solutions.

Comparisons between analytical and exact solutions show that VIM can converge to an accurate and rapid periodic solution for nonlinear systems.

*Corresponding author, Researcher,
E-mail: mbayat14@yahoo.com; mbayat@riau.ac.ir

2. Basic idea of variational Iteration method

In this method, the problems are initially approximated with possible unknowns. Then a corrected functional is constructed using a general Lagrange multiplier, which can be identified optimally via the variational theory. To illustrate the basic idea of the method, we consider the following general nonlinear system (He 1999)

$$Lv + Nv = g\left(x\right) \tag{1}$$

Where, L is a linear operator, and N a nonlinear operator, $g(x)$ an inhomogeneous or forcing term. According to the variational iteration method, we can construct a correct functional as follows

$$v_{(n+1)}(t) = v_n(t) + \int_0^t \lambda\left\{Lv_n(\tau) + N\tilde{v}_n(\tau) - g(\tau)\right\}d\tau. \tag{2}$$

Where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, the subscript n denotes the nth approximation, $\tilde{v}_n$ is considered as a restricted variation, i.e., ~ = 0*n* δ*v*.

For linear problems, its exact solution can be obtained by only one iteration step due to the fact that the Lagrange multiplier can be exactly identified.

http://www.techno-press.org/?journal=scs&subpage=6
ISSN: 1229-9367 (Print), 1598-6233 (Online)

3. Cases of nonlinear two-degree-of-freedom (TDOF) oscillating systems

The proposed approach is applied for two nonlinear two degree of freedom oscillating system to show the effectiveness of analytical procedure.

3.1 Case 1

Fig. 1 represents a two equal mass connected with three linear and nonlinear stiffnesses fixed to the body. The governing equation of the motion is (Cveticanin 2001)

$$\begin{aligned} m\ddot{x}+k_1x+k_2\left(x-y\right)+k_3\left(x-y\right)^3=0 \\ m\ddot{y}+k_1x+k_2\left(y-x\right)+k_3\left(y-x\right)^3=0 \end{aligned} \tag{3}$$

With initial conditions

$$\begin{aligned} x(0)=X_0,\quad \dot{x}(0)=0, \\ y(0)=Y_0,\quad \dot{y}(0)=0, \end{aligned} \tag{4}$$

Where the time derivatives are (.)=d/dt and (..)=d^2/dt^2. By dividing Eq. (3) by mass m yields

$$\begin{aligned} \ddot{x}+\frac{k_1}{m}x+\frac{k_2}{m}\left(x-y\right)+\frac{k_3}{m}\left(x-y\right)^3=0 \\ \ddot{y}+\frac{k_1}{m}x+\frac{k_2}{m}\left(y-x\right)+\frac{k_3}{m}\left(y-x\right)^3=0 \end{aligned} \tag{5}$$

Here, we introduce the intermediate variables u and v as follows (Lai andLim 2007)

$$x:=u \tag{6}$$

$$y-x:=v \tag{7}$$

Transforming of the Eq. (5) by using the above intermediate variables yields

$$\ddot{u}+\delta u-\rho v-\varepsilon v^3=0 \tag{8}$$

$$\ddot{u}+\ddot{v}+\delta u-\delta v+\rho v+\varepsilon v^3=0 \tag{9}$$

Where $\delta=k_1/m$, $\rho=k_2/m$ and $\varepsilon=k_3/m$. Eq. (8) is rearranged as follows

$$\ddot{u}=-\delta u+\rho v+\varepsilon v^3 \tag{10}$$

Substituting Eq. (10) into Eq. (9) yields

$$\ddot{v}+\left(\delta+2\rho\right)v+2\varepsilon v^3=0 \tag{11}$$

With initial conditions

$$v(0)=y(0)-x(0)=Y_0-X_0=A,\qquad \dot{v}(0)=0 \tag{12}$$

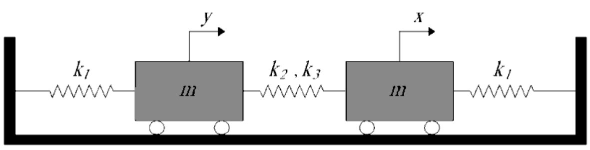


Fig. 1 Two-mass system connected with the fixed bodies

3.1.1 Solution using VIM

Assume that the angular frequency of the Eq. (11) is ω, we have the following linearized equation

$$\ddot{v}+\omega^2 v=0 \tag{13}$$

So we can rewrite Eq. (11) in the form

$$\ddot{v}+\omega^2 v+g(v)=0 \tag{14}$$

Where $g(v)=\left(\delta+2\rho\right)v+2\varepsilon v^3-\omega^2 v$.

Applying the variational iteration method, we can construct the following functional equation

$$v_{n+1}(t)=v_n(t)+\int_0^t \lambda\left(\ddot{v}(\tau)+\omega^2 v_n(\tau)-g(\tau)\right)d\tau \tag{15}$$

Where $\tilde{g}$ is considered as a restricted variation, i.e., $\delta\tilde{g}=0$.

Calculating variation with the respect to v_n and nothing that $\delta\tilde{g}(v_n)=0$. We have the following stationary conditions

$$\begin{cases} \lambda''+\omega^2\lambda(\tau)=0, \\ \lambda(\tau)\big|_{\tau=t}=0, \\ 1-\lambda'(\tau)\big|_{\tau=t}=0. \end{cases} \tag{16}$$

The Lagrange multiplier, therefore, can be identified as

$$\lambda=\frac{1}{\omega}\sin\omega\left(\tau-t\right) \tag{17}$$

Substituting the identified multiplier into Eq. (16) results in the following iteration formula

$$v_{n+1}(t)=v_n(t)+\frac{1}{\omega}\int_0^t \sin\omega\left(\tau-t\right)\times\left(\ddot{v}_n(\tau)+\left(\delta+2\rho\right)v_n(\tau)+2\varepsilon v_n^3(\tau)\right)d\tau \tag{18}$$

Assuming its initial approximate solution has the form

$$v_0=A\cos\left(\omega t\right) \tag{19}$$

And substituting Eq. (19) into Eq. (18) leads to the following residual

$$R_0(t)=\left(-A\omega^2+A\delta+2A\rho+\frac{3}{2}\varepsilon A^3\right)\cos\left(\omega t\right)+\frac{1}{2}\varepsilon A^3\cos\left(3\omega t\right) \tag{20}$$

By the formulation (20), we can obtain

$$u_1(t)=A\cos\left(\omega t\right)+\int_0^t \frac{1}{\omega}\sin\omega\left(\tau-t\right)R_0\left(\tau\right)d\tau. \tag{21}$$

To avoid secular terms appear in u_1, the coefficient of $\cos\left(\omega t\right)$ in Eq. (21) requires being zero, i.e.

$$\omega_{VIM}=\frac{1}{2}\sqrt{4\delta+8\rho+6\varepsilon A^2} \tag{22}$$

According to Eqs. (22) and (19), we can obtain the following approximate solution

$$v(t)=A\cos\left(\frac{1}{2}\sqrt{4\delta+8\rho+6\varepsilon A^2}\,t\right) \tag{23}$$

The first-order analytical approximation for $u(t)$ is

$$u(t)=\frac{-cos(\sqrt{\delta}t)\left(-X_0\delta^2+10X_0\delta\omega^2-9X_0\omega^4+\varepsilon A^3\delta-7\varepsilon A^3\omega^2-9A\rho\omega^2+A\delta\rho\right)}{\delta^2-10\delta\omega^2+9\omega^4}$$
$$-\frac{27A\left(cos(\omega t)\left((\varepsilon A^2+\frac{4}{3}\rho)(\omega^2-\frac{1}{9}\delta)\right)+cos(3\omega t)\left(\frac{1}{27}\varepsilon A^2(\omega^2-\delta)\right)\right)}{4\delta^2-40\delta\omega^2+36\omega^4} \tag{24}$$

Therefore, the first-order analytical approximate displacements $x(t)$ and $y(t)$ are

$$\begin{aligned} x(t)&=u(t)\\ x(t)&=u(t)+A\cos(\omega t)\end{aligned} \tag{25}$$

3.2 Case 2

Fig. 2 represents a two mass-system with a connection of linear and nonlinear stiffnesses. The governing equation of the motion is given as (Cveticanin 2002)

$$\begin{aligned} m\ddot{x}+k_1(x-y)+k_2(x-y)^3=0\\ m\ddot{y}+k_1(y-x)+k_2(y-x)^3=0\end{aligned} \tag{26}$$

With initial conditions

$$\begin{aligned} x(0)=X_0,\quad \dot{x}(0)=0,\\ y(0)=Y_0,\quad \dot{y}(0)=0,\end{aligned} \tag{27}$$

Where the time derivatives are (.)=*d/dt* and (..)=d^2/dt^2. k_1 is linear spring stiffness and k_2 is the nonlinear spring stiffness, respectively. By dividing Eq. (27) by mass m yields

$$\begin{aligned} \ddot{x}+\frac{k_1}{m}(x-y)+\frac{k_2}{m}(x-y)^3=0\\ \ddot{y}+\frac{k_1}{m}(y-x)+\frac{k_2}{m}(y-x)^3=0\end{aligned} \tag{28}$$

Like in Case1, transforming of the Eq. (28) by using the intermediate variables in Eqs. (29)And (30) yields

$$\ddot{u}-\alpha v-\beta v^3=0 \tag{29}$$

$$\ddot{v}+\ddot{u}+\alpha v+\beta v^3=0 \tag{30}$$

Where $\alpha=k_1/m$ and $\beta=k_2/m$. Eq. (29) is rearranged as follows

$$\ddot{u}=\alpha v+\beta v^3. \tag{31}$$

Substituting Eq. (31) into Eq. (30) yields

$$\ddot{v}+2\alpha v+2\beta v^3=0 \tag{32}$$

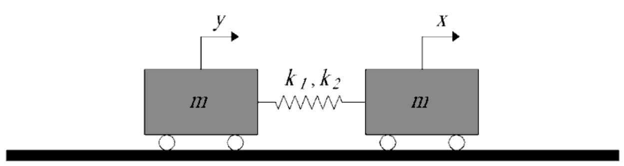


Fig. 2 Two masses connected by linear and nonlinear stiffnesses

With initial conditions

$$v(0)=y(0)-x(0)=Y_0-X_0=A,\qquad \dot{v}(0)=0 \tag{33}$$

3.2.1 Solution using VIM

Assume that the angular frequency of the Eq. (32) is ω, we have the following linearized equation

$$\ddot{v}+\omega^2 v=0 \tag{34}$$

So we can rewrite Eq. (34) in the form

$$\ddot{v}+\omega^2 v+g(v)=0 \tag{35}$$

Where $g(v)=2\alpha v+2\beta v^3-\omega^2 v$.

Applying the variational iteration method, we can construct the following functional equation

$$v_{n+1}(t)=v_n(t)+\int_0^t \lambda\left(\ddot{v}(\tau)+\omega^2 v_n(\tau)-g(\tau)\right)d\tau \tag{36}$$

Where $\tilde{g}$ is considered as a restricted variation, i.e., $\delta\tilde{g}=0$.

Calculating variation with the respect to v_n and nothing that $\delta\tilde{g}(v_n)=0$. We have the following stationary conditions

$$\begin{cases} \lambda''+\omega^2\lambda(\tau)=0,\\ \lambda(\tau)\big|_{\tau=t}=0,\\ 1-\lambda'(\tau)\big|_{\tau=t}=0.\end{cases} \tag{37}$$

The Lagrange multiplier, therefore, can be identified as

$$\lambda=\frac{1}{\omega}\sin\omega(\tau-t) \tag{38}$$

Substituting the identified multiplier into Eq. (36) results in the following iteration formula

$$v_{n+1}(t)=v_n(t)+\frac{1}{\omega}\int_0^t \sin\omega(\tau-t)\times\left(\ddot{v}_n(\tau)+2\alpha v_n(\tau)+2\beta v_n^3(\tau)\right)d\tau \tag{39}$$

Assuming its initial approximate solution has the form

$$v_0=A\cos(\omega t) \tag{40}$$

And substituting Eq. (40) into Eq. (39) leads to the following residual

$$R_0(t)=\left(-A\omega^2+2A\alpha+\frac{3}{2}\beta A^3\right)cos(\omega t)+\frac{1}{2}\beta A^3 cos(3\omega t) \tag{41}$$

By the formulation (39), we can obtain

$$u_1(t)=A\cos(\omega t)+\int_0^t \frac{1}{\omega}\sin\omega(\tau-t)R_0(\tau)d\tau. \tag{42}$$

To avoid secular terms appear in u_1, the coefficient of $\cos(\omega t)$ in Eq. (41) requires being zero, i.e.

$$\omega_{VIM}=\frac{1}{2}\sqrt{8\alpha+6\beta A^2} \tag{43}$$

According to Eqs. (43) and (40), we can obtain the following approximate solution

$$v(t) = A\cos\left(\frac{1}{2}\sqrt{8\alpha + 6\beta A^2}\, t\right) \tag{44}$$

The first-order analytical approximation for $u(t)$ is

$$u(t) = \iint (\alpha v + \beta v^3)\,dt\,dt = -\frac{1}{9\omega^2} A\cos(\omega t)\left(9\alpha + 6\beta A^2 + A\beta\cos^2(\omega t)\right). \tag{45}$$

Therefore, the first-order analytical approximate displacements $x(t)$ and $y(t)$ are

$$\begin{aligned} x(t) &= u(t) \\ x(t) &= u(t) + A\cos(\omega t) \end{aligned} \tag{46}$$

4. Discussion of examples

Vairational Iteration Method (VIM) is applied to 2DOF systems for two different cases to illustrate the accuracy of the proposed method. Tables 1 and 2 give the comparisons of obtained results with the exact ones for different values of *m, k*, k_1, k_2, and initial conditions. The maximum relative error between the Vairational Iteration Method (VIM) results and exact results is less than 2.2124%. It is obvious from the tables that there are an excellent agreement between the results obtained from the Vairational Iteration Method (VIM) and the exact solution. In these cases, Vairational Iteration Method (VIM) prepares accurate results for a wide range of system parameters.

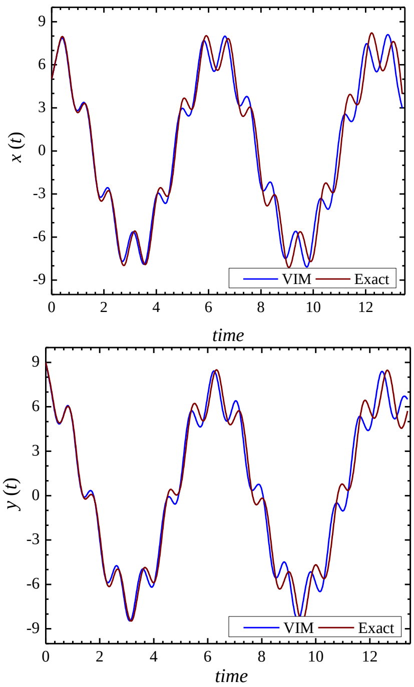


Fig. 3 (Case 1): Comparison of analytical solution of displacement $x(t)$ and $y(t)$ based on time with the exact solution for $m = 2,\ k_1 = 2,\ k_2 = 1,\ k_3 = 4,\ X_0 = -5,\ Y_0 = 1$

Figs. 3 and 6 are the time history oscillatory displacement results for x(t) and y(t) for different fixed values of the problem.

The effects of the linear and nonlinear springs on the frequency of the system are studied in Fig. 4 for three different cases:

(a): $m = 5,\ k_2 = 5,\ k_3 = 5$

(b): $m = 5,\ k_1 = 5,\ k_3 = 5$ (c): $m = 5,\ k_1 = 5,\ k_2 = 5$

In Fig. 5, we have considered a sensitivity analysis of frequency for various parameters of amplitude and stiffness simultaneously.

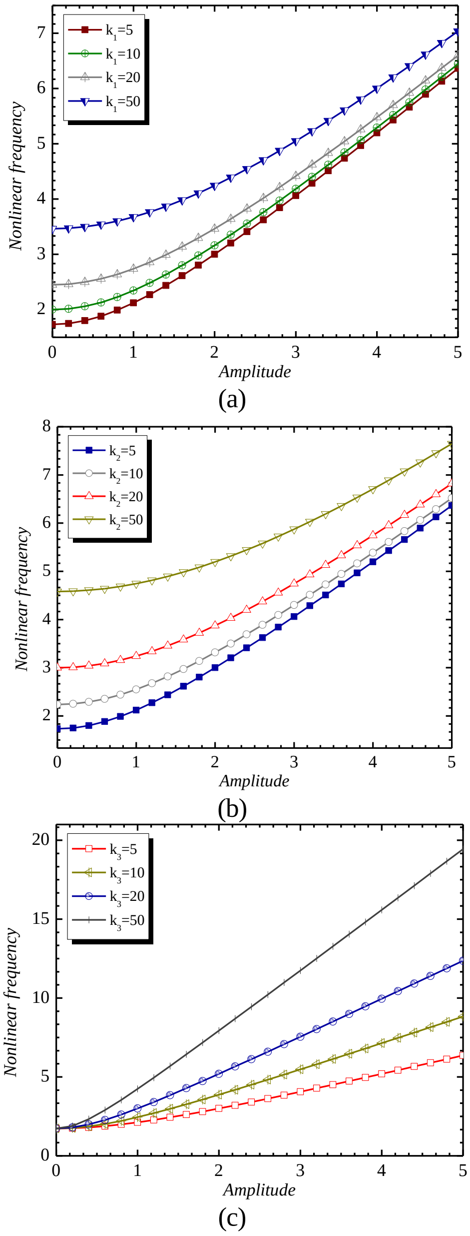


Fig. 4 (Case 1): Influence of linear and nonlinear spring of system on nonlinear frequency base on amplitude for (a) $m = 5,\ k_2 = 5,\ k_3 = 5$ and (b) $m = 5,\ k_1 = 5,\ k_3 = 5$ and (c) $m = 5,\ k_1 = 5,\ k_2 = 5$

For case 2, in Fig. 5 the effects of linear and nonlinear spring on the nonlinear frequency of the system are also studied for two cases:

(a): $m=5,\quad k_2=5,\quad$ (b): $m=5,\quad k_1=5,$

Fig. 6, represent the sensitivity analysis of frequency for various parameters of amplitude and stiffness for case 2.

By applying the first iteration of this method the high accuracy of the solution can be obtained. It has been concluded that excellent agreement with the exact solutions for the nonlinear Duffing equation is provided. Theses presentations are accurate for a wide range of vibration amplitudes and initial conditions. The VIM is quickly convergent and can also be readily generalized to two-degree-of-freedom oscillation systems with quadratic nonlinearity by combining the transformation technique.

This method can be easily extended to any nonlinear oscillator without any difficulty to obtain the nonlinear frequency of the systems with high accuracy.

Table 1 Comparison of frequency corresponding to various parameters of system (Case 1)

Constant parameters						solution		Relativ
m	k_1	k_2	k_3	X_0	Y_0	ω_{VIM}	ω_{Exact}	$\frac{\omega_{VIM}-\omega_{Ex}}{\omega_{Ex}}$
1	0.	0.	0.	1	2	1.500	1.496	0.2331
1	1	1	2	-1	1	3.873	3.820	1.3879
2	2	1	4	-5	1	7.071	6.930	2.0357
2	4	3	5	-3	2	9.937	9.743	1.9882
5	10	5	10	2	10	14.00	13.70	2.1250
5	5	20	10	-15	-5	17.57	17.21	2.0850
10	10	20	5	5	20	13.18	12.91	2.0866
10	20	30	10	-10	15	30.74	30.09	2.1806
20	40	50	20	-20	10	36.83	36.04	2.1961

Table 2 Comparison of frequency corresponding to various parameters of system (Case 2)

Constant parameters					solution		Relative
m	k_1	k_2	X_0	Y_0	ω_{VIM}	ω_{Exact}	$\frac{\omega_{VIM}-\omega_{Ex}}{\omega_{Ex}}$
1	2	3	1	2	2.9155	2.8983	0.5941
1	4	5	1	3	6.1644	6.0823	1.3501
2	5	3	-1	4	7.8262	7.6838	1.8534
2	8	6	-4	4	17.204	16.851	2.0946
5	5	5	5	15	12.328	12.068	2.1587
5	10	15	-5	10	31.882	31.195	2.2019
10	15	20	10	30	34.684	33.934	2.2087
20	40	50	15	40	48.453	47.404	2.2124
50	10	50	-20	10	36.796	36.002	2.2065

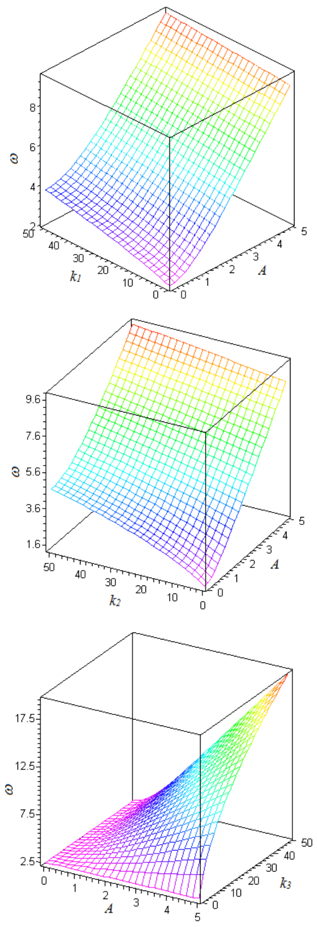


Fig. 5 Sensitivity analysis of frequency for various parameters of amplitude and stiffness

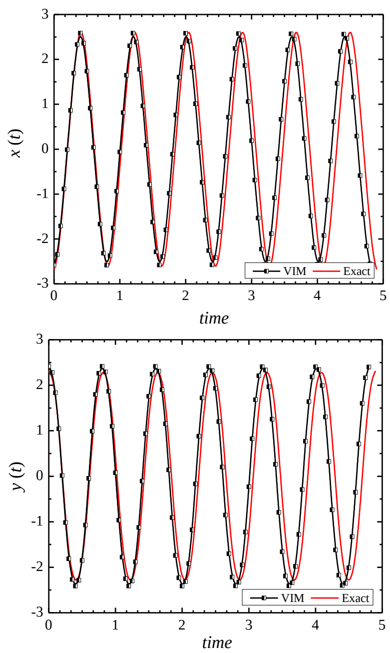
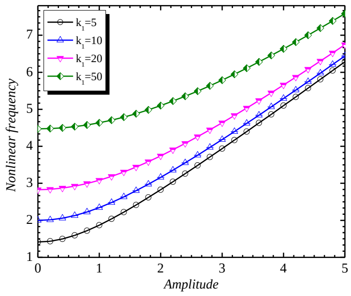
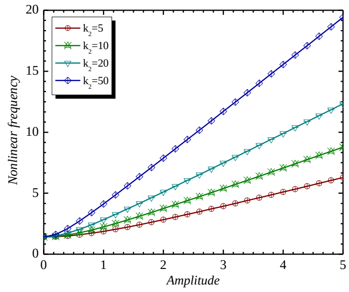


Fig. 6 (Case 2):Comparison of analytical solution of displacement $x(t)$ and $y(t)$ based on time with the exact solution for $m=2,\quad k_1=5,\quad k_2=3,\quad X_0=-1,\quad Y_0=4$

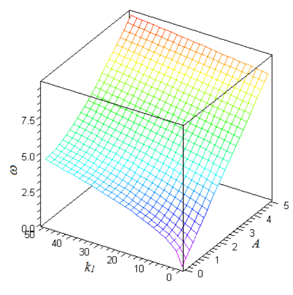


(a)

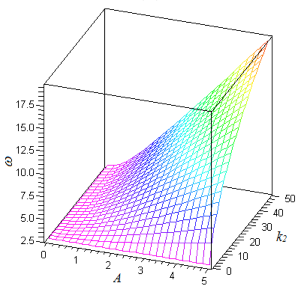


(b)

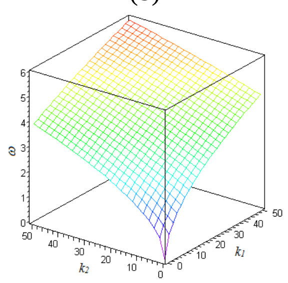
Fig. 7 (Case 2): Influence of linear and nonlinear spring of system on nonlinear frequency base on amplitude for (a) $m=5,\ k_2=5,$ and (b) $m=5,\ k_1=5,$



(a)



(b)



(c)

Fig. 8 (case 2) Sensitivity analysis of frequency for various parameters of amplitude and stiffness

5. Conclusions

Variational Iteration Method (VIM) has been applied to achieve the first-order approximate frequencies and periods for two degree of freedom oscillation systems. Excellent agreements between approximate frequencies the exact one have been demonstrated and discussed. In general, we conclude that this method is efficient for calculating periodic solutions for nonlinear oscillatory systems, and we think that the method has a great potential and could be applied to other strongly nonlinear oscillators.

References

Akgoz, B. and Civalek, O. (2011), “Nonlinear vibration analysis of laminated plates resting on nonlinear two-parameters elastic foundations”, *Steel Compos. Struct.*, **11**(5), 403-421.

Bayat, M., Bayat, M. and Pakar, I. (2015a), “Analytical study of nonlinear vibration of oscillators with damping”, *Earthq. Struct.*, **9**(1), 221-232.

Bayat, M. and Pakar, I. (2015b), “Mathematical solution for nonlinear vibration equations using variational approach”, *Smart Struct. Syst.*, **15**(5), 1311-1327.

Bayat, M., Pakar, I. and Domaiirry, G. (2012), “Recent developments of some asymptotic methods and their applications for nonlinear vibration equations in engineering problems: a review”, *Latin Am. J. Solid. Struct.*, **9**(2), 145-234.

Chen, G. (1987), “Applications of a generalized Galerkin's method to non-linear oscillations of two-degree-of-freedom systems”, *J. Sound Vib.*, **119**, 225-242.

Cveticanin, L. (2001), “Vibrations of a coupled two-degree-of-freedom system”, *J. Sound Vib.*, **247**, 279-292.

Cveticanin, L. (2002). “The motion of a two-mass system with non-linear connection”, *J. Sound Vib.*, **252**, 361-369.

HashemiKachapi, S.M. and Ganji, D.D. (2013), Dynamics and Vibrations: Progress in Nonlinear Analysis, Springer 2013.

He, J.H. (1999), “Variational iteration method: a kind of nonlinear analytical technique: some examples”, *Int. J. Nonlinear Mech.*, **34**(4), 699-708

He, J.H. (2002), “Preliminary report on the energy balance for nonlinear oscillations”, *Mech. Res. Commun.*, **29**, 107-111.

He, J.H. (2010), “Hamiltonian approach to nonlinear oscillators”, *Phys. Lett. A.*, **374**, 2312-2314.

Khavaji, A., Ganji, D.D., Roshan, N., Moheimani, R., Hatami, M. and Hasanpour, A. (2012), “Slope variation effect on large deflection of compliant beam using analytical approach”, *Struct. Eng. Mech.*, **44**(3), 405-416.

Lai ,S.K. and Lim, C.W. (2007), “Nonlinear vibration of a two-mass system with nonlinearstiffnesses”, *Nonlinear Dynam.*, **49**, 233-249.

Lau, S.L., Cheung, Y.K. and Wu, S.Y. (1983), “Incremental harmonic balance method with multiple time scales for aperiodic vibration of nonlinear systems”, *J. Appl. Mech.* - ASME, **50**(4), 871-876.

Masri, S.F. (1972), forced vibration of a class of non-linear two-degree of freedom oscillators”, *Int. J. Nonlinear Mech.*, **7**, 663-674.

Mehdipour, I., Ganji, D.D. and Mozaffari, M. (2010),. “Application of the energy balance method to nonlinear vibrating equations”, *Curr. Appl. Phys.*, **10**(1), 104-112.

Öziş, T. and Yıldırım, A. (2017), “A note on He’s homotopy perturbation method for van der Pol oscillator with very strong nonlinearity”, *Chaos Soliton. Fract.*, **34**(3), 989-991.

Oztur, B. and Coskun, S.B. (2011), “The Homotopy Perturbation Method for free vibration analysis of beam on elastic

foundation'', *Struct. Eng. Mech*., **37**(4), 415-425.
Pakar, I. and Bayat, M. (2015), "Nonlinear vibration of stringer shell: An analytical approach", *Proceedings of the Institution of Mechanical Engineers, Part E: Journal of Process Mechanical Engineering*, **229**(1), 44-51.
Sedighi, H.M. and Bozorgmehri, A. (2016), "Dynamic instability analysis of doubly clamped cylindrical nanowires in the presence of Casimir attraction and surface effects using modified couple stress theory", *Acta Mech*., **227**(6), 1575-1591.
Sedighi, H.M., Koochi, A., Daneshmand, F. and Abadyan, M. (2015), "Non-linear dynamic instability of a double-sided nano-bridge considering centrifugal force and rarefied gas flow", *Int. J. Nonlinear Mech*., **77**, 96-106
Shen, Y.Y. and Mo, L.F. (2009), "The max–min approach to a relativistic equation", *Comput. Math. Appl*., **58**(11), 2131-2133.
Vakakis, A.F. and Rand, R.H. (2004), "Non-linear dynamics of a system of coupled oscillators with essential stifness non-linearities", *Int. J. Nonlinear Mech*., **39**, 1079 -1091.
Wu, G. (2011), "Adomian decomposition method for non-smooth initial value problems", *Math. Comput. Model*., **54**(9-10), 2104-2108.

CC