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Analytical study on non-natural vibration equations

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Abstract. In this paper, two powerful analytical methods called Variational Approach (VA) and Hamiltonian Approach (HA) are used to solve high nonlinear non-Natural vibration problems. The presented approaches are works well for the whole range of amplitude of the oscillator. The first iteration of the approaches leads us to high accurate solution. Numerical results are also presented by using Runge-Kutta's [RK] algorithm. The full comparison between the presented approaches and the numerical ones are shown in figures. The effects of important parameters on the response of nonlinear behavior of the systems are studied completely. Finally, the results show that the Variational Approach and Hamiltonian approach are strong enough to prepare easy analytical solutions.

Keywords: Hamiltonian Approach (HA); Variational Approach (VA); nonlinear vibration; analytical method

1. Introduction

Linear differential equations are very simple to solve than nonlinear differential equations. There is an explicit formula for the solutions to all linear equations but there is no general formula for solutions to all nonlinear equations. There are different types of nonlinear equations using particular methods, and arrived at different formulas for their solutions. It is possible to prove that a large class of nonlinear deferential equations actually have solutions. Nonlinear deferential equations are more difficult to solve that the linear ones. That is why one tries to find information about solutions of deferential equations without having to actually solve the equations. One of the most interesting areas in many physics and engineering problems is nonlinear vibrations. It is very important in mechanical and structural dynamics for the comprehensive understanding and accurate prediction of their motions. In the past few decades many researchers have been working on the analytical and numerical methods in nonlinear vibrations such as: Variational Iteration Method (Wazwaz 2007), Homotopy Perturbation Method (HPM) (Shou 2012) Energy Balance Method (EBM) (Ganji et al. 2011), Max-Min Method (Zeng 2009), Frequency-Amplitude Formulation (Ren and Gui 2011), Parameter Expansion Method (Kaya and Demirbağ 2013), Variational approach (He 2007), Hamiltonian approach (He 2010) other methods (Bayat et al. 2015, 2017, Pakar and Bayat 2015, Pakar et al. 2016, Ganji and Sadighi 2007, Ganji et al. 2007, Tari et al. 2007a, b, Sadighi and Ganji 2007, Jamshidi and Ganji 2007, Samaee et al. 2015, He 2002, Beléndez et al. 2010, Fu et al. 2011, Nayfeh 2973, Pirbodaghi and Hoseini 2010).

Among these methods, Variational Approach (VA) and Hamiltonian Approach (HA) is considered to solve the nonlinear vibration equations in this paper.

The paper has been collocated as follows:

First, we describe the basic concept of Variational Approach (VA) and Hamiltonian Approach (HA). Then the applications of Variational Approach (VA) and Hamiltonian Approach (HA) have been studied to demonstrate the applicability and preciseness of the method for two examples. Some comparisons between analytical and numerical solutions are presented. Eventually we show that VA and HA can converge to a precise cyclic solution for nonlinear systems.

2. Basic idea of Variational Approach (VA)

He suggested a variational approach which is different from the known variational methods in open literature (He 2007). Hereby we give a brief introduction of the method

$$\ddot{u} + f\ (u\ ) = 0 \tag{1}$$

Its variational principle can be easily established utilizing the semi-inverse method (He 2007)

$$J\left(u\right) = \int_0^{T/4}\left(-\frac{1}{2}\dot{u}^2 + F\left(u\right)\right)dt \tag{2}$$

Where Tis period of the nonlinear oscillator, $\dfrac{\partial F}{\partial u} = f$.

Assume that its solution can be expressed as

$$u\left(t\right) = A\cos\left(\omega t\right) \tag{3}$$

Where A and ω are the amplitude and frequency of the oscillator, respectively. Substituting Eq. (3) into Eq. (2)

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results in

$$J(A,\omega)=\int_0^{T/4}\left(-\frac{1}{2}A^2\omega^2\sin^2\omega t+F(A\cos\omega t)\right)dt$$
$$=\frac{1}{\omega}\int_0^{\pi/2}\left(-\frac{1}{2}A^2\omega^2\sin^2\omega t+F(A\cos\omega t)\right)dt \tag{4}$$
$$=-\frac{1}{2}A^2\omega\int_0^{\pi/2}\sin^2 t\,dt+\frac{1}{\omega}\int_0^{\pi/2}F(A\cos t)dt$$

Applying the Ritz method, we require

$$\frac{\partial J}{\partial A}=0 \tag{5}$$

$$\frac{\partial J}{\partial \omega}=0 \tag{6}$$

But with a careful inspection, for most cases we find that

$$\frac{\partial J}{\partial \omega}=-\frac{1}{2}A^2\int_0^{\pi/2}\sin^2 t\,dt-\frac{1}{\omega^2}\int_0^{\pi/2}F(A\cos t)dt<0 \tag{7}$$

Thus, we modify conditions Eqs. (5) and (6) into a simpler form

$$\frac{\partial J}{\partial \omega}=0 \tag{8}$$

From which the relationship between the amplitude and frequency of the oscillator can be obtained.

3. Basic idea of Hamiltonian Approach (HA)

Recently, He (2010) has proposed the Hamiltonian approach to overcome the shortcomings of the energy balance method. This approach is a kind of energy method with a vast application in conservative oscillatory systems. In order to clarify this approach, consider the following general oscillator

$$\ddot{u}+f(u,\dot{u},\ddot{u})=0 \tag{9}$$

With initial conditions

$$u(0)=A,\quad \dot{u}(0)=0. \tag{10}$$

Oscillatory systems contain two important physical parameters, i.e., the frequency ω and the amplitude of oscillation A. It is easy to establish a variational principle for Eq. (9), which reads

$$J(u)=\int_0^{T/4}\left\{-\frac{1}{2}\dot{u}^2+F(u)\right\}dt \tag{11}$$

Where Tis period of the nonlinear oscillator, $\frac{\partial F}{\partial u}=f$.

In the Eq. (11), $\frac{1}{2}\dot{u}^2$ is kinetic energy and $F(u)$ potential energy, so the Eq. (11) is the least Lagrangian action, from which we can immediately obtain its Hamiltonian, which reads

$$H(u)=\frac{1}{2}\dot{u}^2+F(u)=\text{constant} \tag{12}$$

From Eq. (12), we have

$$\frac{\partial H}{\partial A}=0 \tag{13}$$

Introducing a new function, $\overline{H}(u)$, defined as

$$\overline{H}(u)=\int_0^{T/4}\left\{\frac{1}{2}\dot{u}^2+F(u)\right\}dt=\frac{1}{4}TH \tag{14}$$

Eq. (14) is, then, equivalent to the following one

$$\frac{\partial}{\partial A}\left(\frac{\partial \overline{H}}{\partial T}\right)=0 \tag{15}$$

or

$$\frac{\partial}{\partial A}\left(\frac{\partial \overline{H}}{\partial(1/\omega)}\right)=0 \tag{16}$$

From Eq. (16) we can obtain approximate frequency–amplitude relationship of a nonlinear oscillator.

4. Application

In order to assess the advantages and the accuracy of the Variational approach and Hamiltonian Approach, we will consider thefollowing examples:

4.1 Example 1

These systems include shallow arches, ship roll dynamics, some electrical circuits, microperforated panel absorber and heavy symmetric gyroscope. We shall solve the following conservative Helmholtz–Duffing oscillator with VA and HA

$$\ddot{u}+u+(1-\sigma)u^2+\sigma u^3=0, \tag{17}$$

with initial conditions

$$u(0)=A,\quad \dot{u}(0)=0 \tag{18}$$

where σ is an asymmetric parameter representing the extend of asymmetry and an over dot denotes differentiation with respect to t. When σ = 1, Eq. (11) is a classical Duffing oscillator. Eq. (17) becomes a Helmholtz oscillator with a single-well potential when σ = 0.

4.1.1 Solution using VA

Variational formulation can be readily obtained Eq. (11) as follows

$$J(u)=\int_0^t\left(-\tfrac{1}{2}\dot{u}^2+\tfrac{1}{2}u^2+\tfrac{1}{3}u^3-\tfrac{1}{3}\sigma u^3+\tfrac{1}{4}\sigma u^4\right)dt \tag{19}$$

Choosing the trial function $u(t)=A\cos(\omega t)$ into Eq. (13) we obtain

$$J(A)=\int_0^{T/4}\begin{pmatrix}\frac{1}{2}\omega^2A^2\sin^2(\omega t)+\frac{1}{2}A^2\cos^2(\omega t)\\ +\frac{1}{3}A^3\cos^3(\omega t)-\frac{1}{3}\sigma A^3\cos^3(\omega t)\\ +\frac{1}{4}\sigma A^4\cos^4(\omega t)\end{pmatrix}dt \tag{20}$$

The stationary condition with respect to A leads to

$$\frac{\partial J}{\partial A}=\int_0^{T/4}\begin{pmatrix}\omega^2A\sin^2(\omega t)+A\cos^2(\omega t)\\ +A^2\cos^3(\omega t)-\sigma A^2\cos^3(\omega t)\\ +\sigma A^3\cos^4(\omega t)\end{pmatrix}dt=0 \tag{21}$$

Or

$$\frac{\partial J}{\partial A}=\int_0^{\pi/2}\begin{pmatrix}\omega^2A\sin^2 t+A\cos^2 t+A^2\cos^3 t\\ -\sigma A^2\cos^3 t+\sigma A^3\cos^4 t\end{pmatrix}dt=0 \tag{22}$$

Solving Eq. (16), according to ω, we have

$$\omega^2=\frac{\int_0^{\frac{\pi}{2}}\begin{pmatrix}A\cos^2 t+A^2\cos^3 t\\ -\sigma A^2\cos^3 t+\sigma A^3\cos^4 t\end{pmatrix}dt}{\int_0^{\frac{\pi}{2}}A\,sin^2t\,dt} \tag{23}$$

Then we have

$$\omega_{VA}=\sqrt{1+\frac{3}{4}\sigma A^2+\frac{8}{3}\frac{1}{\pi}A-\frac{8}{3}\frac{\sigma}{\pi}A} \tag{24}$$

According to $u(t)=A\cos(\omega t)$ and Eq. (18), we can obtain the following approximate solution

$$u(t)=A\cos\left(\sqrt{1+\frac{3}{4}\sigma A^2+\frac{8}{3}\frac{1}{\pi}A-\frac{8}{3}\frac{\sigma}{\pi}A}\;t\right) \tag{25}$$

4.1.2 Solution using HA

The Hamiltonian of Eq. (9) is constructed as

$$H=-\frac{1}{2}\dot{u}^2+\frac{1}{2}u^2+\frac{1}{3}u^3-\frac{1}{3}\sigma u^3+\frac{1}{4}\sigma u^4 \tag{26}$$

Integrating Eq. (12) with respect to τ from 0 to $T/4$, we have

$$\bar{H}=\int_0^{T/4}\left(-\frac{1}{2}\dot{u}^2+\frac{1}{2}u^2+\frac{1}{3}u^3-\frac{1}{3}\sigma u^3+\frac{1}{4}\sigma u^4\right)dt \tag{27}$$

Assume that the solution can be expressed as

$$u(t)=A\cos(\omega t) \tag{28}$$

Substituting Eq. (14) into Eq. (13), we obtain

$$\bar{H}=\int_0^{T/4}\begin{pmatrix}-\frac{1}{2}\omega^2A^2\sin^2(\omega t)+\frac{1}{2}A^2\cos^2(\omega t)+\frac{1}{3}A^3\cos^3(\omega t)\\ -\frac{1}{3}\sigma A^3\cos^3(\omega t)+\frac{1}{4}\sigma A^4\cos^4(\omega t)\end{pmatrix}dt \tag{29}$$

$$\begin{aligned}&=\int_0^{\pi/2}\begin{pmatrix}-\frac{1}{2}\omega^2A^2\sin^2 t+\frac{1}{2}A^2\cos^2 t+\frac{1}{3}A^3\cos^3 t\\ -\frac{1}{3}\sigma A^3\cos^3 t+\frac{1}{4}\sigma A^4\cos^4 t\end{pmatrix}dt\\ &=\frac{1}{8}A^2\frac{\pi}{\omega}+\frac{2}{9}A^3\frac{1}{\omega}-\frac{2}{9}A^3\frac{\sigma}{\omega}+\frac{3}{64}A^4\frac{\sigma\pi}{\omega}+\frac{1}{8}\omega\pi A^2\end{aligned} \tag{29}$$

Setting

$$\begin{aligned}&\frac{\partial}{\partial A}\left(\frac{\partial\bar{H}}{\partial(1/\omega)}\right)=\\ &\frac{1}{4}\pi A+\frac{2}{3}A^2-\frac{2}{3}\sigma A^2+\frac{3}{16}\sigma\pi A^3+\frac{1}{4}\omega^2\pi A=0\end{aligned} \tag{30}$$

Solving the above equation, an approximate frequency as a function of amplitude equals

$$\omega_{HA}=\sqrt{1+\frac{8}{3}\frac{A}{\pi}-\frac{8}{3}A\frac{\sigma}{\pi}+\frac{3}{4}\sigma A^2} \tag{31}$$

According to Eqs. (14) and (17), we can obtain the following approximate solution

$$u(t)=A\cos\left(\sqrt{1+\frac{8}{3}\frac{A}{\pi}-\frac{8}{3}A\frac{\sigma}{\pi}+\frac{3}{4}\sigma A^2}\;t\right) \tag{32}$$

4.2 Example 2

The motion of a particle on a rotating parabola .The governing equation of motion and initial conditions can be expressed as (Nayfeh 1973)

$$(1+4q^2u^2)\ddot{u}+4q^2u\dot{u}^2+\Delta u=0\quad u(0)=A,\quad \dot{u}(0)=0 \tag{33}$$

Where $q>0$ and $\Delta>0$ are known positive constants.

4.2.1 Solution using VA

Its variational formulation can be readily obtained Eq. (19) as follows

$$J(u)=\int_0^t\left(\frac{1}{2}\dot{u}^2+2q^2u^2\dot{u}^2+\frac{1}{2}\Delta u^2\right)dt \tag{34}$$

Choosing the trial function $u(t)=A\cos(\omega t)$ into Eq. (20) we obtain

$$J(A)=\int_0^{T/4}\begin{pmatrix}\frac{1}{2}A^2\omega^2sin^2(\omega t)+\frac{1}{2}\Delta A^2cos^2(\omega t)\\ +2q^2A^4\omega^2sin^2(\omega t)cos^2(\omega t)\end{pmatrix}dt \tag{35}$$

The stationary condition with respect to A leads to

$$\frac{\partial J}{\partial A}=\int_0^{T/4}\begin{pmatrix}A\,\omega^2sin^2(\omega t)+\Delta A\,cos^2(\omega t)\\ +8qA^3\omega^2sin^2(\omega t)cos^2(\omega t)\end{pmatrix}dt=0 \tag{36}$$

Or

$$\frac{\partial J}{\partial A}=\int_0^{\pi/2}\begin{pmatrix}A\,\omega^2sin^2 t+\Delta A\,cos^2 t\\ +8q\,A^3\omega^2sin^2 t\,cos^2 t\end{pmatrix}dt=0 \tag{37}$$

Solving Eq. (23), according to ω, we have

$$\omega^2 = \frac{\int_0^{\frac{\pi}{2}} \left(\Delta A \cos^2 t\right) dt}{\int_0^{\frac{\pi}{2}} \left(A \sin^2 t + 8q A^3 \sin^2 t \cos^2 t\right) dt} \tag{38}$$

Then we have

$$\omega_{VA} = \sqrt{\frac{\Delta}{1+2A^2q^2}} \tag{39}$$

According to Eqs. (3) and (25), we can obtain the following approximate solution

$$u(t) = A \cos\left(\sqrt{\frac{\Delta}{1+2A^2q^2}}\, t\right) \tag{40}$$

The exact period is

$$\omega_{Exact} = 2\pi \Big/ 4A \int_0^{\pi/2} \frac{\sqrt{1+4q^2A^2\cos^2 t}\ \sin t}{\sqrt{\Delta A^2 \sin^2 t}}\, dt \tag{41}$$

4.2.2 Solution using HA

The Hamiltonian of Eq. (9) is constructed as:

$$H = -\tfrac{1}{2}\dot{u}^2 - 2q^2u^2\dot{u}^2 + \tfrac{1}{2}\Delta u^2 \tag{42}$$

Integrating Eq. (12) with respect to τ from 0 to $T/4$, we have

$$\bar{H} = \int_0^{T/4} \left(-\tfrac{1}{2}\dot{u}^2 - 2q^2u^2\dot{u}^2 + \tfrac{1}{2}\Delta u^2\right) dt \tag{43}$$

Assume that the solution can be expressed as

$$u(t) = A \cos(\omega t) \tag{44}$$

Substituting Eq. (14) into Eq. (13), we obtain

$$\begin{aligned}\bar{H} &= \int_0^{T/4} \begin{pmatrix} -\tfrac{1}{2}\omega^2A^2\sin^2(\omega t) + \tfrac{1}{2}\Delta A^2\cos^2(\omega t) \\ -2\omega^2q^2A^4\sin^2(\omega t)\cos^2(\omega t) \end{pmatrix} dt \\ &= \int_0^{\pi/2} \begin{pmatrix} -\tfrac{1}{2}\omega^2A^2\sin^2 t + \tfrac{1}{2}\Delta A^2\cos^2 t \\ -2\omega^2q^2A^4\sin^2 t\,\cos^2 t \end{pmatrix} dt \\ &= -\tfrac{1}{8}A^2\omega\pi - \tfrac{1}{8}A^4\omega q^2\pi + \tfrac{1}{8}A^2\frac{\Delta\pi}{\omega} \end{aligned} \tag{45}$$

Setting

$$\begin{aligned}&\frac{\partial}{\partial A}\left(\frac{\partial \bar{H}}{\partial(1/\omega)}\right) = \\ &-\tfrac{1}{4}A\,\pi\,\omega^2 - \tfrac{1}{2}A^3\,\pi\,\omega^2 q^2 + \tfrac{1}{4}A\,\pi\,\Delta = 0\end{aligned} \tag{46}$$

Solving the above equation, an approximate frequency as a function of amplitude equals

$$\omega_{HA} = \frac{\sqrt{\Delta}}{\sqrt{1+2A^2q^2}}, \tag{47}$$

According to Eqs. (14) and (17), we can obtain the following approximate solution

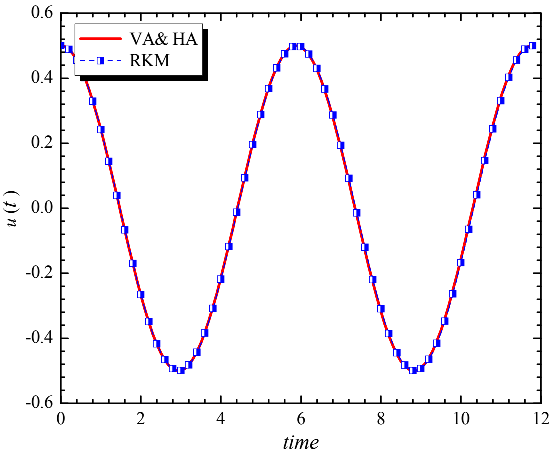
$$u(t) = A \cos\left(\frac{\sqrt{\Delta}}{\sqrt{1+2A^2q^2}}\, t\right) \tag{48}$$

5. Results and discussions

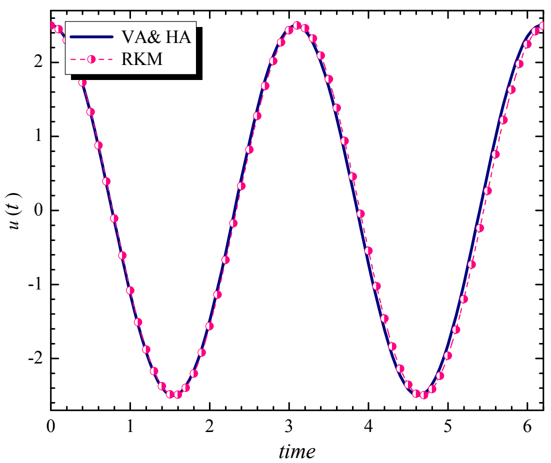
To illustrate and verify the accuracy of this new approximate analytical approaches, some comparison of the time history oscillatory displacement responses with the numerical solutions are presented in Figs. 1-3 for examples 1, and Table 1, and Figs. 4 to 6 for examples 2 and Table 2.

Tables 1 and 2 are the compared solutions of analytical methods and numerical ones. The maximum error is less than 2.1%.

In example 1, Figs. 1(a) and (b) represent the high accuracy of the variational approach and Hamitonian

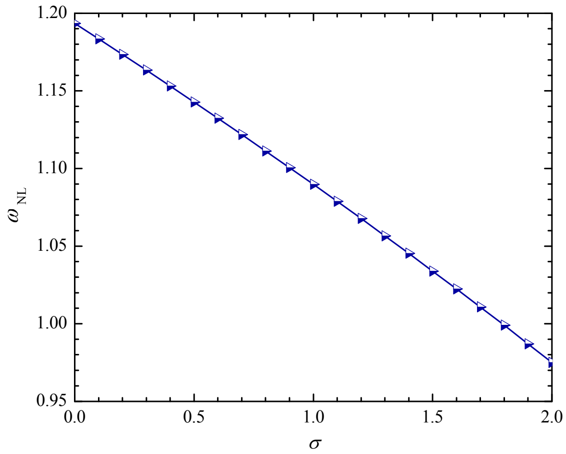


(a) $A = 0.5$, $\sigma = 1.2$

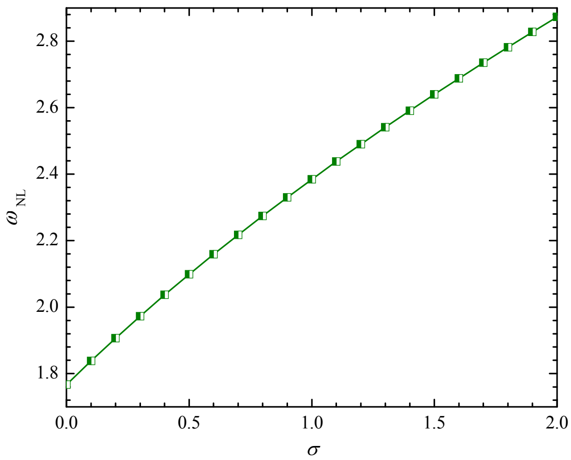


(b) $A = 2.5$, $\sigma = 0.4$

Fig. 1 (Ex1) Comparison of analytical solutions of $u(t)$ based on time with the RKM solution



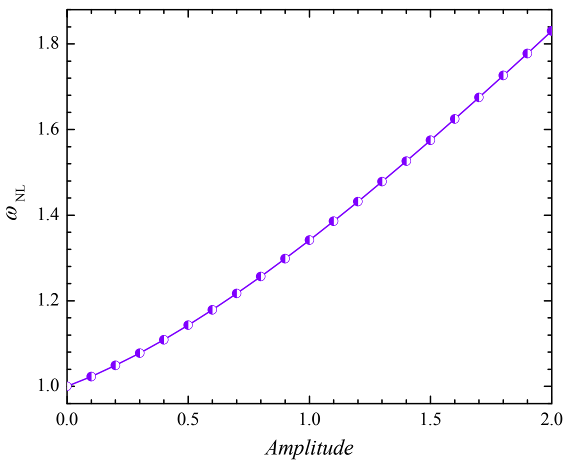
(a)



(b)

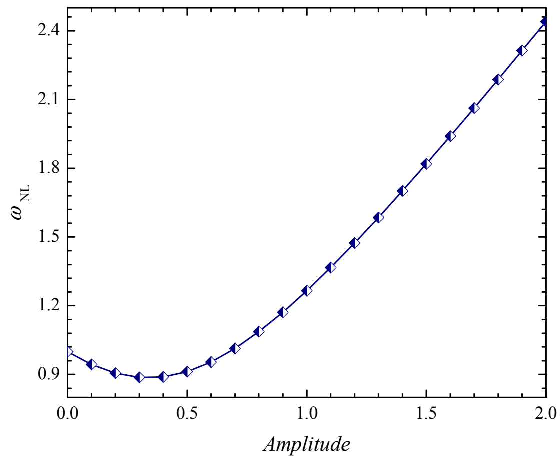
Fig. 2 (Ex1) Effect of asymmetric parameter (σ) on nonlinear frequency

approach with the Runge Kutta's algorithm. Figs. 2 and 3 show the effect of parameters of (σ) and (A) to the nonlinear frequency of the system.



(a)

Fig. 3 (Ex1) Effect of amplitude(A) on nonlinear frequency



(b)

Fig. 3 Continued

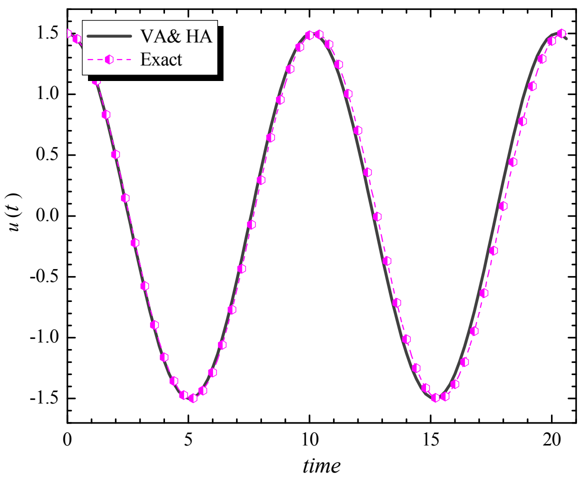
Table 1 Comparison of nonlinear frequency of approximate solution (VA) and (HA) with numerical solution (RKM) corresponding to various parameters of system (example 1)

A	σ	VA&HA	RKM	Error %
0.5	1.2	1.0678	1.0653	0.2339
1	0.8	1.3303	1.3254	0.3714
1.5	1	1.6394	1.6281	0.6838
2	1.5	2.1567	2.1344	1.0329
2.5	0.4	2.0367	2.0185	0.8930
3	0.8	2.6286	2.5919	1.3961
3.5	0.9	3.0929	3.0320	1.9681
4	0.5	2.9492	2.9023	1.5899
4.5	0.6	3.4118	3.3382	2.1566
5	0.2	2.8540	2.8116	1.4849

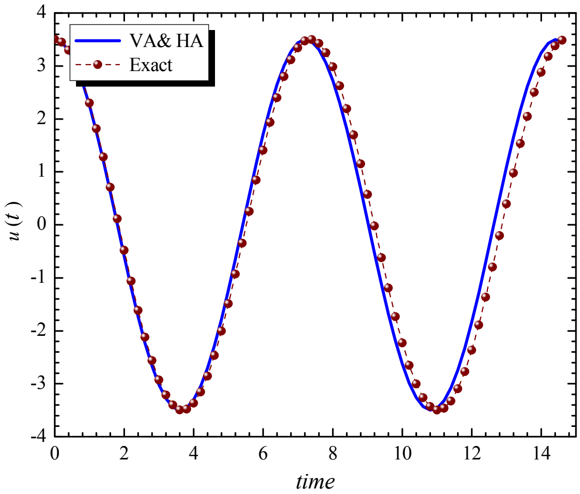
In example 2, Table 2 gives the comparison of the obtained results with the exact solution for different values of A, q, Δ and the maximum relative error is less than 1.8%. Figs. 4(a) and (b) represent comparison of the analytical solution of $u(t)$ based on time with the numerical solution. Figs. 5 and 6 show the effects of Δ and q and amplitude of

Table 2 Comparison of the approximate and exact frequencies corresponding to various parameters in Eq. (25) (Example 2)

A	q	Δ	$\omega_{VA\&HA}$	ω_{Exact}	Error%
0.5	1	0.5	0.5774	0.5721	0.9134
1	0.5	1	0.8165	0.8039	1.5685
1.5	0.8	1.5	0.6218	0.6139	1.2812
2	0.7	0.5	0.3188	0.3163	0.7912
2.5	0.5	2	0.6963	0.6872	1.3211
3	1	3	0.3974	0.3950	0.6014
3.5	0.2	1.5	0.8704	0.8544	1.8675
4	0.4	3	0.7001	0.6928	1.0636



(a)



(b)

Fig. 4 Comparison of analytical solution of time history the exact solution for: (a) A = 1.5, q = 1.5, Δ = 0.8; (b) A = 3.5, q = 0.2, Δ = 1.5

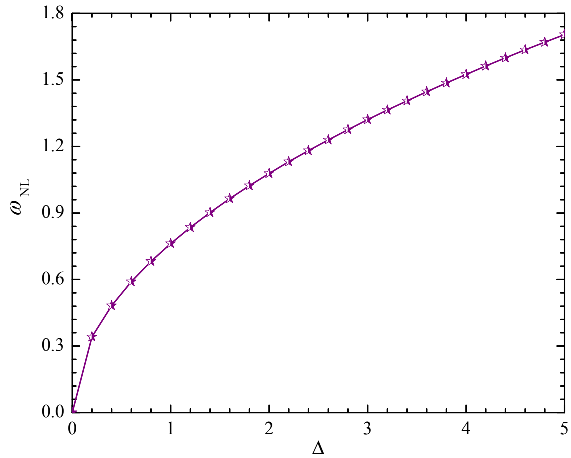


Fig. 5 (Ex 2) Effect of Δ and q on nonlinear frequency

the oscillation. The comparison of analytical solutions based on time with the exact solution shows an excellent agreement of the proposed methods.

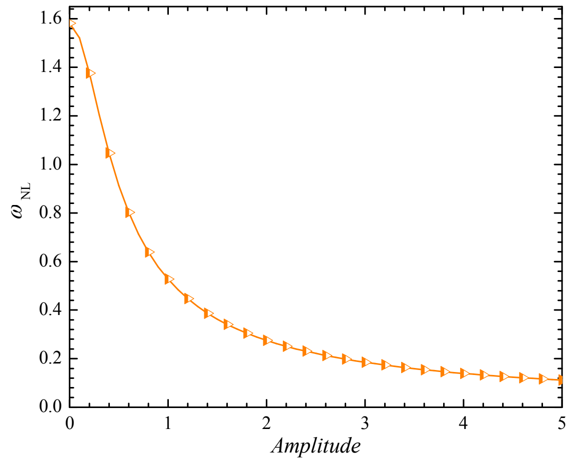


Fig. 6 (Ex 2) Effect of amplitude (A) on nonlinear frequency

It is evident that VA and HA show high accuracy with the numerical solution and is quickly convergent and valid for a wide range of vibration amplitudes and initial conditions. The accuracy of the results shows that the VA and HA could be potentiality used for the analysis of strongly nonlinear oscillation problems.

6. Conclusions

In this study, new powerful approaches have been introduced and applied to high nonlinear vibration equations. Two strong examples have been studied and the effects of important parameters on the nonlinear frequency of the systems are figured. The accuracy of the proposed approaches are demonstrated by comparing with numerical solutions. It has been proven that the variational approach and Hamiltonian approach (HA) are very efficient, comfortable and sufficiently exact in engineering problems. The proposed approaches can be simply extended to any nonlinear equation for the analysis of nonlinear systems. The obtained results from the approximate analytical solutions are in excellent agreement with the corresponding exact solutions.

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Appendix A
Basic idea of Runge-Kutta's method (RK)

For the numerical approach to verify the analytic solution, the fourth RK (Runge-Kutta) method has been used. This iterative algorithm is written in the form of the following formulae for the second-order differential equation

$$\begin{aligned} \dot{u}_{i+1} &= \dot{u}_i + \frac{\Delta t}{6}\left(h_1 + 2h_2 + 2h_3 + h_4\right) \\ u_{i+1} &= u_i + \Delta t\left(\dot{u}_i + \frac{\Delta t}{6}\left(h_1 + h_2 + h_3\right)\right) \end{aligned} \tag{A1}$$

Where, Δt is the increment of the time and h_1, h_2, h_3 and h_4 are determined from the following formulae

$$\begin{aligned} h_1 &= f\left(\dot{u}, u_i, \dot{u}_i\right), \\ h_2 &= f\left(t_i + \frac{\Delta t}{2},\ u_i + \frac{\Delta t}{2}\dot{u}_i,\ \dot{u}_i + \frac{\Delta t}{2}h_1\right), \\ h_3 &= f\left(t_i + \frac{\Delta t}{2},\ u_i + \frac{\Delta t}{2}\dot{u}_i,\ \frac{1}{4}\Delta t^2 h_1,\ \dot{u}_i + \frac{\Delta t}{2}h_2\right), \\ h_4 &= f\left(t_i + \Delta t,\ u_i + \Delta t\,\dot{u}_i,\ \frac{1}{2}\Delta t^2 h_2,\ \dot{u}_i + \Delta t\, h_3\right). \end{aligned} \tag{A2}$$

The numerical solution starts from the boundary at the initial time, where the first value of the displacement function and its first-order derivative are determined from initial condition. Then, with a small time increment Δt, the displacement function and its first-order derivative at the new position can be obtained using Eq. (2). This process continues to the end of the time limit.