Steel and Composite Structures, *Vol. 24, No. 2 (2017) 151-176*
DOI: https://doi.org/10.12989/scs.2017.24.2.151

General equations for free vibrations of thick doubly curved sandwich panels with compressible and incompressible core using higher order shear deformation theory

M. Nasihatgozar [1a], S.M.R. Khalili [*2,3] and K. Malekzadeh Fard [4b]

[1] *Department of Mechanical and Aerospace Engineering, Science and Research Branch, Islamic Azad University, Tehran, Iran*
[2] *Faculty of Mechanical Engineering, K.N. Toosi University of Technology, Tehran, Iran*
[3] *Department of Applied Mechanics, Indian Institute of Technolog Delhi, New Delhi, 110016, India*
[4] *Department of Structural Analysis and Simulation, Space Research Institute, Tehran, Iran*

(*Received October 25, 2016, Revised February 28, 2017, Accepted March 19, 2017*)

Abstract. This paper deals with general equations of motion for free vibration analysis response of thick three-layer doubly curved sandwich panels (DCSP) under simply supported boundary conditions (BCs) using higher order shear deformation theory. In this model, the face sheets are orthotropic laminated composite that follow the first order shear deformation theory (FSDT) based on Rissners-Mindlin (RM) kinematics field. The core is made of orthotropic material and its in-plane transverse displacements are modeled using the third order of the Taylor's series extension. It provides the potentiality for considering both compressible and incompressible cores. To find these equations and boundary conditions, Hamilton's principle is used. Also, the effect of trapezoidal shape factor for cross-section of curved panel element $(1 \pm z/R)$ is considered. The natural frequency parameters of DCSP are obtained using Galerkin Method. Convergence studies are performed with the appropriate formulas in general form for three-layer sandwich plate, cylindrical and spherical shells (both deep and shallow). The influences of core stiffness, ratio of core to face sheets thickness and radii of curvatures are investigated. Finally, for the first time, an optimum range for the core to face sheet stiffness ratio by considering the existence of in-plane stress which significantly affects the natural frequencies of DCSP are presented.

Keywords: sandwich panel; natural frequency; doubly curved; compressible core; incompressible core

1. Introduction

Lightweight and stiff, sandwich panels are a vital element of many modern aircraft interior designs. Reinforcing and edge finishing of such panels can be costly and time consuming, but it is essential. Thin and thick panels and shells have been studied during the past decades for different: (a) geometric configurations such as flat, single curved (cylindrical, conical, etc), doubly curved (spherical, etc.) (Reddy 2003, Qatu 2004, Qatu and Asadi 2012); (b) materials such as conventional and modern composites (Vinson and Sierakowski 2006); (c) loading conditions (statically, dynamically and thermally) by various theories and models (Amabili 2008, Leissa and Qatu 2011) indicating extensive applications of this structural element in a wide variety of engineering fields. These structures are subjected to vibrations in different loading conditions and consequently susceptible to lose their strength and safety. The main purpose of this paper is therefore to analyze the free vibration of doubly curved sandwich panels (DCSP) for their optimum design. Such an optimization requires accurate models and theories in such a way that all of the governing conditions on DCSP such as continuity conditions of displacements at the face sheets - core interfaces and boundary conditions are satisfied. Selecting a suitable model is a fundamental step in DCSP analysis and extremely depends on mechanical (face-to-core stiffness ratio) and geometrical parameters. A typical DCSP consists of two thin high-density face sheets that are very stiff with a high strength and usually are made of metallic or laminated composite material. A considerable amount of discussions about the main aspects considered in the design, analysis and construction of sandwich structures are existed in the literature (Noor *et al*. 1996).

To select a suitable model for analysis of sandwich shells, two main parameters of the complex mechanical behavior of sandwich shells as well as the presence of $\left(1 \pm \frac{z}{R}\right)$ term in the basic equations should be considered. $\left(1 \pm \frac{z}{R}\right)$ term appears in both the strain displacement and the stress resultant equations, since the curvature of each parallel surface through the thickness of the shell is different. The analysis by Bhimaraddi (1984) accounted for the $\left(1 \pm \frac{z}{R}\right)$ terms in the stress-resultant, but truncated the terms beyond the order of $\left(\frac{h^3}{R^3}\right)$. Later, Chang (1992) and Leissa and Chang (1996) considered this term, but truncated it using a geometric series expansion and neglected the

*Corresponding author, Professor,
E-mail: smrkhalili2017@gmail.com
[a] Ph.D. Student
[b] Professor

http://www.techno-press.org/?journal=scs&subpage=6
ISSN: 1229-9367 (Print), 1598-6233 (Online)

terms beyond the order of $\left(\frac{h}{R}\right)$. They showed that by considering the $\left(1 \pm \frac{z}{R}\right)$ terms with only FSDT gives more accurate results than higher-order theories in which the term is neglected as reported in Refs (Librescu *et al*. 1989). Liew and Lim (1996) developed a zigzag deformation higher-order theory for vibration of isotropic thick doubly curved shallow shells. Taking into account the $\left(1 \pm \frac{z}{R}\right)$ terms and neglecting the terms beyond the order of $\left(\frac{h}{R}\right)$, they accounted for the cubic distribution of transverse shear strains through the shell thickness in contrast with existing parabolic shear distribution theories (PSDTs). Khalili *et al*. (2012) used the term in free vibration analysis of homogenous isotropic circular cylindrical shells based on a new 3D refined higher order theory. Qatu (2004) incorporated the $\left(1 \pm \frac{z}{R}\right)$ terms in the framework of FSDT for the free vibration analysis of laminated deep thick shells. He did not truncate the series expansion of the $\left(1 \pm \frac{z}{R}\right)$ terms (in the denominator of the stress-resultant integrands) and calculated the integrals of the stress-resultant accurately by exact integration through the thickness of the shell. He showed that the accurate stress-resultants are needed for laminated composite deep thick shells, especially if the shell is not spherical. He concluded that using the plate approximation equations for stiffness parameters of isotropic thick shells leads to an error of 2%. Nevertheless, to the best of the authors' knowledge, this term has not yet been used for three-layer thick laminated DCSP. The current article therefore employs this term to increase the accuracy of the analysis.

In general, there are three main approaches to analyze sandwich structures: (a) 3D elasticity approaches in which the equations of motion expressed without considering any assumption for the displacement field are solved and the stresses, strains and displacement components are obtained. Therefore, this theory is the most exact theory for analyzing mechanical behavior of constructions. (b) Equivalent single layer (ESL) theories in which all the unknown displacement field functions do not depend on the considered layer. It means that all layers have the same degrees of freedom (DOF); and (c) Layer wise (LW) theories in which the unknown displacement field functions depend on the considered layer.

There are few exact 3D elasticity solutions for static and dynamic analysis of the composite sandwich plates (Pagano 1970, Kardomateas 2005). The 3D elasticity approaches are perfect, but 2D models are preferred in sandwich structures because of their required computational efforts. ESL models are Classical Laminated Plate Theory (CLPT), FSDT and HSDT. The classical laminated plate theory (CLPT) based on Love–Kirchhoff yields sufficiently accurate results when: (1) length to thickness ratio is large; (2) the material anisotropy is not severe; (3) the dynamic excitation are within the low- frequency range (Toorani and Lakis 2000). FSDT based on Rissner–Mindlin (RM) kinematics field does not satisfy the transverse shear stresses boundary conditions on the top and the bottom surfaces of the shells or plates (Librescu and Khdeir 1989, Thai *et al*. 2012, Valizadeh *et al*. 2013, Kapoor and Kapania 2012). For this reason, in application of such theories based on these kinematic relations, shear correction factors for equilibrium considerations are needed (Reissner and Wan 1982). Hence, some researchers (Frostig *et al*. 2004, Jedari Salami 2016, Kant and Swaminathan 2001, Wu *et al*. 2008) applied third model of ESL, i.e., HSDT to avoid using shear correction factors. ESL models also can predict global behavior of thin and thick laminates, but they are not able to distinguish some of dynamic and static behaviors such as local modes of buckling (wrinkling), high mode of vibration and local bending. So, these models cannot account for the discontinuities in the displacement field and transverse strains at the interfaces between the layers with different stiffness properties. LW theories improved ESL disadvantages and were used in many research works (Hause and Librescu 2006, Ferreira 2005), but the main problem in using LW theories is that the amount of unknown quantities increases by increasing layer number and so finding an analytical solution for them becomes impossible. In this case, it seems that using theories such as Frostig theory (Frostig 1992) which divides the whole structure into three layers and has constant unknown quantities is helpful. Fares and Youssif (2001) studied a refined ESL model of doubly curved shells using an extension of Reissner's mixed variation of formula based on Maupertuis' principle. In their study, the stresses were continuous through the shell thickness and were consistent with the surface conditions and none of shear correction factors were used. Singh used Rayleigh–Ritz method to obtain the natural frequencies of doubly curved open deep sandwich shells with ESL model in which the displacement fields are defined by Bezier surface patches (Singh 1999). The dynamic analysis of anisotropic and multi-layered shells and panels with different curvatures by using HSDT in which the displacement field having a fixed nine degrees of freedom was investigated by Viola *et al*. (2013). The equations have been solved numerically using the Generalized Differential Quadrature (GDQ) technique. Free and forced vibrations of cross-ply laminated composite arches under various boundary conditions were investigated by Khedeir and Reddy (Khdeir and Reddy 1997). Their formulation included ESL third, second, first and classical theories. Hohe *et al*. (2006) investigated the dynamic buckling and the post buckling analysis of the flat and curved sandwich panels with transversely compressible core in which the standard Kirchhoff-Love hypothesis for the face sheets and a first/second order power series expansion for the core were used. They neglected the transverse shear strains of the core layer. Biglari and Jafari (2010) presented a complex three-layer theory for the free vibration and bending analysis of doubly-curved sandwich structures with flexible core. In their model, Donell's shallow shell theory was used for the face sheets. Malekzadeh Fard *et al*. (2014) studied bending analysis of doubly curved sandwich panels subjected to multiple loading conditions using improved high order sandwich panel theory and second computational Frostig's model (2004). In their formulation, the in-plane hoop stresses of the core and the trapezoidal shape factor $\left(1 + \frac{z}{R}\right)$ were considered.

To the best of the authors' knowledge, no research work on free vibrations of thick doubly curved sandwich panels and shells with compressible/incompressible core using higher order shear deformation theory is reported adequately in the existed literature. The current research work presents the free vibration of simply-supported three-layer thick doubly-curved orthotropic sandwich panel using a new type of high-order sandwich panel theory. In this model, face sheets are orthotropic laminated composite and the FSDT is applied to them. Additionally all stress components, except normal stress for face sheets are considered. The core is made of compressible and incompressible orthotropic material and a third order pattern for both the in-plane and the vertical displacement was used. Also, all six stress components of the core were considered. Different radii of curvatures for the face sheets and the core (R_α, R_β) were taken into account using the terms $\left(1+\frac{z}{R_\alpha}\right)$ and $\left(1+\frac{z}{R_\beta}\right)$ due to their effects on accuracy of stress resultants. These coefficients have significant role in free vibration analysis of thick DCSP. In order to validate the present model and formulations, the obtained numerical results of the analysis are compared with those available in the literature. Also, parametric study including the effect of radius of curvature, core to face sheet thickness ratio and flexibility of the core are carried out.

2. Analytical model for thick DCSP

2.1 Structural model

A three-layer DCSP is considered as shown in Fig. 1. The DCSP is composed of two orthotropic laminated composite face sheets separated by an orthotropic thick compressible or incompressible core. The global coordinate system (α, β, z) is orthogonal curvilinear shown in Fig. 1. The origin of the coordinate system (α, β, z) is located on one corner of the mid plane of the sandwich panel. The α and β curves are lines of curvature on the sandwich panel mid surface, $z = 0$. The z-axis is a straight line normal to shell mid surface. The thickness of the top face, core and bottom face layers are t^t, t^c, t^b respectively and H is the total thickness of DCSP and R^i_α, R^i_β $(i = t, c, b)$ denote the radii of curvature to mid surface of the top, core and bottom layers in the α and β directions. $R^i_{\alpha\beta}$ $(i = t, c, b)$ is the radii of twist of the surface. When the direction of α and β coordinate axes coincide with principle directions, then R^i_α, R^i_β are called as the radii of principle curvature and $R^i_{\alpha\beta}$ is infinity. The DCSP may be circular cylindrical shell with $R^i_\alpha = R^i$ and $R^i_\beta = R^i_{\alpha\beta} = \infty$ or $\frac{R^i_\alpha}{R^i_\beta} = \frac{R^i_\alpha}{R^i_{\alpha\beta}} = 0$, a spherical panel with $R^i_\alpha = R^i_\beta = R^i$ or $\frac{R^i_\alpha}{R^i_\beta} = 1$. Noted that the curvature effect of layers is considered in this paper.

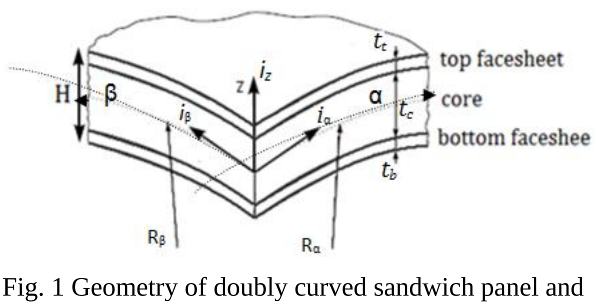

Fig. 1 Geometry of doubly curved sandwich panel and curvilinear coordinate

2.2. Basic assumptions

(a) As the face sheets and the core deflections are small and the strains are infinitesimal, they are assumed to be linearly elastic.
(b) The face sheets are made of orthotropic laminated composite and the core is made of incompressible material such as metallic honeycomb or balsa wood and a compressible core such as foam.
(c) The interfaces between the layers and the face-core interfaces are perfectly bonded, so there is no delamination or interlayer slip between the layers.
(d) Face sheets are sufficiently thin (compared to the core) to be treated as thin plate or shells and follow the FSDT assumption.
(e) The face sheets and the core are of constant thicknesses and uniform throughout the entire DCSP.

Considering a differential element of DCSP (see Fig. 1), the square of linear element "dS" between the points (α, β, z) and $(\alpha, \beta, z + dz)$ is given by

$$(dS)^2 = A_1^2(d\alpha)^2 + A_2^2(d\beta)^2 + A_3^2(dz)^2 \tag{1}$$

where A_1, A_2 and A_3 are referred to geometrical scale factor quantities

$$A_1 = A\left(1+\frac{z}{R_\alpha}\right),\quad A_2 = B\left(1+\frac{z}{R_\beta}\right),\quad A_3 = 1 \tag{2}$$

In Eq. (2), A and B are Lame' parameters. An infinitesimal rectangular area of the surface at $+z$ is given by

$$dA_z = A_1 A_2 d\alpha d\beta \tag{3}$$

The volume of an infinitesimal element at $+z$ is given by

$$dV = A_1 A_2 d\alpha d\beta dz \tag{4}$$

It is to be noted that $A = B = 1$, when the shell curvature is constant for example cylindrical, spherical and hyperbolic paraboloid.

2.3 Definition of the 3D displacement field in the face sheets and the core

Displacement field for an arbitrary point within top and bottom face sheets based on Mindlin–Reissner shell theory can be written as (Reddy 2003)

$$\begin{aligned} u^i(\alpha,\beta,z,t) &= u^i_0(\alpha,\beta,t) + z^i\theta^i_\alpha(\alpha,\beta,t) \\ v^i(\alpha,\beta,z,t) &= v^i_0(\alpha,\beta,t) + z^i\theta^i_\beta(\alpha,\beta,t) \\ w^i(\alpha,\beta,z,t) &= w^i(\alpha,\beta,t) \end{aligned} \tag{5}$$

Where u^i, v^i and w^i $(i = t, b)$ denote the displacement components of the face sheets of DCSP. In Eq. (5), u_0^i, v_0^i and w^i are the displacements at the mid surface in the α, β and z directions. θ_α^i and θ_β^i are rotations of a transverse normal around α and β curvilinear coordinates, respectively. The displacement components u^c, v^c and w^c of a generic point in the core are related to midsurface displacement (u_0^c, v_0^c, w_0^c) by

$$\begin{aligned} u^c(\alpha,\beta,z,t) &= u_0^c(\alpha,\beta,t) + z u_1^c(\alpha,\beta,t) \\ &\quad + z^2 u_2^c(\alpha,\beta,t) + z^3 u_3^c(\alpha,\beta,t) \\ v^c(\alpha,\beta,z,t) &= v_0^c(\alpha,\beta,t) + z v_1^c(\alpha,\beta,t) \\ &\quad + z^2 v_2^c(\alpha,\beta,t) + z^3 v_3^c(\alpha,\beta,t) \\ w^c(\alpha,\beta,z,t) &= w_0^c(\alpha,\beta,t) + z w_1^c(\alpha,\beta,t) \\ &\quad + z^2 w_2^c(\alpha,\beta,t) + z^3 w_3^c(\alpha,\beta,t) \end{aligned} \tag{6}$$

In Eq. (6), u_1^c, v_1^c and w_1^c functions are rotational, the parameters $u_2^c, u_3^c, v_2^c, v_3^c, w_2^c$ and w_3^c are the higher-order terms in the Taylor's series expansion.

2.4 Strain-displacement equations

Considering DCSP (see Fig. 1) as an element, the mid surface vector $\vec{U}$ at any point within the DCSP is introduced by the following relation

$$\vec{U} = u\vec{\iota}_\alpha + v\vec{\iota}_\beta + w\vec{\iota}_{n_z} \tag{7}$$

Where $\vec{\iota}_\alpha$ and $\vec{\iota}_\beta$ are the tangent unit vectors and $\vec{\iota}_{n_z}$ is the normal unit vector to mid surface as shown in Fig. 1. The strain-displacement equations of a 3D DCSP in curvilinear coordinate with small displacements assumption and using Mainardi-Codazzi equations are (Qatu 2004)

$$\begin{aligned} \frac{\partial}{\partial\beta}\left(\frac{A}{R_\alpha}\right) &= \frac{1}{R_\beta}\frac{\partial A}{\partial\beta} + \frac{1}{B}\frac{\partial}{\partial\alpha}\left(\frac{B^2}{R_{\alpha\beta}}\right), \\ \frac{\partial}{\partial\alpha}\left(\frac{B}{R_\beta}\right) &= \frac{1}{R_\alpha}\frac{\partial B}{\partial\alpha} + \frac{1}{A}\frac{\partial}{\partial\beta}\left(\frac{A^2}{R_{\alpha\beta}}\right) \end{aligned} \tag{8}$$

Gauss characteristic equation is

$$\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial B}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial A}{\partial\beta}\right) = -\frac{AB}{R_\alpha R_\beta} + \frac{AB}{{R_{\alpha\beta}}^2} \tag{9}$$

The strains are found to be (Qatu 2004)

$$\begin{aligned} \varepsilon_\alpha &= \frac{1}{\left(1 + C_0\frac{z}{R_\alpha}\right)}\left(\frac{1}{A}\frac{\partial u}{\partial\alpha} + \frac{v}{AB}\frac{\partial A}{\partial\beta} + \frac{w}{R_\alpha}\right) \\ \varepsilon_\beta &= \frac{1}{\left(1 + C_1\frac{z}{R_\beta}\right)}\left(\frac{1}{B}\frac{\partial v}{\partial\beta} + \frac{u}{AB}\frac{\partial B}{\partial\alpha} + \frac{w}{R_\beta}\right) \\ \varepsilon_z &= \frac{\partial w}{\partial z} \\ \gamma_{\alpha\beta} &= \frac{1}{\left(1 + C_0\frac{z}{R_\alpha}\right)}\left(\frac{1}{A}\frac{\partial v}{\partial\alpha} - \frac{u}{AB}\frac{\partial A}{\partial\beta} + \frac{w}{R_{\alpha\beta}}\right) \\ &\quad + \frac{1}{\left(1 + C_1\frac{z}{R_\beta}\right)}\left(\frac{1}{B}\frac{\partial u}{\partial\beta} - \frac{v}{AB}\frac{\partial B}{\partial\alpha} + \frac{w}{R_{\alpha\beta}}\right) \\ \gamma_{\alpha z} &= \frac{1}{A\left(1 + C_0\frac{z}{R_\alpha}\right)}\frac{\partial w}{\partial\alpha} \\ &\quad + A\left(1 + C_0\frac{z}{R_\alpha}\right)\frac{\partial}{\partial z}\left(\frac{u}{A\left(1 + C_0\frac{z}{R_\alpha}\right)}\right) \\ &\quad - \frac{v}{R_{\alpha\beta}\left(1 + C_0\frac{z}{R_\alpha}\right)} \\ \gamma_{\beta z} &= \frac{1}{B\left(1 + C_1\frac{z}{R_\beta}\right)}\frac{\partial w}{\partial\beta} \\ &\quad + B\left(1 + C_1\frac{z}{R_\beta}\right)\frac{\partial}{\partial z}\left(\frac{v}{B\left(1 + C_1\frac{z}{R_\beta}\right)}\right) \\ &\quad - \frac{u}{R_{\alpha\beta}\left(1 + C_1\frac{z}{R_\beta}\right)} \end{aligned} \tag{10}$$

The above equations can be easily applied for flat plate, cylindrical, spherical shells, etc. The kinematic relations for the top and bottom face sheets and the core in terms of mid surface displacement are obtained by substituting displacement field from Eqs. (5) and (6) into Eq. (10) yields the Eqs. (11a) and (11b) and Eqs. (12a) and (12b) for the face sheets and the core, respectively, as follows

$$\begin{aligned} \varepsilon_\alpha^i &= \frac{1}{\left(1 + C_0\frac{z^i}{R_\alpha}\right)}\left(\varepsilon_{0\alpha}^i + z^i\kappa_\alpha^i\right) \\ \varepsilon_{\alpha\beta}^i &= \frac{1}{\left(1 + C_0\frac{z^i}{R_\alpha}\right)}\left(\varepsilon_{0\alpha\beta}^i + z^i\chi_{\alpha\beta}^i\right) \\ \gamma_{\alpha\beta}^i &= \varepsilon_{\alpha\beta}^i + \varepsilon_{\beta\alpha}^i \\ \gamma_{\alpha z}^i &= \frac{\gamma_{o\alpha z}^i}{\left(1 + C_0\frac{z^i}{R_\alpha}\right)} \\ \varepsilon_{\beta\alpha}^i &= \frac{1}{\left(1 + C_1\frac{z^i}{R_\beta}\right)}\left(\varepsilon_{0\beta\alpha}^i + z^i\chi_{\beta\alpha}^i\right) \\ \gamma_{\beta z}^i &= \frac{\gamma_{o\beta z}^i}{\left(1 + C_1\frac{z^i}{R_\beta}\right)} \end{aligned} \tag{11a}$$

In the above equations i stands for face sheets, $i = t$ means the top face sheets and $i = b$ means the bottom face sheet where

$$\begin{aligned} \varepsilon_{0\alpha}^i &= \frac{1}{A}\frac{\partial u_0^i}{\partial\alpha} + \frac{v_0^i}{AB}\frac{\partial A}{\partial\beta} + \frac{w_0^i}{R_\alpha} \\ \varepsilon_{0\beta}^i &= \frac{1}{B}\frac{\partial v_0^i}{\partial\beta} + \frac{u_0^i}{AB}\frac{\partial B}{\partial\alpha} + \frac{w_0^i}{R_\beta} \end{aligned} \tag{11b}$$

$$
\begin{aligned}
\varepsilon_{0\alpha\beta}^{i} &= \frac{1}{A}\frac{\partial v_0^i}{\partial \alpha} - \frac{u_0^i}{AB}\frac{\partial A}{\partial \beta} + \frac{w_0^i}{R_{\alpha\beta}} \\
\varepsilon_{0\beta\alpha}^{i} &= \frac{1}{B}\frac{\partial u_0^i}{\partial \beta} - \frac{v_0^i}{AB}\frac{\partial B}{\partial \alpha} + \frac{w_0^i}{R_{\alpha\beta}} \\
\gamma_{0\alpha\beta}^{i} &= \varepsilon_{0\alpha\beta}^{i} + \varepsilon_{0\beta\alpha}^{i} \\
\gamma_{0\alpha z}^{i} &= \frac{1}{A}\frac{\partial w_0^i}{\partial \alpha} - \frac{u_0^i}{R_\alpha} - \frac{v_0^i}{R_{\alpha\beta}} + \theta_\alpha^i \\
\gamma_{0\beta z}^{i} &= \frac{1}{B}\frac{\partial w_0^i}{\partial \beta} - \frac{v_0^i}{R_\beta} - \frac{u_0^i}{R_{\alpha\beta}} + \theta_\beta^i \\
\kappa_\alpha^i &= \frac{1}{A}\frac{\partial \theta_\alpha^i}{\partial \alpha} + \frac{\theta_\beta^i}{AB}\frac{\partial A}{\partial \beta} \\
\kappa_\beta^i &= \frac{1}{B}\frac{\partial \theta_\beta^i}{\partial \beta} + \frac{\theta_\alpha^i}{AB}\frac{\partial B}{\partial \alpha} \\
\chi_{\alpha\beta}^i &= \frac{1}{A}\frac{\partial \theta_\beta^i}{\partial \alpha} - \frac{\theta_\alpha^i}{AB}\frac{\partial A}{\partial \beta} \\
\chi_{\beta\alpha}^i &= \frac{1}{B}\frac{\partial \theta_\alpha^i}{\partial \beta} - \frac{\theta_\beta^i}{AB}\frac{\partial B}{\partial \alpha}
\end{aligned}
\tag{11b}
$$

For the core

$$
\begin{aligned}
\varepsilon_\alpha^c &= \frac{1}{1 + c_o\frac{z^c}{R_\alpha}}\left[\varepsilon_{0\alpha}^c + z^c\kappa_\alpha^c + {z^c}^2\varepsilon_{0\alpha}^{*c} + {z^c}^3\kappa_\alpha^{*c}\right] \\
\varepsilon_\beta^c &= \frac{1}{1 + c_1\frac{z^c}{R_\beta}}\left[\varepsilon_{0\beta}^c + z^c\kappa_\beta^c + {z^c}^2\varepsilon_{0\beta}^{*c} + {z^c}^3\kappa_\beta^{*c}\right] \\
\varepsilon_z^c &= \varepsilon_{0z}^c + z^c\kappa_z^c + {z^c}^2\varepsilon_{0\alpha}^{*c} \\
\varepsilon_{\alpha\beta}^c &= \frac{1}{1 + c_o\frac{z^c}{R_\alpha}}\left[\varepsilon_{0\alpha\beta}^c + z^c\chi_{\alpha\beta}^c + {z^c}^2\varepsilon_{0\alpha\beta}^{*c} + {z^c}^3\chi_{\alpha\beta}^{*c}\right] \\
\varepsilon_{\beta\alpha}^c &= \frac{1}{1 + c_1\frac{z^c}{R_\beta}}\left[\varepsilon_{0\beta\alpha}^c + z^c\chi_{\beta\alpha}^c + {z^c}^2\varepsilon_{0\beta\alpha}^{*c} + {z^c}^3\chi_{\beta\alpha}^{*c}\right] \\
\gamma_{\alpha z}^c &= \frac{1}{1 + c_o\frac{z^c}{R_\alpha}}\left[\varepsilon_{0\alpha z}^c + z^c\chi_{\alpha z}^c + {z^c}^2\varepsilon_{0\alpha z}^{*c} + {z^c}^3\chi_{\alpha z}^{*c}\right] \\
&\quad + \left[\varepsilon_{0z\alpha}^c + z^c\chi_{z\alpha}^c + {z^c}^2\varepsilon_{0z\alpha}^{*c}\right] \\
\gamma_{\beta z}^c &= \frac{1}{1 + c_1\frac{z^c}{R_\beta}}\left[\varepsilon_{0\beta z}^c + z^c\chi_{\beta z}^c + {z^c}^2\varepsilon_{0\beta z}^{*c} + {z^c}^3\chi_{\beta z}^{*c}\right] \\
&\quad + \left[\varepsilon_{0z\beta}^c + z^c\chi_{z\beta}^c + {z^c}^2\varepsilon_{0z\beta}^{*c}\right]
\end{aligned}
\tag{12a}
$$

where

$$
\begin{aligned}
&\varepsilon_{0\alpha}^c = \frac{1}{A}u_{0,\alpha}^c + \frac{v_0^c}{AB}A_{,\beta} + \frac{w_0^c}{R_\alpha} \qquad
\varepsilon_{0\beta}^c = \frac{1}{B}v_{0,\beta}^c + \frac{u_0^c}{AB}B_{,\alpha} + \frac{w_0^c}{R_\beta} \qquad
\varepsilon_{0z}^c = w_1^c \qquad \kappa_z^c = 2w_2^c \\
&\kappa_\alpha^c = \frac{1}{A}u_{1,\alpha}^c + \frac{v_1^c}{AB}A_{,\beta} + \frac{w_1^c}{R_\alpha} \qquad
\kappa_\beta^c = \frac{1}{B}v_{1,\beta}^c + \frac{u_1^c}{AB}B_{,\alpha} + \frac{w_1^c}{R_\beta} \\
&\varepsilon_{0\alpha}^{*c} = \frac{1}{A}u_{2,\alpha}^c + \frac{v_2^c}{AB}A_{,\beta} + \frac{w_2^c}{R_\alpha} \qquad
\varepsilon_{0\beta}^{*c} = \frac{1}{B}v_{2,\beta}^c + \frac{u_2^c}{AB}B_{,\alpha} + \frac{w_2^c}{R_\beta} \qquad
\varepsilon_{0z}^{*c} = 3w_3^c \\
&\kappa_\alpha^{*c} = \frac{1}{A}u_{3,\alpha}^c + \frac{v_3^c}{AB}A_{,\beta} + \frac{w_3^c}{R_\alpha} \qquad
\kappa_\beta^{*c} = \frac{1}{B}v_{3,\beta}^c + \frac{u_3^c}{AB}B_{,\alpha} + \frac{w_3^c}{R_\beta}
\end{aligned}
\tag{12b}
$$

$$
\begin{aligned}
&\varepsilon_{0\alpha\beta}^c = \frac{1}{A}v_{0,\alpha}^c - \frac{u_0^c}{AB}A_{,\beta} + \frac{w_0^c}{R_{\alpha\beta}} \qquad
\varepsilon_{0\beta\alpha}^c = \frac{1}{B}u_{0,\beta}^c - \frac{v_0^c}{AB}B_{,\alpha} + \frac{w_0^c}{R_\beta} \qquad
\varepsilon_{0\alpha z}^c = \frac{1}{A}w_{0,\alpha}^c - \frac{u_0^c}{R_\alpha} - \frac{v_0^c}{R_{\alpha\beta}} \\
&\chi_{\alpha\beta}^c = \frac{1}{A}v_{1,\alpha}^c - \frac{u_1^c}{AB}A_{,\beta} + \frac{w_1^c}{R_{\alpha\beta}} \qquad
\chi_{\beta\alpha}^c = \frac{1}{B}u_{1,\beta}^c - \frac{v_1^c}{AB}B_{,\alpha} + \frac{w_1^c}{R_\beta} \qquad
\chi_{\alpha z}^c = \frac{1}{A}w_{1,\alpha}^c - \frac{u_1^c}{R_\alpha} - \frac{v_1^c}{R_{\alpha\beta}} \\
&\varepsilon_{0\alpha\beta}^{*c} = \frac{1}{A}v_{2,\alpha}^c - \frac{u_2^c}{AB}A_{,\beta} + \frac{w_2^c}{R_{\alpha\beta}} \qquad
\varepsilon_{0\beta\alpha}^{*c} = \frac{1}{B}u_{2,\beta}^c - \frac{v_2^c}{AB}B_{,\alpha} + \frac{w_2^c}{R_\beta} \qquad
\varepsilon_{0\alpha z}^{*c} = \frac{1}{A}w_{2,\alpha}^c - \frac{u_2^c}{R_\alpha} - \frac{v_2^c}{R_{\alpha\beta}} \\
&\chi_{\alpha\beta}^{*c} = \frac{1}{A}v_{3,\alpha}^c - \frac{u_3^c}{AB}A_{,\beta} + \frac{w_3^c}{R_{\alpha\beta}} \qquad
\chi_{\beta\alpha}^{*c} = \frac{1}{B}u_{3,\beta}^c - \frac{v_3^c}{AB}B_{,\alpha} + \frac{w_3^c}{R_\beta} \qquad
\chi_{\alpha z}^{*c} = \frac{1}{A}w_{3,\alpha}^c - \frac{u_3^c}{R_\alpha} - \frac{v_3^c}{R_{\alpha\beta}}
\end{aligned}
\tag{12b}
$$

$$
\begin{aligned}
&\varepsilon_{0\beta z}^c = \frac{1}{B}w_{0,\beta}^c - \frac{v_0^c}{R_\beta} - \frac{u_0^c}{R_{\alpha\beta}} \\
&\varepsilon_{0z\alpha}^c = u_1^c \qquad \chi_{\beta z}^c = \frac{1}{B}w_{1,\beta}^c - \frac{v_1^c}{R_\beta} - \frac{u_1^c}{R_{\alpha\beta}} \qquad \varepsilon_{0\beta z}^c = v_1^c \\
&\chi_{z\alpha}^c = 2u_2^c \qquad \chi_{\beta z}^c = 2v_2^c \\
&\varepsilon_{0z\alpha}^{*c} = 3u_3^c \qquad \varepsilon_{0\beta z}^{*c} = \frac{1}{B}w_{2,\beta}^c - \frac{v_2^c}{R_\beta} - \frac{u_2^c}{R_{\alpha\beta}} \qquad \varepsilon_{0\beta z}^{*c} = 3v_3^c \\
&\chi_{\beta z}^{*c} = \frac{1}{B}w_{3,\beta}^c - \frac{v_3^c}{R_\beta} - \frac{u_3^c}{R_{\alpha\beta}}
\end{aligned}
$$

2.5 The continuity conditions of the interface displacements

Reminding that there is no slipping between the face sheets and the core, the following relations can be written (Kheirikhah *et al*. 2012)

$$
\begin{cases} u^c|_{z=\frac{t^c}{2}} = u^t|_{z=\frac{t^c}{2}} \\ u^c|_{z=-\frac{t^c}{2}} = u^b|_{z=-\frac{t^c}{2}} \end{cases},\quad
\begin{cases} v^c|_{z=\frac{t^c}{2}} = v^t|_{z=\frac{t^c}{2}} \\ v^c|_{z=-\frac{t^c}{2}} = v^b|_{z=-\frac{t^c}{2}} \end{cases},\quad
\begin{cases} w^c|_{z=\frac{t^c}{2}} = w^t|_{z=\frac{t^c}{2}} \\ w^c|_{z=-\frac{t^c}{2}} = w^b|_{z=-\frac{t^c}{2}} \end{cases}
\tag{13}
$$

By substituting Eqs. (5) and (6) into Eq. (13), u_0^c, u_1^c, v_0^c, v_1^c, w_0^c, w_1^c are obtained as follows

$$\begin{aligned}
u_0^c &= \frac{u_0^t + u_0^b}{2} + \frac{1}{4}\left(t^b\theta_\alpha^b - t^t\theta_\alpha^t\right) - \left(\frac{t^c}{2}\right)^2 u_2^c \\
u_1^c &= \frac{u_0^t - u_0^b}{t^c} - \frac{1}{2t^c}\left(t^b\theta_\alpha^b + t^t\theta_\alpha^t\right) - \left(\frac{t^c}{2}\right)^2 u_3^c \\
v_0^c &= \frac{v_0^t + v_0^b}{2} + \frac{1}{4}\left(t^b\theta_\beta^b - t^t\theta_\beta^t\right) - \left(\frac{t^c}{2}\right)^2 v_2^c \\
v_1^c &= \frac{v_0^t - v_0^b}{t^c} - \frac{1}{2t^c}\left(t^b\theta_\beta^b + t^t\theta_\beta^t\right) - \left(\frac{t^c}{2}\right)^2 v_3^c \\
w_0^c &= \frac{w^t + w^b}{2} - \left(\frac{t^c}{2}\right)^2 w_2^c, \\
w_1^c &= \frac{w^t - w^b}{t^c} - \left(\frac{t^c}{2}\right)^2 w_3^c
\end{aligned} \tag{14}$$

2.6 Constitutive equations

Since the face sheets and the core are assumed to have linear elastic behavior, the stress-strain relation according to the Hook's law is

$$\{\sigma\} = [Q]\{\varepsilon\} \tag{15}$$

If the principle axes (1, 2, 3) (local axes) coincides the geometric axes $(\alpha,\ \beta,\ z)$ (global axes) (i.e., Fig. 1), the constitutive equation for a fiber–reinforced composite lamina can be written as follows

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{Bmatrix}^i = \begin{bmatrix} C_{11} & C_{21} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}^i \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{23} \end{Bmatrix}^i \tag{16}$$

where the $[Q]$ matrix refers to the elastic stiffness in the principle material axes (1, 2, 3) and C_{ij}'s elements are defined as follows (Garg *et al*. 2006)

$$\begin{gathered}
C_{11} = \frac{E_{11}(1 - \nu_{23}\nu_{32})}{\nu^*}, \quad C_{12} = \frac{E_{11}(\nu_{21} - \nu_{31}\nu_{23})}{\nu^*}, \\
C_{13} = \frac{E_{11}(\nu_{31} + \nu_{21}\nu_{32})}{\nu^*} \\
C_{22} = \frac{E_{22}(1 - \nu_{13}\nu_{31})}{\nu^*}, \quad C_{23} = \frac{E_{22}(\nu_{32} + \nu_{12}\nu_{31})}{\nu^*}, \\
C_{33} = \frac{E_{33}(1 - \nu_{12}\nu_{21})}{\nu^*} \\
C_{44} = G_{12}, \qquad C_{55} = G_{13}, \qquad C_{66} = G_{23} \\
\nu^* = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{13}\nu_{31} - 2\nu_{32}\nu_{13}\nu_{21}
\end{gathered} \tag{17}$$

In general, the principle axes of materials may not necessarily coincide the geometric axes. Since the loading is defined in geometric directions, it is required to consider the relationship between the two axes. The stress-strain relation in coordinate axes (α,β,z) can be written as

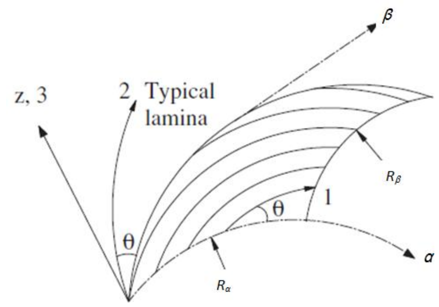

Fig. 2 Lamina reference axes, (α,β,z) (Garg *et al*. 2006)

$$\{\sigma_{ij}\} = [\bar{Q}]\{\varepsilon_{ij}\} \tag{18}$$

Where (Reddy 2003)

$$[\bar{Q}] = [T]^{-1}[Q][T]^{-T} \tag{19}$$

$[\bar{Q}]$ refers to reduced elastic stiffness matrix of the orthotropic material. It corresponds with Kth lamina and is expressed in terms of the orientation θ and material properties. Superscript T denotes transformation matrix $[T]$ and is defined as

$$[T] = \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & c^2 - s^2 \end{bmatrix} \tag{20}$$

Where $c = \cos\theta$, $s = \sin\theta$ and are measured counter-clockwise from the 1-axis (Fig. 2).

2.7 Stress (*Force and Moment*) Resultant

By substituting Eqs. (11a), (11b), (12a) and (12b) in constitutive Eq. (18) and integrating through the thickness, Eq. (18) can be expressed by vectors of mid-surface strains $\bar{\varepsilon}$ and stress resultant $\bar{\sigma}$ as follows

$$\{\bar{\sigma}\} = [D]\{\bar{\varepsilon}\} \tag{21a}$$

in which

$$\begin{gathered}
[D] = \begin{bmatrix} [D_f^i]_{a\times a} & 0 \\ 0 & k_o[D_s^i]_{b\times b} \end{bmatrix}, \\
[D_f^i] = \begin{bmatrix} [A]_{c\times c}^i & [B]_{c\times d}^i \\ [E]_{d\times c}^i & [D]_{d\times d}^i \end{bmatrix}
\end{gathered} \tag{21b}$$

Dimensions of the matrix D for the core are: $a = 19$, $b = 14$, $c = 10$, $d = 9$, and for the face sheets are: $a = 14$, $b = 6$, $c = 4$, $d = 4$. Also, the elements of $[D]$ for the core and the face sheets are given in Appendix A.

Where k_o parameter is called as shear correction factor of FSDT which is equal to $\frac{5}{6}$ (Reissner 1953). Components of $\bar{\varepsilon}$ and $\bar{\sigma}$ for the face sheets and the core are defined as (Garg *et al*. 2006)

$$\begin{aligned}
\{\bar{\varepsilon}_c\} = \{&\varepsilon_{0\alpha}^c, \varepsilon_{0\beta}^c, \varepsilon_{0\beta\alpha}^c, \varepsilon_{0\alpha\beta}^c, \varepsilon_{0\alpha}^{*c}, \varepsilon_{0\beta}^{*c}, \varepsilon_{0\beta\alpha}^{*c}, \varepsilon_{0\alpha\beta}^{*c}, \varepsilon_{0z}^c, \\
&\varepsilon_{0z}^{*c}, \kappa_\alpha^c, \kappa_\beta^c, \chi_{\beta\alpha}^c, \chi_{\alpha\beta}^c, \kappa_\alpha^{*c}, \kappa_\beta^{*c}, \chi_{\beta\alpha}^{*c}, \chi_{\alpha\beta}^{*c}, \\
&\kappa_z^c, \varepsilon_{0z\alpha}^c, \varepsilon_{0\alpha z}^c, \varepsilon_{0z\beta}^c, \varepsilon_{0\beta z}^c, \varepsilon_{0z\alpha}^{*c},
\end{aligned} \tag{22a}$$

$$\varepsilon^{*c}_{0z\beta}, \varepsilon^{*c}_{0\alpha z}, \varepsilon^{*c}_{0\beta z}, \chi^{c}_{z\alpha}, \chi^{c}_{\alpha z}, \chi^{c}_{z\beta}, , \chi^{c}_{\beta z}, \chi^{*c}_{\alpha z}, \chi^{*c}_{\beta z}\}^T \quad (22a)$$

$$\begin{aligned}\{\bar{\sigma}_c\} = \{&N^c_\alpha, N^c_\beta, N^c_{\beta\alpha}, N^c_{\alpha\beta}, N^{*c}_\alpha, N^{*c}_\beta, N^{*c}_{\beta\alpha}, N^{*c}_{\alpha\beta}, N^c_z, N^{*c}_z,\\ &M^c_\alpha, M^c_\beta, M^c_{\beta\alpha}, M^c_{\alpha\beta}, M^{*c}_\alpha, M^{*c}_\beta, M^{*c}_{\beta\alpha}, M^{*c}_{\alpha\beta}, M^c_z,\\ &Q^c_{z\alpha}, Q^c_{\alpha z}, Q^c_{z\beta}, Q^c_{\beta z}, Q^{*c}_{z\alpha}, Q^{*c}_{z\beta}, Q^{*c}_{\alpha z}, Q^{*c}_{\beta z},\\ &S^c_{z\alpha}, S^c_{\alpha z}, S^c_{z\beta}, S^c_{\beta z}, S^{*c}_{\alpha z}, S^{*c}_{\beta z}\}^T\end{aligned} \quad (22b)$$

$$\begin{aligned}\{\bar{\varepsilon}_i\} = \{&\varepsilon^i_{0\alpha}, \varepsilon^i_{0\beta}, \varepsilon^i_{0\beta\alpha}, \varepsilon^i_{0\alpha\beta}, \kappa^i_\alpha, \kappa^i_\beta, \chi^i_{\beta\alpha}, \chi^i_{\alpha\beta},\\ &\varepsilon^i_{0z\alpha}, \varepsilon^i_{0\alpha z}, \varepsilon^i_{0z\beta}, \varepsilon^i_{0\beta z}, \chi^i_{\alpha z}, \chi^i_{z\beta}\}^T \quad \text{that} \quad i = t, b\end{aligned} \quad (22c)$$

$$\begin{aligned}\{\bar{\sigma}_i\} = \{&N^i_\alpha, N^i_\beta, N^i_{\beta\alpha}, N^i_{\alpha\beta}, M^i_\alpha, M^i_\beta, M^i_{\beta\alpha}, M^i_{\alpha\beta},\\ &Q^i_{z\alpha}, Q^i_{\alpha z}, Q^i_{z\beta}, Q^i_{\beta z}, S^i_{\alpha z}, S^i_{z\beta}\}^T \quad \text{that} \quad i = t, b\end{aligned} \quad (22d)$$

Superscript T denotes here as transpose. The components of stress resultant vectors $\{\bar{\sigma}\}$ are forces and moments per unit of length which act along the lines of constant α or β in the face sheets and the core of DCSP (Garg *et al.* 2006)

$$\begin{bmatrix} N^c_\alpha & N^{*c}_\alpha & M^c_\alpha & M^{*c}_\alpha \\ N^c_\beta & N^{*c}_\beta & M^c_\beta & M^{*c}_\beta \\ N^c_z & N^{c*}_z & M^c_z & 0 \\ N^c_{\alpha\beta} & N^{*c}_{\alpha\beta} & M^c_{\alpha\beta} & M^{*c}_{\alpha\beta} \\ N^c_{\beta\alpha} & N^{*c}_{\beta\alpha} & M^c_{\beta\alpha} & M^{*c}_{\beta\alpha} \end{bmatrix} = \int_z \begin{bmatrix} k^c_2 & 0 & 0 & 0 & 0 \\ 0 & k^c_1 & 0 & 0 & 0 \\ 0 & 0 & k^c_1 k^c_2 & 0 & 0 \\ 0 & 0 & 0 & k^c_2 & 0 \\ 0 & 0 & 0 & 0 & k^c_1 \end{bmatrix} \begin{Bmatrix} \sigma_\alpha \\ \sigma_\beta \\ \sigma_z \\ \tau_{\alpha\beta} \\ \tau_{\beta\alpha} \end{Bmatrix}^c (1, z^2, z, z^3) dz \quad (23a)$$

$$\begin{bmatrix} Q^c_{\beta z} & Q^{*c}_{\beta z} & S^c_{\beta z} & S^{*c}_{\beta z} \\ Q^c_{z\beta} & Q^{*c}_{z\beta} & S^c_{z\beta} & 0 \\ Q^c_{\alpha z} & Q^{*c}_{\alpha z} & S^c_{\alpha z} & S^{*c}_{\alpha z} \\ Q^c_{z\alpha} & Q^{*c}_{z\alpha} & S^c_{z\alpha} & 0 \end{bmatrix} = \int_z \begin{bmatrix} k^c_1 & 0 & 0 & 0 \\ 0 & k^c_1 k^c_2 & 0 & 0 \\ 0 & 0 & k^c_2 & 0 \\ 0 & 0 & 0 & k^c_1 k^c_2 \end{bmatrix} \begin{Bmatrix} \tau_{\beta z} \\ \tau_{z\beta} \\ \tau_{\alpha z} \\ \tau_{z\alpha} \end{Bmatrix}^c (1, z^2, z, z^3) dz \quad (23b)$$

$$k^c_1 = \left(1 + \frac{z}{R^c_\alpha}\right), \quad k^c_2 = \left(1 + \frac{z}{R^c_\beta}\right) \quad (23c)$$

$$\begin{bmatrix} N^i_\alpha & M^i_\alpha \\ N^i_\beta & M^i_\beta \\ N^i_{\alpha\beta} & M^i_{\alpha\beta} \\ N^i_{\beta\alpha} & M^i_{\beta\alpha} \end{bmatrix} = \int_{z^i} \begin{bmatrix} k^i_2 & 0 & 0 & 0 \\ 0 & k^i_1 & 0 & 0 \\ 0 & 0 & k^i_2 & 0 \\ 0 & 0 & 0 & k^i_1 \end{bmatrix} \begin{Bmatrix} \sigma_\alpha \\ \sigma_\beta \\ \tau_{\alpha\beta} \\ \tau_{\beta\alpha} \end{Bmatrix}^i (1, z^i) dz^i \quad (24a)$$

$$\begin{bmatrix} Q^i_{\beta z} & S^i_{\beta z} \\ Q^i_{z\beta} & 0 \\ Q^i_{\alpha z} & S^i_{\alpha z} \\ Q^i_{z\alpha} & 0 \end{bmatrix} = \int_{z^i} \begin{bmatrix} k^i_1 & 0 & 0 & 0 \\ 0 & k^i_1 k^i_2 & 0 & 0 \\ 0 & 0 & k^i_2 & 0 \\ 0 & 0 & 0 & k^i_1 k^i_2 \end{bmatrix} \begin{Bmatrix} \tau_{\beta z} \\ \tau_{z\beta} \\ \tau_{\alpha z} \\ \tau_{z\alpha} \end{Bmatrix}^i (1, z^i) dz^i \quad (24b)$$

$$k^i_1 = \left(1 + \frac{z^i}{R^i_\alpha}\right), \quad k^i_2 = \left(1 + \frac{z^i}{R^i_\beta}\right), \quad \text{that} \quad i = t, b \quad (24c)$$

It is worth mentioning that Eqs. (23a) and (24a) clearly show that the symmetric property of stress tensor $\tau_{\alpha\beta} = \tau_{\beta\alpha}$ doesn't imply the symmetry of stress resultants $N_{\alpha\beta} \neq N_{\beta\alpha}$, $M_{\alpha\beta} \neq M_{\beta\alpha}$, because in general $R_\alpha \neq R_\beta$, except for structures such as plate and sphere in which $R_\alpha = R_\beta$ and a thin panel or shell of any shape.

2.8 Equations of motion

Governing equations of motion for the free vibration analysis of DCSP and the boundary conditions are obtained using Hamilton's principle (Reddy 2003)

$$\delta \int_{t_1}^{t_2} (L)\, dt = \delta \int_{t_1}^{t_2} [E - (U + W)]\, dt = 0 \quad (25)$$

Where δ is the first variation operator, $\boldsymbol{E}$ is the kinetic energy, $\boldsymbol{U}$ and $\boldsymbol{W}$ denote the total strain energy due to the deformation and the potential of the external loads, respectively, and $\boldsymbol{t}$ is the time coordinate. For free vibration analysis, there is no damping and external forces on the system. Therefore, Hamilton's principle in Eq. (25) can be written as follows

$$\delta \int_{t_1}^{t_2} [E - U]\, dt = 0 \quad (26)$$

The kinetic energy for DCSP is given by (Reddy 2003)

$$E = \frac{1}{2} \sum_i^{t,b,c} \int_{V^i} \rho^i ({\dot{u}^i}^2 + {\dot{v}^i}^2 + {\dot{w}^i}^2) dV^i \quad (27)$$

Where $\rho^i (i = t, b, c)$ is the mass per unit volume of the top and the bottom face sheets and the core respectively. $\dot{u}^i$, $\dot{v}^i$, $\dot{w}^i (i = t, b, c)$ are the velocities in the α, β and z direction respectively, "." denotes the first time derivative, $V^i (i = t, b, c)$ is the volume of the top and the bottom face sheets and the core, respectively and dV^i is the volume of an infinitesimal element (i.e., Eq. (4)). The first variation of the kinetic energy and integration by parts with respect to the time coordinate for DCSP are given by Eqs. (29) and (30), respectively as follows

$$\delta E = \sum_i^{t,b,c} \int_{V^i} \rho^i (\dot{u}^i \delta \dot{u}^i + \dot{v}^i \delta \dot{v}^i + \dot{w}^i \delta \dot{w}^i) dV^i \quad (28)$$

$$\begin{aligned}\int_0^t \delta E dt = -\sum_i^{t,b,c} \Bigg\{ &\int_0^t \int_{V^i} [\rho^i (\ddot{u}^i \delta u^i + \ddot{v}^i \delta v^i + \ddot{w}^i \delta w^i) \\ &dV^i] dt + \int_v \rho^i (\dot{u}^i \delta \dot{u}^i + \dot{v}^i \delta \dot{v}^i + \dot{w}^i \delta \dot{w}^i) dV^i \Big|_{t=0}^{t} \Bigg\}\end{aligned} \quad (29)$$

Where $\ddot{u}^i$, $\ddot{v}^i$, $\ddot{w}^i (i = t, b, c)$ are the accelerations in the α, β and z direction, respectively. Also, in Eq. (29), the second integral according to the initial assuming in Hamilton's principle is equal to zero, then Eq. (30) can be rewritten as follows

$$\int_0^t \delta E dt = -\sum_i^{t,b,c}\left\{\int_0^t \int_{V^i}\left[\rho^i(\ddot{u}^i\delta u^i + \ddot{v}^i\delta v^i + \ddot{w}^i\delta w^i)dV^i\right]dt\right\} \tag{30}$$

By substituting the displacement field in Eqs. (5) and (6) into Eq. (30) and integration by parts with respect to the time coordinate, the variation of the kinetic energy is obtained as

$$\begin{aligned}
\int_0^t \delta E dt &= \sum_i^{t,b}\int_0^t \delta E^i dt + \int_0^t \delta E^c dt \\
&= \sum_i^{t,b}\left\{-\int_0^t \int_A \left[\left(I_0^i\ddot{u}_0^i + I_1^i\ddot{\theta}_\alpha^i\right)\delta u_0^i + \left(I_1^i\ddot{u}_0^i + I_2^i\ddot{\theta}_\alpha^i\right)\delta\theta_\alpha^i \right.\right.\\
&\quad + \left(I_0^i\ddot{v}_0^i + I_1^i\ddot{\theta}_\beta^i\right)\delta v_0^i + \left(I_1^i\ddot{u}_0^i + I_2^i\ddot{\theta}_\beta^i\right)\delta\theta_\beta^i \\
&\quad \left.\left. + (I_0^i\ddot{w}^i)\delta w^i\right](A^iB^i d\alpha d\beta)dt\right\} \\
&\quad - \int_0^t \int_A [(I_0^c\ddot{u}_0^c + I_1^c\ddot{u}_1^c + I_2^c\ddot{u}_2^c + I_3^c\ddot{u}_3^c)\delta u_0^c \\
&\quad + (I_1^c\ddot{u}_0^c + I_2^c\ddot{u}_1^c + I_3^c\ddot{u}_2^c + I_4^c\ddot{u}_3^c)\delta u_1^c \\
&\quad + (I_2^c\ddot{u}_0^c + I_3^c\ddot{u}_1^c + I_4^c\ddot{u}_2^c + I_5^c\ddot{u}_3^c)\delta u_2^c \\
&\quad + (I_3^c\ddot{u}_0^c + I_4^c\ddot{u}_1^c + I_5^c\ddot{u}_2^c + I_6^c\ddot{u}_3^c)\delta u_3^c \\
&\quad + (I_0^c\ddot{v}_0^c + I_1^c\ddot{v}_1^c + I_2^c\ddot{v}_2^c + I_3^c\ddot{v}_3^c)\delta v_0^c \\
&\quad + (I_1^c\ddot{v}_0^c + I_2^c\ddot{v}_1^c + I_3^c\ddot{v}_2^c + I_4^c\ddot{v}_3^c)\delta v_1^c \\
&\quad + (I_2^c\ddot{v}_0^c + I_3^c\ddot{v}_1^c + I_4^c\ddot{v}_2^c + I_5^c\ddot{v}_3^c)\delta v_2^c \\
&\quad + (I_3^c\ddot{v}_0^c + I_4^c\ddot{v}_1^c + I_5^c\ddot{v}_2^c + I_6^c\ddot{v}_3^c)\delta v_3^c \\
&\quad + (I_0^c\ddot{w}_0^c + I_1^c\ddot{w}_1^c + I_2^c\ddot{w}_2^c + I_3^c\ddot{w}_3^c)\delta w_0^c \\
&\quad + (I_1^c\ddot{w}_0^c + I_2^c\ddot{w}_1^c + I_3^c\ddot{w}_2^c + I_4^c\ddot{w}_3^c)\delta w_1^c \\
&\quad + (I_2^c\ddot{w}_0^c + I_3^c\ddot{w}_1^c + I_4^c\ddot{w}_2^c + I_5^c\ddot{w}_3^c)\delta w_2^c \\
&\quad + (I_3^c\ddot{w}_0^c + I_4^c\ddot{w}_1^c + I_5^c\ddot{w}_2^c + I_6^c\ddot{w}_3^c)\delta w_3^c] \\
&\quad (A^cB^c d\alpha d\beta)dt
\end{aligned} \tag{31}$$

that

$$I_n^i = \int_z \rho^i\left(1+\frac{z}{R_\alpha^i}\right)\left(1+\frac{z}{R_\beta^i}\right)(z^n)dz, \quad i = t,b,c \quad \text{and} \quad n = 1 \text{ to } 6 \tag{32}$$

The first variation of the strain energy for DCSP during the elastic deformation is

$$\begin{aligned}
&\int_{t_1}^{t_2}\delta U^c dt + \sum_i^{t,b}\int_{t_1}^{t_2}\delta U^i dt \\
&= \int_0^t \int_z \int_A (h_\alpha\sigma_\alpha^c\delta\varepsilon_\alpha^c + h_\beta\sigma_\beta^c\delta\varepsilon_\beta^c + \sigma_z^c\delta\varepsilon_z^c \\
&\quad + h_{\alpha\beta}\tau_{\alpha\beta}^c\delta\gamma_{\alpha\beta}^c + \tau_{\alpha\beta}^c\delta\gamma_{\alpha\beta}^c + \tau_{\beta z}^c\delta\gamma_{\beta z}^c)dA^c\, dz\, dt \\
&\quad + \sum_i^{t,b}\left\{\int_0^t \int_{z^i}\int_A (\sigma_\alpha^i\delta\varepsilon_\alpha^i + \sigma_\beta^i\delta\varepsilon_\beta^i + \tau_{\alpha\beta}^i\delta\gamma_{\alpha\beta}^i\right. \\
&\quad + \tau_{\alpha z}^i\delta\gamma_{\alpha z}^i + \tau_{\beta z}^i\delta\gamma_{\beta z}^i)dA^i\, dz^i\, dt
\end{aligned} \tag{33}$$

Noted that in the above relation, $h_\alpha = h_\beta = h_{\alpha\beta} = 0$ indicates the compressible core and $h_\alpha = h_\beta = h_{\alpha\beta} = 1$ presents the incompressible core. Eqs. (11), (12) and (4) are substituted into Eq. (33) and integration by parts is carried out with respect to α and β. For example variation of strain energy related to $\gamma_{\alpha\beta}^i$ for the face sheet is

$$\begin{aligned}
&\int_0^t \int_{A^i}\int_{z^i}\tau_{\alpha\beta}^i\delta\gamma_{\alpha\beta}^i\, dA^i dz^i dt \\
&= \int_0^t \int_{A^i}\int_{z^i}\tau_{\alpha\beta}^i\left[\frac{1}{k_1^i}\left(\delta\varepsilon_{0\alpha\beta}^i + z^i\delta\chi_{\alpha\beta}^i\right)\right. \\
&\quad \left. + \frac{1}{k_1^i}\left(\delta\varepsilon_{0\beta\alpha}^i + z^i\delta\chi_{\beta\alpha}^i\right)\right]dz^i\left(A^iB^ik_1^ik_2^i\right)d\alpha d\beta dt \\
&= \int_0^t \int_{A^i}\int_{z^i}\tau_{\alpha\beta}^i\left\{\frac{1}{k_1^i}\left[\left(\frac{1}{A^i}\delta\left(\frac{\partial v_0^i}{\partial\alpha}\right) - \frac{\left(A_{,\beta}^i\right)}{A^iB^i}\delta u_0^i\right.\right.\right. \\
&\quad \left. + \frac{1}{R_{\alpha\beta}^i}\delta w_0^i\right) + z^i\left.\left(\frac{1}{A^i}\delta\left(\frac{\partial\theta_\beta^i}{\partial\alpha}\right) - \frac{\left(A_{,\beta}^i\right)}{A^iB^i}\delta\theta_\alpha^i\right)\right] \\
&\quad + \frac{1}{k_2^i}\left[\left(\frac{1}{B^i}\delta\left(\frac{\partial u_0^i}{\partial\beta}\right) - \frac{\left(B_{,\alpha}^i\right)}{A^iB^i}\delta v_0^i + \frac{1}{R_{\alpha\beta}^i}\delta w_0^i\right)\right. \\
&\quad \left.\left. + z^i\left(\frac{1}{B^i}\delta\left(\frac{\partial\theta_\alpha^i}{\partial\beta}\right) - \frac{\left(B_{,\alpha}^i\right)}{A^iB^i}\delta\theta_\beta^i\right)\right]\right\} \\
&\quad dz^i\left(A^iB^ik_1^ik_2^i\right)d\alpha d\beta dt \\
&= \int_0^t -\left\{\int_{A^i}\left(\left[\frac{\partial}{\partial\alpha}\left(B^iN_{\alpha\beta}^i\right)\left(\delta v_0^i\right)\right.\right.\right. \\
&\quad \left. + \left(N_{\alpha\beta}^i\left(A_{,\beta}^i\right)\right)\left(\delta u_0^i\right) - \left(\frac{A^iB^i}{R_{\alpha\beta}^i}N_{\alpha\beta}^i\right)\left(\delta w_0^i\right)\right] \\
&\quad + \left[\frac{\partial}{\partial\alpha}\left(B^iM_{\alpha\beta}^i\right)\left(\delta\theta_\beta^i\right) + \left(M_{\alpha\beta}^i\left(A_{,\beta}^i\right)\right)\left(\delta\theta_\alpha^i\right)\right] \\
&\quad + \left[\frac{\partial}{\partial\beta}\left(A^iN_{\beta\alpha}^i\right)\left(\delta u_0^i\right) + \left(N_{\beta\alpha}^i\left(B_{,\alpha}^i\right)\right)\left(\delta v_0^i\right)\right. \\
&\quad \left. - \left(\frac{A^iB^i}{R_{\alpha\beta}^i}N_{\beta\alpha}^i\right)\left(\delta w_0^i\right)\right] + \left[\frac{\partial}{\partial\beta}\left(A^iM_{\beta\alpha}^i\right)\left(\delta\theta_\alpha^i\right)\right. \\
&\quad \left.\left.\left. + \left(M_{\beta\alpha}^i\left(B_{,\alpha}^i\right)\right)\left(\delta\theta_\beta^i\right)\right]\right)\right\}d\alpha d\beta dt \\
&\quad + \int_0^t \int_\beta \left[\left(B^iN_{\alpha\beta}^i\right)\left(\delta v_0^i\right) + \left(B^iM_{\alpha\beta}^i\right)\left(\delta\theta_\beta^i\right)\right]_{\alpha_1}^{\alpha_2} d\beta dt \\
&\quad + \int_0^t \int_\alpha \left[\left(A^iN_{\beta\alpha}^i\right)\left(\delta u_0^i\right) + \left(A^iM_{\beta\alpha}^i\right)\left(\delta\theta_\alpha^i\right)\right]_{\beta_1}^{\beta_2} d\alpha dt
\end{aligned} \tag{34}$$

Finally, by substituting Eqs. (31) and (33) into Eq. (26) and considering the Eq. (14) and collecting the coefficients of independent variations in δu_0^t, δv_0^t, δw^t, $\delta\theta_\alpha^t$, $\delta\theta_\beta^t$,

δu_2^c, δv_2^c, δw_2^c, δu_3^c, δv_3^c, δw_3^c, δu_0^b, δv_0^b, δw^b, $\delta\theta_\alpha^b$, $\delta\theta_\beta^b$ sixteen equations of motion for thick DCSP and sixteen boundary conditions (B.C.) are obtained. The equations of motion for thick DCSP are

$$
\begin{aligned}
&\left(\frac{h_\alpha B_{,\alpha}^c}{2}\right)\{N_\alpha^c\}+\left(\frac{h_\alpha B^c}{2}\right)\{N_{\alpha,\alpha}^c\}+\left(\frac{h_\alpha B_{,\alpha}^c}{t^c}\right)\{M_\alpha^c\}\\
&+\left(\frac{h_\alpha B^c}{t^c}\right)\{M_{\alpha,\alpha}^c\}-\left(\frac{h_\beta B_{,\alpha}^c}{2}\right)\{N_\beta^c\}-\left(\frac{h_\beta B_{,\alpha}^c}{t^c}\right)\{M_\beta^c\}\\
&+\left(\frac{h_{\alpha\beta} A_{,\beta}^c}{2}\right)\{N_{\alpha\beta}^c\}+\left(\frac{h_{\alpha\beta} A_{,\beta}^c}{2}\right)\{N_{\beta\alpha}^c\}\\
&+\left(\frac{h_{\alpha\beta} A^c}{2}\right)\{N_{\beta\alpha,\beta}^c\}+\left(\frac{h_{\alpha\beta} A_{,\beta}^c}{t^c}\right)\{M_{\alpha\beta}^c\}\\
&+\left(\frac{h_{\alpha\beta} A_{,\beta}^c}{t^c}\right)\{M_{\beta\alpha}^c\}+\left(\frac{A^c h_{\alpha\beta}}{t^c}\right)\{M_{\beta\alpha,\beta}^c\}\\
&+\left(\frac{A^cB^c}{2R_\alpha^c}\right)\{Q_{\alpha z}^c\}+\left(\frac{A^cB^c}{t^cR_\alpha^c}\right)\{S_{\alpha z}^c\}-\left(\frac{A^cB^c}{t^c}\right)\{Q_{z\alpha}^c\}\\
&+\left(\frac{A^cB^c}{2R_{\alpha\beta}^c}\right)\{Q_{\beta z}^c\}+\left(\frac{A^cB^c}{t^cR_{\alpha\beta}^c}\right)\{S_{\beta z}^c\}+\left(B_{,\alpha}^t\right)\{N_\alpha^t\}\\
&+(B^t)\{N_{\alpha,\alpha}^t\}-\left(B_{,\alpha}^t\right)\{N_\beta^t\}+\left(A_{,\beta}^t\right)\{N_{\alpha\beta}^t\}\\
&+\left(A_{,\beta}^t\right)\{N_{\beta\alpha}^t\}+(A^t)\{N_{\beta\alpha,\beta}^t\}+\left(\frac{A^tB^tk_s}{R_\alpha^t}\right)\{Q_{\alpha z}^t\}\\
&+\left(\frac{A^tB^tk_s}{R_{\alpha\beta}^t}\right)\{Q_{\beta z}^t\}\\
&=\left[(A^cB^c)\left(\frac{1}{4}I_0^c+\frac{1}{t^c}I_1^c+\frac{1}{t^{c\,2}}I_2^c\right)+(A^tB^t)I_0^t\right]\ddot{u}_0^t\\
&\quad+\left[(A^cB^c)\left(-\frac{t^t}{8}I_0^c-\frac{t^t}{2t^c}I_1^c-\frac{t^t}{2t^{c\,2}}I_2^c\right)\right.\\
&\quad\left.+(A^tB^t)I_1^t\right]\ddot{\theta}_\alpha^t+(A^cB^c)\left[-\frac{t^{c\,2}}{8}I_0^c-\frac{t^c}{4}I_1^c+\frac{1}{2}I_2^c\right.\\
&\quad\left.+\frac{1}{t^c}I_3^c\right]\ddot{u}_2^c+(A^cB^c)\left[-\frac{t^{c\,2}}{8}I_1^c-\frac{t^c}{4}I_2^c\right.\\
&\quad\left.+\frac{1}{2}I_3^c+\frac{1}{t^c}I_4^c\right]\ddot{u}_3^c+(A^cB^c)\left[\frac{1}{4}I_0^c-\frac{1}{t^{c\,2}}I_2^c\right]\ddot{u}_0^b\\
&\quad+(A^cB^c)\left[\frac{t^b}{8}I_0^c-\frac{t^b}{2t^{c\,2}}I_2^c\right]\ddot{\theta}_\alpha^b
\end{aligned}
\qquad (35a)
$$

$$
\begin{aligned}
&-\left(\frac{h_\alpha A_{,\beta}^c}{2}\right)\{N_\alpha^c\}-\left(\frac{h_\alpha A_{,\beta}^c}{t^c}\right)\{M_\alpha^c\}+\left(\frac{h_\beta A_{,\beta}^c}{2}\right)\{N_\beta^c\}\\
&+\left(\frac{h_\beta A^c}{2}\right)\{N_{\beta,\beta}^c\}+\left(\frac{h_\beta A_{,\beta}^c}{t^c}\right)\{M_\beta^c\}\\
&+\left(\frac{h_\beta A^c}{t^c}\right)\{M_{\beta,\beta}^c\}+\left(\frac{h_{\alpha\beta} B_{,\alpha}^c}{2}\right)\{N_{\alpha\beta}^c\}\\
&+\left(\frac{h_{\alpha\beta} B^c}{2}\right)\{N_{\alpha\beta,\alpha}^c\}+\left(\frac{h_{\alpha\beta} B_{,\alpha}^c}{2}\right)\{N_{\beta\alpha}^c\}\\
&+\left(\frac{h_{\alpha\beta} B_{,\alpha}^c}{t^c}\right)\{M_{\alpha\beta}^c\}+\left(\frac{h_{\alpha\beta} B^c}{t^c}\right)\{M_{\alpha\beta,\alpha}^c\}\\
&+\left(\frac{h_{\alpha\beta} B_{,\alpha}^c}{t^c}\right)\{M_{\beta\alpha}^c\}+\left(\frac{A^cB^c}{2R_{\alpha\beta}^c}\right)\{Q_{\alpha z}^c\}\\
&+\left(\frac{A^cB^c}{t^cR_{\alpha\beta}^c}\right)\{S_{\alpha z}^c\}+\left(\frac{A^cB^c}{2R_\beta^c}\right)\{Q_{\beta z}^c\}\\
&+\left(\frac{A^cB^c}{t^cR_\beta^c}\right)\{S_{\beta z}^c\}-\left(\frac{A^cB^c}{t^c}\right)\{Q_{z\beta}^c\}\\
&-\left(A_{,\beta}^t\right)\{N_\alpha^t\}+\left(A_{,\beta}^t\right)\{N_\beta^t\}+(A^t)\{N_{\beta,\beta}^t\}\\
&+\left(B_{,\alpha}^t\right)\{N_{\alpha\beta}^t\}+(B^t)\{N_{\alpha\beta,\alpha}^t\}+\left(B_{,\alpha}^t\right)\{N_{\beta\alpha}^t\}\\
&+\left(\frac{k_sA^tB^t}{R_{\alpha\beta}^t}\right)\{Q_{\alpha z}^t\}+\left(\frac{k_sA^tB^t}{R_\beta^t}\right)\{Q_{\beta z}^t\}\\
&=\left[(A^cB^c)\left(\frac{1}{4}I_0^c+\frac{1}{t^c}I_1^c+\frac{1}{t^{c\,2}}I_2^c\right)+(A^tB^t)I_0^t\right]\ddot{v}_0^t\\
&\quad+\left[(A^cB^c)\left(-\frac{t^t}{8}I_0^c-\frac{t^t}{2t^c}I_1^c-\frac{t^t}{2t^{c\,2}}I_2^c\right)\right.\\
&\quad\left.+(A^tB^t)I_1^t\right]\ddot{\theta}_\beta^t+(A^cB^c)\left[-\frac{t^{c\,2}}{8}I_0^c-\frac{t^c}{4}I_1^c\right.\\
&\quad\left.+\frac{1}{2}I_2^c+\frac{1}{t^c}I_3^c\right]\ddot{v}_2^c+(A^cB^c)\left[-\frac{t^{c\,2}}{8}I_1^c-\frac{t^c}{4}I_2^c\right.\\
&\quad\left.+\frac{1}{2}I_3^c+\frac{1}{t^c}I_4^c\right]\ddot{v}_3^c+(A^cB^c)\left[\frac{1}{4}I_0^c-\frac{1}{t^{c\,2}}I_2^c\right]\ddot{v}_0^b\\
&\quad+(A^cB^c)\left[\frac{t^b}{8}I_0^c-\frac{t^b}{2t^{c\,2}}I_2^c\right]\ddot{\theta}_\beta^b
\end{aligned}
\qquad (35b)
$$

$$
\begin{aligned}
&-\left(\frac{h_\alpha A^cB^c}{2R_\alpha^c}\right)\{N_\alpha^c\}-\left(\frac{h_\alpha A^cB^c}{t^cR_\alpha^c}\right)\{M_\alpha^c\}\\
&-\left(\frac{h_\beta A^cB^c}{2R_\beta^c}\right)\{N_\beta^c\}-\left(\frac{h_\beta A^cB^c}{t^cR_\beta^c}\right)\{M_\beta^c\}\\
&-\left(\frac{A^cB^c}{t^c}\right)\{N_z^c\}-\left(\frac{h_{\alpha\beta} A^cB^c}{2R_{\alpha\beta}^c}\right)\{N_{\alpha\beta}^c\}\\
&-\left(\frac{h_{\alpha\beta} A^cB^c}{2R_{\alpha\beta}^c}\right)\{N_{\beta\alpha}^c\}-\left(\frac{h_{\alpha\beta} A^cB^c}{t^cR_{\alpha\beta}^c}\right)\{M_{\alpha\beta}^c\}\\
&-\left(\frac{h_{\alpha\beta} A^cB^c}{t^cR_{\alpha\beta}^c}\right)\{M_{\beta\alpha}^c\}+\left(\frac{B_{,\alpha}^c}{2}\right)\{Q_{\alpha z}^c\}\\
&+\left(\frac{B^c}{2}\right)\{Q_{\alpha z,\alpha}^c\}+\left(\frac{B_{,\alpha}^c}{t^c}\right)\{S_{\alpha z}^c\}+\left(\frac{B^c}{t^c}\right)\{S_{\alpha z,\alpha}^c\}\\
&+\left(\frac{A_{,\beta}^c}{2}\right)\{Q_{\beta z}^c\}+\left(\frac{A^c}{2}\right)\{Q_{\beta z,\beta}^c\}+\left(\frac{A_{,\beta}^c}{t^c}\right)\{S_{\beta z}^c\}\\
&+\left(\frac{A^c}{t^c}\right)\{S_{\beta z,\beta}^c\}-\left(\frac{A^tB^t}{R_\beta^t}\right)\{N_\alpha^t\}-\left(\frac{A^tB^t}{R_\beta^t}\right)\{N_\beta^t\}\\
&-\left(\frac{A^tB^t}{R_{\alpha\beta}^t}\right)\{N_{\alpha\beta}^t\}-\left(\frac{A^tB^t}{R_{\alpha\beta}^t}\right)\{N_{\beta\alpha}^t\}+k_s\left(B_{,\alpha}^t\right)\{Q_{\alpha z}^t\}\\
&+(k_sB^t)\{Q_{\alpha z,\alpha}^t\}+k_s\left(A_{,\beta}^t\right)\{Q_{\beta z}^t\}+(k_sA^t)\{Q_{\beta z,\beta}^t\}\\
&=\left[(A^cB^c)\left(\frac{1}{4}I_0^c+\frac{1}{t^c}I_1^c+\frac{1}{t^{c\,2}}I_2^c\right)+(A^tB^t)I_0^t\right]\ddot{w}^t\\
&\quad+(A^cB^c)\left[-\frac{t^{c\,2}}{8}I_0^c-\frac{t^c}{4}I_1^c+\frac{1}{2}I_2^c+\frac{1}{t^c}I_3^c\right]\ddot{w}_2^c\\
&\quad+(A^cB^c)\left[-\frac{t^{c\,2}}{8}I_1^c-\frac{t^c}{4}I_2^c+\frac{1}{2}I_3^c+\frac{1}{t^c}I_4^c\right]\ddot{w}_3^c\\
&\quad+(A^cB^c)\left[\frac{1}{4}I_0^c-\frac{1}{t^{c\,2}}I_2^c\right]\ddot{w}^b
\end{aligned}
\qquad (35c)
$$

$$-\left(\frac{h_\alpha t^t B^c_{,\alpha}}{4}\right)\{N^c_\alpha\}-\left(\frac{h_\alpha t^t B^c}{4}\right)\{N^c_{\alpha,\alpha}\}$$

$$-\left(\frac{h_\alpha t^t B^c_{,\alpha}}{2t^c}\right)\{M^c_\alpha\}-\left(\frac{h_\alpha t^t B^c}{2t^c}\right)\{M^c_{\alpha,\alpha}\}$$

$$+\left(\frac{h_\beta t^t B^c_{,\alpha}}{4}\right)\{N^c_\beta\}+\left(\frac{h_\beta t^t B^c_{,\alpha}}{2t^c}\right)\{M^c_\beta\}$$

$$-\left(\frac{h_{\alpha\beta} t^t A^c_{,\beta}}{4}\right)\{N^c_{\alpha\beta}\}-\left(\frac{h_{\alpha\beta} A^c_{,\beta} t^t}{4}\right)\{N^c_{\beta\alpha}\}$$

$$-\left(\frac{h_{\alpha\beta} t^t A^c}{4}\right)\{N^c_{\beta\alpha,\beta}\}-\left(\frac{h_{\alpha\beta} t^t A^c_{,\beta}}{2t^c}\right)\{M^c_{\alpha\beta}\}$$

$$-\left(\frac{h_{\alpha\beta} t^t A^c_{,\beta}}{2t^c}\right)\{M^c_{\beta\alpha}\}-\left(\frac{h_{\alpha\beta} t^t A^c}{2t^c}\right)\{M^c_{\beta\alpha,\beta}\}$$

$$-\left(\frac{t^t A^c B^c}{4R^c_\alpha}\right)\{Q^c_{\alpha z}\}-\left(\frac{t^t A^c B^c}{2t^c R^c_\alpha}\right)\{S^c_{\alpha z}\}$$

$$+\left(\frac{t^t A^c B^c}{2t^c}\right)\{Q^c_{z\alpha}\}-\left(\frac{t^t A^c B^c}{4R^c_{\alpha\beta}}\right)\{Q^c_{\beta z}\}$$

$$-\left(\frac{t^t A^c B^c}{2t^c R^c_{\alpha\beta}}\right)\{S^c_{\beta z}\}+(B^t_{,\alpha})\{M^t_\alpha\}+(B^t)\{M^t_{\alpha,\alpha}\}$$

$$-(B^t_{,\alpha})\{M^t_\beta\}+(A^t_{,\beta})\{M^t_{\alpha\beta}\}+(A^t_{,\beta})\{M^t_{\beta\alpha}\} \tag{35d}$$

$$+(A^t)\{M^t_{\beta\alpha,\beta}\}+\left(\frac{k_s}{R^t_\alpha}A^tB^t\right)\{S^t_{\alpha z}\}$$

$$-k_sA^tB^t\{Q^t_{z\alpha}\}+\left(\frac{k_s e_\alpha}{R^t_{\alpha\beta}}A^tB^t\right)\{S^t_{\beta z}\}$$

$$=\Big[(A^cB^c)\left(-\frac{t^t}{8}I^c_0-\frac{t^t}{2t^c}I^c_1-\frac{t^t}{2t^{c2}}I^c_2\right)$$

$$+(A^tB^t)I^t_1\Big]\ddot{u}^t_0+\Big[(A^cB^c)\left(\frac{t^{t2}}{16}I^c_0+\frac{t^{t2}}{4t^c}I^c_1+\frac{t^{t2}I^c_2}{4t^{c2}}\right)$$

$$+(A^tB^t)I^t_2\Big]\ddot{\theta}^t_\alpha+(A^cB^c)\Big[\frac{t^tt^{c2}}{16}I^c_0+\frac{t^tt^c}{8}I^c_1$$

$$-\frac{t^t}{4}I^c_2-\frac{t^t}{2t^c}I^c_3\Big]\ddot{u}^c_2+(A^cB^c)\Big[\frac{t^tt^{c2}}{16}I^c_1+\frac{t^tt^c}{8}I^c_2$$

$$-\frac{t^t}{4}I^c_3-\frac{t^t}{2t^c}I^c_4\Big]\ddot{u}^c_3+(A^cB^c)\Big[-\frac{t^t}{8}I^c_0+\frac{t^t}{2t^{c2}}I^c_2\Big]\ddot{u}^b_0$$

$$+(A^cB^c)\Big[-\frac{t^tt^b}{16}I^c_0+\frac{t^tt^b}{4t^{c2}}I^c_2\Big]\ddot{\theta}^b_\alpha$$

$$-\left(\frac{h_\beta t^t A^c_{,\beta}}{4}\right)\{N^c_\beta\}-\left(\frac{h_\beta t^t A^c}{4}\right)\{N^c_{\beta,\beta}\}$$

$$-\left(\frac{h_\beta t^t A^c_{,\beta}}{2t^c}\right)\{M^c_\beta\}-\left(\frac{h_\beta t^t A^c}{2t^c}\right)\{M^c_{\beta,\beta}\} \tag{35e}$$

$$+\left(\frac{h_\alpha t^t A^c_{,\beta}}{4}\right)\{N^c_\alpha\}+\left(\frac{h_\alpha t^t A^c_{,\beta}}{2t^c}\right)\{M^c_\alpha\}$$

$$-\left(\frac{h_{\alpha\beta} t^t B^c_{,\alpha}}{4}\right)\{N^c_{\beta\alpha}\}-\left(\frac{h_{\alpha\beta} B^c_{,\alpha} t^t}{4}\right)\{N^c_{\alpha\beta}\}$$

$$-\left(\frac{h_{\alpha\beta} t^t B^c}{4}\right)\{N^c_{\alpha\beta,\alpha}\}-\left(\frac{h_{\alpha\beta} t^t B^c_{,\alpha}}{2t^c}\right)\{M^c_{\beta\alpha}\}$$

$$-\left(\frac{h_{\alpha\beta} t^t B^c_{,\alpha}}{2t^c}\right)\{M^c_{\alpha\beta}\}-\left(\frac{h_{\alpha\beta} t^t B^c}{2t^c}\right)\{M^c_{\alpha\beta,\alpha}\}$$

$$-\left(\frac{t^t}{4R^c_\beta}A^cB^c\right)\{Q^c_{\beta z}\}-\left(\frac{t^t}{2t^cR^c_\beta}A^cB^c\right)\{S^c_{\beta z}\}$$

$$+\left(\frac{t^t}{2t^c}A^cB^c\right)\{Q^c_{z\beta}\}-\left(\frac{t^t}{4R^c_{\alpha\beta}}A^cB^c\right)\{Q^c_{\alpha z}\}$$

$$-\left(\frac{t^t}{2t^cR^c_{\alpha\beta}}A^cB^c\right)\{S^c_{\alpha z}\}$$

$$+(A^t_{,\beta})\{M^t_\beta\}+(A^t)\{M^t_{\beta,\beta}\}-(A^t_{,\beta})\{M^t_\alpha\}$$

$$+(B^t_{,\alpha})\{M^t_{\beta\alpha}\}+(B^t_{,\alpha})\{M^t_{\alpha\beta}\}+(B^t)\{M^t_{\alpha\beta,\alpha}\}$$

$$+\left(\frac{k_s}{R^t_\beta}A^tB^t\right)\{S^t_{\beta z}\}-k_sA^tB^t\{Q^t_{z\beta}\}$$

$$+\left(\frac{k_se_\beta}{R^t_{\alpha\beta}}A^tB^t\right)\{S^t_{\alpha z}\}$$

$$=\Big[(A^cB^c)\left(-\frac{t^t}{8}I^c_0-\frac{t^t}{2t^c}I^c_1-\frac{t^t}{2t^{c2}}I^c_2\right)$$

$$+(A^tB^t)I^t_1\Big]\ddot{v}^t_0+\Big[(A^cB^c)\left(\frac{t^{t2}}{16}I^c_0+\frac{t^{t2}}{4t^c}I^c_1+\frac{t^{t2}I^c_2}{4t^{c2}}\right)$$

$$+(A^tB^t)I^t_2\Big]\ddot{\theta}^t_\beta+(A^cB^c)\Big[\frac{t^tt^{c2}}{16}I^c_0+\frac{t^tt^c}{8}I^c_1 \tag{35e}$$

$$-\frac{t^t}{4}I^c_2-\frac{t^t}{2t^c}I^c_3\Big]\ddot{v}^c_2+(A^cB^c)\Big[\frac{t^tt^{c2}}{16}I^c_1+\frac{t^tt^c}{8}I^c_2$$

$$-\frac{t^t}{4}I^c_3-\frac{t^t}{2t^c}I^c_4\Big]\ddot{v}^c_3+(A^cB^c)\Big[-\frac{t^t}{8}I^c_0+\frac{t^t}{2t^{c2}}I^c_2\Big]\ddot{v}^b_0$$

$$+(A^cB^c)\Big[-\frac{t^tt^b}{16}I^c_0+\frac{t^tt^b}{4t^{c2}}I^c_2\Big]\ddot{\theta}^b_\beta$$

$$-\left(\frac{h_\beta t^t A^c_{,\beta}}{4}\right)\{N^c_\beta\}-\left(\frac{h_\beta t^t A^c}{4}\right)\{N^c_{\beta,\beta}\}$$

$$-\left(\frac{h_\beta t^t A^c_{,\beta}}{2t^c}\right)\{M^c_\beta\}-\left(\frac{h_\beta t^t A^c}{2t^c}\right)\{M^c_{\beta,\beta}\}$$

$$+\left(\frac{h_\alpha t^t A^c_{,\beta}}{4}\right)\{N^c_\alpha\}+\left(\frac{h_\alpha t^t A^c_{,\beta}}{2t^c}\right)\{M^c_\alpha\}$$

$$-\left(\frac{h_{\alpha\beta} t^t B^c_{,\alpha}}{4}\right)\{N^c_{\beta\alpha}\}-\left(\frac{h_{\alpha\beta} B^c_{,\alpha} t^t}{4}\right)\{N^c_{\alpha\beta}\}$$

$$-\left(\frac{h_{\alpha\beta} t^t B^c}{4}\right)\{N^c_{\alpha\beta,\alpha}\}-\left(\frac{h_{\alpha\beta} t^t B^c_{,\alpha}}{2t^c}\right)\{M^c_{\beta\alpha}\}$$

$$-\left(\frac{h_{\alpha\beta} t^t B^c_{,\alpha}}{2t^c}\right)\{M^c_{\alpha\beta}\}-\left(\frac{h_{\alpha\beta} t^t B^c}{2t^c}\right)\{M^c_{\alpha\beta,\alpha}\}$$

$$-\left(\frac{t^t}{4R^c_\beta}A^cB^c\right)\{Q^c_{\beta z}\}-\left(\frac{t^t}{2t^cR^c_\beta}A^cB^c\right)\{S^c_{\beta z}\}$$

$$+\left(\frac{t^t}{2t^c}A^cB^c\right)\{Q^c_{z\beta}\}-\left(\frac{t^t}{4R^c_{\alpha\beta}}A^cB^c\right)\{Q^c_{\alpha z}\}$$

$$
\begin{aligned}
&-\left(\frac{t^t}{2t^c R^c_{\alpha\beta}} A^c B^c\right)\{S^c_{\alpha z}\} + \left(A^t_{,\beta}\right)\{M^t_\beta\} \\
&+(A^t)\{M^t_{\beta,\beta}\} - \left(A^t_{,\beta}\right)\{M^t_\alpha\} + \left(B^t_{,\alpha}\right)\{M^t_{\beta\alpha}\} \\
&+\left(B^t_{,\alpha}\right)\{M^t_{\alpha\beta}\} + (B^t)\{M^t_{\alpha\beta,\alpha}\} + \left(\frac{k_s}{R^t_\beta} A^t B^t\right)\{S^t_{\beta z}\} \\
&-k_s A^t B^t\{Q^t_{z\beta}\} + \left(\frac{k_s e_\beta}{R^t_{\alpha\beta}} A^t B^t\right)\{S^t_{\alpha z}\} \\
&= \Bigg[(A^c B^c)\left(-\frac{t^t}{8} I^c_0 - \frac{t^t}{2t^c} I^c_1 - \frac{t^t}{2{t^c}^2} I^c_2\right) \\
&\quad + (A^t B^t) I^t_1\Bigg] \ddot{v}^t_0 + \Big[(A^c B^c) \\
&\left(\frac{{t^t}^2}{16} I^c_0 + \frac{{t^t}^2}{4t^c} I^c_1 + \frac{{t^t}^2 I^c_2}{4{t^c}^2}\right) + (A^t B^t) I^t_2\Bigg] \ddot{\theta}^t_\beta \\
&+(A^c B^c)\left[\frac{t^t {t^c}^2}{16} I^c_0 + \frac{t^t t^c}{8} I^c_1 - \frac{t^t}{4} I^c_2 - \frac{t^t}{2t^c} I^c_3\right] \ddot{v}^c_2 \\
&+(A^c B^c)\left[\frac{t^t {t^c}^2}{16} I^c_1 + \frac{t^t t^c}{8} I^c_2 - \frac{t^t}{4} I^c_3 - \frac{t^t}{2t^c} I^c_4\right] \ddot{v}^c_3 \\
&+(A^c B^c)\left[-\frac{t^t}{8} I^c_0 + \frac{t^t}{2{t^c}^2} I^c_2\right] \ddot{v}^b_0 \\
&+(A^c B^c)\left[-\frac{t^t t^b}{16} I^c_0 + \frac{t^t t^b}{4{t^c}^2} I^c_2\right] \ddot{\theta}^b_\beta
\end{aligned}
\tag{35e}
$$

$$
\begin{aligned}
&-\left(\frac{h_\alpha {t^c}^2 B^c_{,\alpha}}{4}\right)\{N^c_\alpha\} - \left(\frac{h_\alpha {t^c}^2 B^c}{4}\right)\{N^c_{\alpha,\alpha}\} \\
&+\left(h_\alpha B^c_{,\alpha}\right)\{N^{*c}_\alpha\} + (h_\alpha B^c)\{N^{*c}_{\alpha,\alpha}\} \\
&+\left(\frac{h_\beta {t^c}^2 B^c_{,\alpha}}{4}\right)\{N^c_\beta\} - \left(h_\beta B^c_{,\alpha}\right)\{N^{*c}_\beta\} \\
&-\left(\frac{h_{\alpha\beta} {t^c}^2 A^c_{,\beta}}{4}\right)\{N^c_{\alpha\beta}\} - \left(\frac{h_{\alpha\beta} {t^c}^2 A^c_{,\beta}}{4}\right)\{N^c_{\beta\alpha}\} \\
&-\left(\frac{h_{\alpha\beta} {t^c}^2 A^c}{4}\right)\{N^c_{\beta\alpha,\beta}\} + \left(h_{\alpha\beta} A^c_{,\beta}\right)\{N^{*c}_{\alpha\beta}\} \\
&+\left(h_{\alpha\beta} A^c_{,\beta}\right)\{N^{*c}_{\beta\alpha}\} + \left(h_{\alpha\beta} A^c\right)\{N^{*c}_{\beta\alpha,\beta}\} \\
&-\left(\frac{{t^c}^2}{4R^c_{\alpha\beta}} A^c B^c\right)\{Q^c_{\beta z}\} + \left(\frac{A^c B^c}{R^c_{\alpha\beta}}\right)\{Q^{*c}_{\beta z}\} \\
&-\left(\frac{{t^c}^2}{4R^c_{\alpha}} A^c B^c\right)\{Q^c_{\alpha z}\} + \left(\frac{A^c B^c}{R^c_{\alpha}}\right)\{Q^{*c}_{\alpha z}\} \\
&-2A^c B^c\{S^c_{z\alpha}\} \\
&= (A^c B^c)\left[-\frac{{t^c}^2}{8} I^c_0 - \frac{t^c}{4} I^c_1 + \frac{1}{2} I^c_2 + \frac{1}{t^c} I^c_3\right] \ddot{u}^t_0 \\
&+(A^c B^c)\left[\frac{{t^c}^2 t^t}{16} I^c_0 + \frac{t^c t^t}{8} I^c_1 - \frac{t^t}{4} I^c_2 - \frac{t^t}{2t^c} I^c_3\right] \ddot{\theta}^t_\alpha \\
&+(A^c B^c)\left[\frac{{t^c}^4}{16} I^c_0 - \frac{{t^c}^2}{2} I^c_2 + I^c_4\right] \ddot{u}^c_2
\end{aligned}
\tag{35f}
$$

$$
\begin{aligned}
&+(A^c B^c)\left[\frac{{t^c}^4}{16} I^c_1 - \frac{{t^c}^2}{2} I^c_3 + I^c_5\right] \ddot{u}^c_3 \\
&+(A^c B^c)\left[-\frac{{t^c}^2}{8} I^c_0 + \frac{t^c}{4} I^c_1 + \frac{1}{2} I^c_2 - \frac{1}{t^c} I^c_3\right] \ddot{u}^b_0 \\
&+(A^c B^c)\left[-\frac{{t^c}^2 t^b}{16} I^c_0 + \frac{t^c t^b}{8} I^c_1 + \frac{t^b}{4} I^c_2 - \frac{t^b}{2t^c} I^c_3\right] \ddot{\theta}^b_\alpha
\end{aligned}
\tag{35f}
$$

$$
\begin{aligned}
&\left(\frac{h_\alpha {t^c}^2 A^c_{,\beta}}{4}\right)\{N^c_\alpha\} - \left(h_\alpha A^c_{,\beta}\right)\{N^{*c}_\alpha\} \\
&-\left(\frac{h_\beta {t^c}^2 A^c_{,\beta}}{4}\right)\{N^c_\beta\} - \left(\frac{h_\beta {t^c}^2 A^c}{4}\right)\{N^c_{\beta,\beta}\} \\
&+\left(h_\beta A^c_{,\beta}\right)\{N^{*c}_\beta\} + \left(h_\beta A^c\right)\{N^{*c}_{\beta,\beta}\} \\
&-\left(\frac{h_{\alpha\beta} {t^c}^2 B^c_{,\alpha}}{4}\right)\{N^c_{\beta\alpha}\} - \left(\frac{h_{\alpha\beta} {t^c}^2 B^c_{,\alpha}}{4}\right)\{N^c_{\alpha\beta}\} \\
&-\left(\frac{h_{\alpha\beta} {t^c}^2 B^c}{4}\right)\{N^c_{\alpha\beta,\alpha}\} + \left(h_{\alpha\beta} B^c_{,\alpha}\right)\{N^{*c}_{\beta\alpha}\} \\
&+\left(h_{\alpha\beta} B^c_{,\alpha}\right)\{N^{*c}_{\alpha\beta}\} + \left(h_{\alpha\beta} B^c\right)\{N^{*c}_{\alpha\beta,\alpha}\} \\
&-\left(\frac{{t^c}^2 A^c B^c}{4R^c_{\alpha\beta}}\right)\{Q^c_{\alpha z}\} + \left(\frac{A^c B^c}{R^c_{\alpha\beta}}\right)\{Q^{*c}_{\alpha z}\} \\
&-\left(\frac{{t^c}^2 A^c B^c}{4R^c_{\beta}}\right)\{Q^c_{\beta z}\} + \left(\frac{A^c B^c}{R^c_{\beta}}\right)\{Q^{*c}_{\beta z}\} \\
&-2A^c B^c\{S^c_{z\beta}\} \\
&= (A^c B^c)\left[-\frac{{t^c}^2}{8} I^c_0 - \frac{t^c}{4} I^c_1 + \frac{1}{2} I^c_2 + \frac{1}{t^c} I^c_3\right] \ddot{v}^t_0 \\
&+(A^c B^c)\left[\frac{{t^c}^2 t^t}{16} I^c_0 + \frac{t^c t^t}{8} I^c_1 - \frac{t^t}{4} I^c_2 - \frac{t^t}{2t^c} I^c_3\right] \ddot{\theta}^t_\beta \\
&+(A^c B^c)\left[\frac{{t^c}^4}{16} I^c_0 - \frac{{t^c}^2}{2} I^c_2 + I^c_4\right] \ddot{v}^c_2 \\
&+(A^c B^c)\left[\frac{{t^c}^4}{16} I^c_1 - \frac{{t^c}^2}{2} I^c_3 + I^c_5\right] \ddot{v}^c_3 \\
&+(A^c B^c)\left[-\frac{{t^c}^2}{8} I^c_0 + \frac{t^c}{4} I^c_1 + \frac{1}{2} I^c_2 - \frac{1}{t^c} I^c_3\right] \ddot{v}^b_0 \\
&+(A^c B^c)\left[-\frac{{t^c}^2 t^b}{16} I^c_0 + \frac{t^c t^b}{8} I^c_1 + \frac{t^b}{4} I^c_2 - \frac{t^b}{2t^c} I^c_3\right] \ddot{\theta}^b_\beta
\end{aligned}
\tag{35g}
$$

$$
\begin{aligned}
&\left(\frac{h_\alpha {t^c}^2 A^c B^c}{4R^c_\alpha}\right)\{N^c_\alpha\} - \left(\frac{h_\alpha A^c B^c}{R^c_\alpha}\right)\{N^{*c}_\alpha\} \\
&+\left(\frac{h_\beta {t^c}^2 A^c B^c}{4R^c_\beta}\right)\{N^c_\beta\} - \left(\frac{h_\beta A^c B^c}{R^c_\beta}\right)\{N^{*c}_\beta\} \\
&-2A^c B^c\{M^c_z\} + \left(\frac{h_{\alpha\beta} {t^c}^2 A^c B^c}{4R^c_{\alpha\beta}}\right)\{N^c_{\alpha\beta}\} \\
&+\left(\frac{h_{\alpha\beta} {t^c}^2 A^c B^c}{4R^c_{\alpha\beta}}\right)\{N^c_{\beta\alpha}\} - \left(\frac{h_{\alpha\beta} A^c B^c}{R^c_{\alpha\beta}}\right)\{N^{*c}_{\alpha\beta}\}
\end{aligned}
\tag{35h}
$$

$$
\begin{aligned}
&-\left(\frac{h_{\alpha\beta}A^cB^c}{R^c_{\alpha\beta}}\right)\{N^{*c}_{\beta\alpha}\}-\left(\frac{B^c_{,\alpha}t^{c2}}{4}\right)\{Q^c_{\alpha z}\}\\
&-\left(\frac{t^{c2}B^c}{4}\right)\{Q^c_{\alpha z,\alpha}\}+\left(B^c_{,\alpha}\right)\{Q^{*c}_{\alpha z}\}+(B^c)\{Q^{*c}_{\alpha z,\alpha}\}\\
&-\left(\frac{A^c_{,\beta}t^{c2}}{4}\right)\{Q^c_{\beta z}\}-\left(\frac{t^{c2}A^c}{4}\right)\{Q^c_{\beta z,\beta}\}\\
&+\left(A^c_{,\beta}\right)\{Q^{*c}_{\beta z}\}+(A^c)\{Q^{*c}_{\beta z,\beta}\}\\
&=(A^cB^c)\left[-\frac{t^{c2}}{8}I^c_0-\frac{t^c}{4}I^c_1+\frac{1}{2}I^c_2+\frac{1}{t^c}I^c_3\right]\ddot{w}^t\\
&+(A^cB^c)\left[\frac{t^{c4}}{16}I^c_0-\frac{t^{c2}}{2}I^c_2+I^c_4\right]\ddot{w}^c_2\\
&+(A^cB^c)\left[\frac{t^{c4}}{16}I^c_1-\frac{t^{c2}}{2}I^c_3+I^c_5\right]\ddot{w}^c_3\\
&+(A^cB^c)\left[-\frac{t^{c2}}{8}I^c_0+\frac{t^c}{4}I^c_1+\frac{1}{2}I^c_2-\frac{1}{t^c}I^c_3\right]\ddot{w}^b
\end{aligned}
\tag{35h}
$$

$$
\begin{aligned}
&-\left(\frac{h_\alpha t^{c2}B^c_{,\alpha}}{4}\right)\{M^c_\alpha\}-\left(\frac{h_\alpha t^{c2}B^c}{4}\right)\{M^c_{\alpha,\alpha}\}\\
&+\left(h_\alpha B^c_{,\alpha}\right)\{M^{*c}_\alpha\}+(h_\alpha B^c)\{M^{*c}_{\alpha,\alpha}\}\\
&+\left(\frac{h_\beta t^{c2}B^c_{,\alpha}}{4}\right)\{M^c_\beta\}-\left(h_\beta B_{,\alpha}\right)\{M^{*c}_\beta\}\\
&-\left(\frac{h_{\alpha\beta}t^{c2}A^c_{,\beta}}{4}\right)\{M^c_{\alpha\beta}\}-\left(\frac{h_{\alpha\beta}t^{c2}A^c_{,\beta}}{4}\right)\{M^c_{\beta\alpha}\}\\
&-\left(\frac{h_{\alpha\beta}t^{c2}A^c}{4}\right)\{M^c_{\beta\alpha,\beta}\}+\left(h_{\alpha\beta}A^c_{,\beta}\right)\{M^{*c}_{\alpha\beta}\}\\
&+\left(h_{\alpha\beta}A^c_{,\beta}\right)\{M^{*c}_{\beta\alpha}\}+\left(h_{\alpha\beta}A^c\right)\{M^{*c}_{\beta\alpha,\beta}\}\\
&-\left(\frac{t^{c2}A^cB^c}{4R^c_{\alpha\beta}}\right)\{S^c_{\beta z}\}+\left(\frac{A^cB^c}{R^c_{\alpha\beta}}\right)\{S^{*c}_{\beta z}\}\\
&-\left(\frac{t^{c2}A^cB^c}{4R^c_\alpha}\right)\{S^c_{\alpha z}\}+\left(\frac{t^{c2}A^cB^c}{4}\right)\{Q^c_{z\alpha}\}\\
&+\left(\frac{A^cB^c}{R^c_\alpha}\right)\{S^{*c}_{\alpha z}\}-3A^cB^c\{Q^{*c}_{z\alpha}\}\\
&=(A^cB^c)\left[-\frac{t^{c2}}{8}I^c_1-\frac{t^c}{4}I^c_2+\frac{1}{2}I^c_3+\frac{1}{t^c}I^c_4\right]\ddot{u}^t_0\\
&+(A^cB^c)\left[\frac{t^{c2}t^t}{16}I^c_1+\frac{t^ct^t}{8}I^c_2-\frac{t^t}{4}I^c_3-\frac{t^t}{2t^c}I^c_4\right]\ddot{\theta}^t_\alpha\\
&+(A^cB^c)\left[\frac{t^{c4}}{16}I^c_1-\frac{t^{c2}}{2}I^c_3+I^c_5\right]\ddot{u}^c_2\\
&+(A^cB^c)\left[\frac{t^{c4}}{16}I^c_2-\frac{t^{c2}}{2}I^c_4+I^c_6\right]\ddot{u}^c_3\\
&+(A^cB^c)\left[-\frac{t^{c2}}{8}I^c_1+\frac{t^c}{4}I^c_2+\frac{1}{2}I^c_3-\frac{1}{t^c}I^c_4\right]\ddot{u}^b_0\\
&+(A^cB^c)\left[-\frac{t^{c2}t^b}{16}I^c_1+\frac{t^ct^b}{8}I^c_2+\frac{t^b}{4}I^c_3-\frac{t^b}{2t^c}I^c_4\right]\ddot{\theta}^b_\alpha
\end{aligned}
\tag{35i}
$$

$$
\begin{aligned}
&\left(\frac{h_\alpha t^{c2}A^c_{,\beta}}{4}\right)\{M^c_\alpha\}-\left(h_\alpha A^c_{,\beta}\right)\{M^{*c}_\alpha\}\\
&-\left(\frac{h_\beta t^{c2}A^c_{,\beta}}{4}\right)\{M^c_\beta\}-\left(\frac{h_\beta t^{c2}A^c}{4}\right)\{M^c_{\beta,\beta}\}\\
&+\left(h_\beta A^c_{,\beta}\right)\{M^{*c}_\beta\}+\left(h_\beta A^c\right)\{M^{*c}_{\beta,\beta}\}\\
&-\left(\frac{h_{\alpha\beta}t^{c2}B^c_{,\alpha}}{4}\right)\{M^c_{\beta\alpha}\}-\left(\frac{h_{\alpha\beta}t^{c2}B^c_{,\alpha}}{4}\right)\{M^c_{\alpha\beta}\}\\
&-\left(\frac{h_{\alpha\beta}t^{c2}B^c}{4}\right)\{M^c_{\alpha\beta,\alpha}\}+\left(h_{\alpha\beta}B^c_{,\alpha}\right)\{M^{*c}_{\beta\alpha}\}\\
&+\left(h_{\alpha\beta}B^c_{,\alpha}\right)\{M^{*c}_{\alpha\beta}\}+\left(h_{\alpha\beta}B^c\right)\{M^{*c}_{\alpha\beta,\alpha}\}\\
&-\left(\frac{t^{c2}A^cB^c}{4R^c_{\alpha\beta}}\right)\{S^c_{\alpha z}\}+\left(\frac{A^cB^c}{R^c_{\alpha\beta}}\right)\{S^{*c}_{\alpha z}\}\\
&-\left(\frac{t^{c2}A^cB^c}{4R^c_\beta}\right)\{S^c_{\beta z}\}+\left(\frac{t^{c2}A^cB^c}{4}\right)\{Q^c_{z\beta}\}\\
&+\left(\frac{A^cB^c}{R^c_\beta}\right)\{S^{*c}_{\beta z}\}-3A^cB^c\{Q^{*c}_{z\beta}\}\\
&=(A^cB^c)\left[-\frac{t^{c2}}{8}I^c_1-\frac{t^c}{4}I^c_2+\frac{1}{2}I^c_3+\frac{1}{t^c}I^c_4\right]\ddot{v}^t_0\\
&+(A^cB^c)\left[\frac{t^{c2}t^t}{16}I^c_1+\frac{t^ct^t}{8}I^c_2-\frac{t^t}{4}I^c_3-\frac{t^t}{2t^c}I^c_4\right]\ddot{\theta}^t_\beta\\
&+(A^cB^c)\left[\frac{t^{c4}}{16}I^c_1-\frac{t^{c2}}{2}I^c_3+I^c_5\right]\ddot{v}^c_2\\
&+(A^cB^c)\left[\frac{t^{c4}}{16}I^c_2-\frac{t^{c2}}{2}I^c_4+I^c_6\right]\ddot{v}^c_3\\
&+(A^cB^c)\left[-\frac{t^{c2}}{8}I^c_1+\frac{t^c}{4}I^c_2+\frac{1}{2}I^c_3-\frac{1}{t^c}I^c_4\right]\ddot{v}^b_0\\
&+(A^cB^c)\left[-\frac{t^{c2}t^b}{16}I^c_1+\frac{t^ct^b}{8}I^c_2+\frac{t^b}{4}I^c_3-\frac{t^b}{2t^c}I^c_4\right]\ddot{\theta}^b_\beta
\end{aligned}
\tag{35j}
$$

$$
\begin{aligned}
&\left(\frac{h_\alpha t^{c2}A^cB^c}{4R^c_\alpha}\right)\{M^c_\alpha\}-\left(\frac{h_\alpha A^cB^c}{R^c_\alpha}\right)\{M^{*c}_\alpha\}\\
&+\left(\frac{h_\beta t^{c2}A^cB^c}{4R^c_\beta}\right)\{M^c_\beta\}-\left(\frac{h_\beta A^cB^c}{R^c_\beta}\right)\{M^{*c}_\beta\}\\
&+\left(\frac{t^{c2}A^cB^c}{4}\right)\{N^c_z\}-3A^cB^c\{N^{*c}_z\}\\
&+\left(\frac{h_{\alpha\beta}t^{c2}A^cB^c}{4R^c_{\alpha\beta}}\right)\{M^c_{\alpha\beta}\}+\left(\frac{h_{\alpha\beta}t^{c2}A^cB^c}{4R^c_{\alpha\beta}}\right)\{M^c_{\beta\alpha}\}\\
&-\left(\frac{h_{\alpha\beta}A^cB^c}{R^c_{\alpha\beta}}\right)\{M^{*c}_{\alpha\beta}\}-\left(\frac{h_{\alpha\beta}A^cB^c}{R^c_{\alpha\beta}}\right)\{M^{*c}_{\beta\alpha}\}\\
&-\left(\frac{A^c_{,\beta}t^{c2}}{4}\right)\{S^c_{\beta z}\}-\left(\frac{t^{c2}A^c}{4}\right)\{S^c_{\beta z,\beta}\}\\
&+\left(A^c_{,\beta}\right)\{S^{*c}_{\beta z}\}+(A^c)\{S^{*c}_{\beta z,\beta}\}-\left(\frac{t^{c2}B^c_{,\alpha}}{4}\right)\{S^c_{\alpha z}\}
\end{aligned}
\tag{35k}
$$

$$\begin{aligned}
&-\left(\frac{t^{c2}B^c}{4}\right)\{S^c_{\alpha z,\alpha}\} + \left(B^c_{,\alpha}\right)\{S^{*c}_{\alpha z}\} + (B^c)\{S^{*c}_{\alpha z,\alpha}\} \\
&= (A^cB^c)\left[-\frac{t^{c2}}{8}I_1^c - \frac{t^c}{4}I_2^c + \frac{1}{2}I_3^c + \frac{1}{t^c}I_4^c\right]\ddot{w}^t \\
&\quad +(A^cB^c)\left[\frac{t^{c4}}{16}I_1^c - \frac{t^{c2}}{2}I_3^c + I_5^c\right]\ddot{w}_2^c \\
&\quad +(A^cB^c)\left[\frac{t^{c4}}{16}I_2^c - \frac{t^{c2}}{2}I_4^c + I_6^c\right]\ddot{w}_3^c \\
&\quad +(A^cB^c)\left[-\frac{t^{c2}}{8}I_1^c + \frac{t^c}{4}I_2^c + \frac{1}{2}I_3^c - \frac{1}{t^c}I_4^c\right]\ddot{w}^b
\end{aligned}\tag{35k}$$

$$\begin{aligned}
&\left(\frac{h_\alpha B^c_{,\alpha}}{2}\right)\{N^c_\alpha\} + \left(\frac{h_\alpha B^c}{2}\right)\{N^c_{\alpha,\alpha}\} - \left(\frac{h_\alpha B^c_{,\alpha}}{t^c}\right)\{M^c_\alpha\} \\
&-\left(\frac{h_\alpha B^c}{t^c}\right)\{M^c_{\alpha,\alpha}\} - \left(\frac{h_\beta B^c_{,\alpha}}{2}\right)\{N^c_\beta\} + \left(\frac{h_\beta B^c_{,\alpha}}{t^c}\right)\{M^c_\beta\} \\
&+\left(\frac{h_{\alpha\beta} A^c_{,\beta}}{2}\right)\{N^c_{\alpha\beta}\} + \left(\frac{h_{\alpha\beta} A^c_{,\beta}}{2}\right)\{N^c_{\beta\alpha}\} \\
&+\left(\frac{h_{\alpha\beta} A^c}{2}\right)\{N^c_{\beta\alpha,\beta}\} - \left(\frac{h_{\alpha\beta} A^c_{,\beta}}{t^c}\right)\{M^c_{\alpha\beta}\} \\
&-\left(\frac{h_{\alpha\beta} A^c_{,\beta}}{t^c}\right)\{M^c_{\beta\alpha}\} - \left(\frac{h_{\alpha\beta} A^c}{t^c}\right)\{M^c_{\beta\alpha,\beta}\} \\
&+\left(\frac{A^cB^c}{2R^c_\alpha}\right)\{Q^c_{\alpha z}\} - \left(\frac{A^cB^c}{t^cR^c_\alpha}\right)\{S^c_{\alpha z}\} + \left(\frac{A^cB^c}{t^c}\right)\{Q^c_{z\alpha}\} \\
&+\left(\frac{A^cB^c}{2R^c_{\alpha\beta}}\right)\{Q^c_{\beta z}\} - \left(\frac{A^cB^c}{t^cR^c_{\alpha\beta}}\right)\{S^c_{\beta z}\} \\
&+\left(B^b_{,\alpha}\right)\{N^b_\alpha\} + (B^b)\{N^b_{\alpha,\alpha}\} - \left(B^b_{,\alpha}\right)\{N^b_\beta\} \\
&+\left(A^b_{,\beta}\right)\{N^b_{\alpha\beta}\} + \left(A^b_{,\beta}\right)\{N^b_{\beta\alpha}\} + (A^b)\{N^b_{\beta\alpha,\beta}\} \\
&+\left(\frac{k_sA^bB^b}{R^b_\alpha}\right)\{Q^b_{\alpha z}\} + \left(\frac{k_sA^bB^b}{R^b_{\alpha\beta}}\right)\{Q^b_{\beta z}\} \\
&= (A^cB^c)\left[\frac{1}{4}I_0^c - \frac{1}{t^{c2}}I_2^c\right]\ddot{u}_0^t \\
&\quad +(A^cB^c)\left[-\frac{t^t}{8}I_0^c + \frac{t^t}{2t^{c2}}I_2^c\right]\ddot{\theta}_\alpha^t \\
&\quad +(A^cB^c)\left[-\frac{t^{c2}}{8}I_0^c + \frac{t^c}{4}I_1^c + \frac{1}{2}I_2^c - \frac{1}{t^c}I_3^c\right]\ddot{u}_2^c \\
&\quad +(A^cB^c)\left[-\frac{t^{c2}}{8}I_1^c + \frac{t^c}{4}I_2^c + \frac{1}{2}I_3^c - \frac{1}{t^c}I_4^c\right]\ddot{u}_3^c \\
&\quad +\left[(A^cB^c)\left(\frac{1}{4}I_0^c - \frac{1}{t^c}I_1^c + \frac{1}{t^{c2}}I_2^c\right) + (A^bB^b)I_0^b\right]\ddot{u}_0^b \\
&\quad +\left[(A^cB^c)\left(\frac{t^b}{8}I_0^c - \frac{t^b}{2t^c}I_1^c + \frac{t^b}{2t^{c2}}I_2^c\right)\right. \\
&\qquad \left. + (A^bB^b)I_1^b\right]\ddot{\theta}_\alpha^b
\end{aligned}\tag{35l}$$

$$\begin{aligned}
&-\left(\frac{h_\alpha A^c_{,\beta}}{2}\right)\{N^c_\alpha\} + \left(\frac{h_\alpha A^c_{,\beta}}{t^c}\right)\{M^c_\alpha\} \\
&+\left(\frac{h_\beta A^c_{,\beta}}{2}\right)\{N^c_\beta\} + \left(\frac{h_\beta A^c}{2}\right)\{N^c_{\beta,\beta}\} \\
&-\left(\frac{h_\beta A^c_{,\beta}}{t^c}\right)\{M^c_\beta\} - \left(\frac{h_\beta A^c}{t^c}\right)\{M^c_{\beta,\beta}\} \\
&+\left(\frac{h_{\alpha\beta} B^c_{,\alpha}}{2}\right)\{N^c_{\alpha\beta}\} + \left(\frac{h_{\alpha\beta} B^c}{2}\right)\{N^c_{\alpha\beta,\alpha}\} \\
&+\left(\frac{h_{\alpha\beta} B^c_{,\alpha}}{2}\right)\{N^c_{\beta\alpha}\} - \left(\frac{h_{\alpha\beta} B^c_{,\alpha}}{t^c}\right)\{M^c_{\alpha\beta}\} \\
&-\left(\frac{h_{\alpha\beta} B^c}{t^c}\right)\{M^c_{\alpha\beta,\alpha}\} - \left(\frac{h_{\alpha\beta} B^c_{,\alpha}}{t^c}\right)\{M^c_{\beta\alpha}\} \\
&+\left(\frac{A^cB^c}{2R^c_{\alpha\beta}}\right)\{Q^c_{\alpha z}\} - \left(\frac{A^cB^c}{t^cR^c_{\alpha\beta}}\right)\{S^c_{\alpha z}\} \\
&+\left(\frac{A^cB^c}{2R^c_\beta}\right)\{Q^c_{\beta z}\} - \left(\frac{A^cB^c}{t^cR^c_\beta}\right)\{S^c_{\beta z}\} \\
&+\left(\frac{A^cB^c}{t^c}\right)\{Q^c_{z\beta}\} - \left(A^b_{,\beta}\right)\{N^b_\alpha\} + \left(A^b_{,\beta}\right)\{N^b_\beta\} \\
&+(A^b)\{N^b_{\beta,\beta}\} + \left(B^b_{,\alpha}\right)\{N^b_{\alpha\beta}\} + (B^b)\{N^b_{\alpha\beta,\alpha}\} \\
&+\left(B^b_{,\alpha}\right)\{N^b_{\beta\alpha}\} + \left(\frac{k_sA^bB^b}{R^b_{\alpha\beta}}\right)\{Q^b_{\alpha z}\} \\
&+\left(\frac{k_sA^bB^b}{R^b_\beta}\right)\{Q^b_{\beta z}\} \\
&= (A^cB^c)\left[\frac{1}{4}I_0^c - \frac{1}{t^{c2}}I_2^c\right]\ddot{v}_0^t \\
&\quad +(A^cB^c)\left[-\frac{t^t}{8}I_0^c + \frac{t^t}{2t^{c2}}I_2^c\right]\ddot{\theta}_\beta^t \\
&\quad +(A^cB^c)\left[-\frac{t^{c2}}{8}I_0^c + \frac{t^c}{4}I_1^c + \frac{1}{2}I_2^c - \frac{1}{t^c}I_3^c\right]\ddot{v}_2^c \\
&\quad +(A^cB^c)\left[-\frac{t^{c2}}{8}I_1^c + \frac{t^c}{4}I_2^c + \frac{1}{2}I_3^c - \frac{1}{t^c}I_4^c\right]\ddot{v}_3^c \\
&\quad +\left[(A^cB^c)\left(\frac{1}{4}I_0^c - \frac{1}{t^c}I_1^c + \frac{1}{t^{c2}}I_2^c\right)\right. \\
&\qquad \left. + (A^bB^b)I_0^b\right]\ddot{v}_0^b \\
&\quad +\left[(A^cB^c)\left(\frac{t^b}{8}I_0^c - \frac{t^b}{2t^c}I_1^c + \frac{t^b}{2t^{c2}}I_2^c\right)\right. \\
&\qquad \left. + (A^bB^b)I_1^b\right]\ddot{\theta}_\beta^b
\end{aligned}\tag{35m}$$

$$\begin{aligned}
&-\left(\frac{h_\alpha A^cB^c}{2R^c_\alpha}\right)\{N^c_\alpha\} + \left(\frac{h_\alpha A^cB^c}{t^cR^c_\alpha}\right)\{M^c_\alpha\} \\
&-\left(\frac{h_\beta A^cB^c}{2R^c_\beta}\right)\{N^c_\beta\} + \left(\frac{h_\beta A^cB^c}{t^cR^c_\beta}\right)\{M^c_\beta\} \\
&+\left(\frac{A^cB^c}{t^c}\right)\{N^c_z\} - \left(\frac{h_{\alpha\beta} A^cB^c}{2R^c_{\alpha\beta}}\right)\{N^c_{\alpha\beta}\}
\end{aligned}\tag{35n}$$

$$-\left(\frac{h_{\alpha\beta}A^cB^c}{2R^c_{\alpha\beta}}\right)\{N^c_{\beta\alpha}\}+\left(\frac{h_{\alpha\beta}A^cB^c}{t^cR^c_{\alpha\beta}}\right)\{M^c_{\alpha\beta}\}$$
$$+\left(\frac{h_{\alpha\beta}A^cB^c}{t^cR^c_{\alpha\beta}}\right)\{M^c_{\beta\alpha}\}+\left(\frac{B^c_{,\alpha}}{2}\right)\{Q^c_{\alpha z}\}$$
$$+\left(\frac{B^c}{2}\right)\{Q^c_{\alpha z,\alpha}\}-\left(\frac{B^c_{,\alpha}}{t^c}\right)\{S^c_{\alpha z}\}-\left(\frac{B^c}{t^c}\right)\{S^c_{\alpha z,\alpha}\}$$
$$+\left(\frac{A^c_{,\beta}}{2}\right)\{Q^c_{\beta z}\}+\left(\frac{A^c}{2}\right)\{Q^c_{\beta z,\beta}\}-\left(\frac{A^c_{,\beta}}{t^c}\right)\{S^c_{\beta z}\}$$
$$-\left(\frac{A^c}{t^c}\right)\{S^c_{\beta z,\beta}\}-\left(\frac{A^bB^b}{R^b_\alpha}\right)\{N^b_\alpha\}-\left(\frac{A^bB^b}{R^b_\beta}\right)\{N^b_\beta\}$$
$$-\left(\frac{A^bB^b}{R^b_{\alpha\beta}}\right)\{N^b_{\alpha\beta}\}-\left(\frac{A^bB^b}{R^b_{\alpha\beta}}\right)\{N^b_{\beta\alpha}\}+\left(k_sB^b_{,\alpha}\right)\{Q^b_{\alpha z}\} \quad (35n)$$
$$+(k_sB^b)\{Q^b_{\alpha z,\alpha}\}+\left(k_sA^b_{,\beta}\right)\{Q^b_{\beta z}\}+(k_sA^b)\{Q^b_{\beta z,\beta}\}$$
$$=(A^cB^c)\left[\frac{1}{4}I^c_0-\frac{1}{{t^c}^2}I^c_2\right]\ddot{w}^t$$
$$+(A^cB^c)\left[-\frac{{t^c}^2}{8}I^c_0+\frac{t^c}{4}I^c_1+\frac{1}{2}I^c_2-\frac{1}{t^c}I^c_3\right]\ddot{w}^c_2$$
$$+(A^cB^c)\left[-\frac{{t^c}^2}{8}I^c_1+\frac{t^c}{4}I^c_2+\frac{1}{2}I^c_3-\frac{1}{t^c}I^c_4\right]\ddot{w}^c_3$$
$$+\left[(A^cB^c)\left(\frac{1}{4}I^c_0-\frac{1}{t^c}I^c_1+\frac{1}{{t^c}^2}I^c_2\right)+(A^bB^b)I^b_0\right]\ddot{w}^b$$

$$\left(\frac{h_\alpha t^bB^c_{,\alpha}}{4}\right)\{N^c_\alpha\}+\left(\frac{h_\alpha t^bB^c}{4}\right)\{N^c_{\alpha,\alpha}\}$$
$$-\left(\frac{h_\alpha t^bB^c_{,\alpha}}{2t^c}\right)\{M^c_\alpha\}-\left(\frac{h_\alpha t^bB^c}{2t^c}\right)\{M^c_{\alpha,\alpha}\}$$
$$-\left(\frac{h_\beta t^bB^c_{,\alpha}}{4}\right)\{N^c_\beta\}+\left(\frac{h_\beta t^bB^c_{,\alpha}}{2t^c}\right)\{M^c_\beta\}$$
$$+\left(\frac{h_{\alpha\beta}t^bA^c_{,\beta}}{4}\right)\{N^c_{\alpha\beta}\}+\left(\frac{h_{\alpha\beta}A^c_{,\beta}t^b}{4}\right)\{N^c_{\beta\alpha}\}$$
$$+\left(\frac{h_{\alpha\beta}t^bA^c}{4}\right)\{N^c_{\beta\alpha,\beta}\}-\left(\frac{h_{\alpha\beta}t^bA^c_{,\beta}}{2t^c}\right)\{M^c_{\alpha\beta}\}$$
$$-\left(\frac{h_{\alpha\beta}t^bA^c_{,\beta}}{2t^c}\right)\{M^c_{\beta\alpha}\}-\left(\frac{h_{\alpha\beta}t^bA^c}{2t^c}\right)\{M^c_{\beta\alpha,\beta}\}$$
$$+\left(\frac{t^bA^cB^c}{4R^c_\alpha}\right)\{Q^c_{\alpha z}\}-\left(\frac{t^bA^cB^c}{2t^cR^c_\alpha}\right)\{S^c_{\alpha z}\} \quad (35o)$$
$$+\left(\frac{t^bA^cB^c}{2t^c}\right)\{Q^c_{z\alpha}\}+\left(\frac{t^bA^cB^c}{4R^c_{\alpha\beta}}\right)\{Q^c_{\beta z}\}$$
$$-\left(\frac{t^bA^cB^c}{2t^cR^c_{\alpha\beta}}\right)\{S^c_{\beta z}\}+\left(B^b_{,\alpha}\right)\{M^b_\alpha\}+(B^b)\{M^b_{\alpha,\alpha}\}$$
$$-\left(B^b_{,\alpha}\right)\{M^b_\beta\}+\left(A^b_{,\beta}\right)\{M^b_{\alpha\beta}\}+\left(A^b_{,\beta}\right)\{M^b_{\beta\alpha}\}$$
$$+(A^b)\{M^b_{\beta\alpha,\beta}\}+\left(\frac{k_sA^bB^b}{R^b_\alpha}\right)\{S^b_{\alpha z}\}$$
$$-k_sA^bB^b\{Q^b_{z\alpha}\}+\left(\frac{k_se_\alpha A^bB^b}{R^b_{\alpha\beta}}\right)\{S^b_{\beta z}\}$$
$$=(A^cB^c)\left[\frac{t^b}{8}I^c_0-\frac{t^b}{2{t^c}^2}I^c_2\right]\ddot{u}^t_0$$
$$+(A^cB^c)\left[-\frac{t^tt^b}{16}I^c_0+\frac{t^tt^b}{4{t^c}^2}I^c_2\right]\ddot{\theta}^t_\alpha$$
$$+(A^cB^c)\left[-\frac{t^b{t^c}^2}{16}I^c_0+\frac{t^bt^c}{8}I^c_1+\frac{t^b}{4}I^c_2-\frac{t^b}{2t^c}I^c_3\right]\ddot{u}^c_2$$
$$+(A^cB^c)\left[-\frac{t^b{t^c}^2}{16}I^c_1+\frac{t^bt^c}{8}I^c_2+\frac{t^b}{4}I^c_3-\frac{t^b}{2t^c}I^c_4\right]\ddot{u}^c_3 \quad (35o)$$
$$+\left[(A^cB^c)\left(\frac{t^b}{8}I^c_0-\frac{t^b}{2t^c}I^c_1+\frac{t^b}{2{t^c}^2}I^c_2\right)+(A^bB^b)I^b_1\right]\ddot{u}^b_0$$
$$+\left[(A^cB^c)\left(\frac{{t^b}^2}{16}I^c_0-\frac{{t^b}^2}{4t^c}I^c_1+\frac{{t^b}^2I^c_2}{4{t^c}^2}\right)+(A^bB^b)I^b_2\right]\ddot{\theta}^b_\alpha$$

$$\left(\frac{h_\beta t^bA^c_{,\beta}}{4}\right)\{N^c_\beta\}+\left(\frac{h_\beta t^bA^c}{4}\right)\{N^c_{\beta,\beta}\}$$
$$-\left(\frac{h_\beta t^bA^c_{,\beta}}{2t^c}\right)\{M^c_\beta\}-\left(\frac{h_\beta t^bA^c}{2t^c}\right)\{M^c_{\beta,\beta}\}$$
$$-\left(\frac{h_\alpha t^bA^c_{,\beta}}{4}\right)\{N^c_\alpha\}+\left(\frac{h_\alpha t^bA^c_{,\beta}}{2t^c}\right)\{M^c_\alpha\}$$
$$+\left(\frac{h_{\alpha\beta}t^bB^c_{,\alpha}}{4}\right)\{N^c_{\beta\alpha}\}+\left(\frac{h_{\alpha\beta}B^c_{,\alpha}t^b}{4}\right)\{N^c_{\alpha\beta}\}$$
$$+\left(\frac{h_{\alpha\beta}t^bB^c}{4}\right)\{N^c_{\alpha\beta,\alpha}\}-\left(\frac{h_{\alpha\beta}t^bB^c_{,\alpha}}{2t^c}\right)\{M^c_{\beta\alpha}\}$$
$$-\left(\frac{h_{\alpha\beta}t^bB^c_{,\alpha}}{2t^c}\right)\{M^c_{\alpha\beta}\}-\left(\frac{h_{\alpha\beta}t^bB^c}{2t^c}\right)\{M^c_{\alpha\beta,\alpha}\}$$
$$+\left(\frac{t^bA^cB^c}{4R^c_\beta}\right)\{Q^c_{\beta z}\}-\left(\frac{t^bA^cB^c}{2t^cR^c_\beta}\right)\{S^c_{\beta z}\} \quad (35p)$$
$$+\left(\frac{t^bA^cB^c}{2t^c}\right)\{Q^c_{z\beta}\}+\left(\frac{t^bA^cB^c}{4R^c_{\alpha\beta}}\right)\{Q^c_{\alpha z}\}$$
$$-\left(\frac{t^bA^cB^c}{2t^cR^c_{\alpha\beta}}\right)\{S^c_{\alpha z}\}+\left(A^b_{,\beta}\right)\{M^b_\beta\}+(A^c)\{M^b_{\beta,\beta}\}$$
$$-\left(A^b_{,\beta}\right)\{M^b_\alpha\}+\left(B^b_{,\alpha}\right)\{M^b_{\beta\alpha}\}+\left(B^b_{,\alpha}\right)\{M^b_{\alpha\beta}\}$$
$$+(B^b)\{M^b_{\alpha\beta,\alpha}\}+\left(\frac{k_sA^bB^b}{R^b_\beta}\right)\{S^b_{\beta z}\}$$
$$-k_sA^bB^b\{Q^b_{z\beta}\}+\left(\frac{k_se_\beta A^bB^b}{R^b_{\alpha\beta}}\right)\{S^b_{\alpha z}\}$$
$$=(A^cB^c)\left[\frac{t^b}{8}I^c_0-\frac{t^b}{2{t^c}^2}I^c_2\right]\ddot{v}^t_0$$
$$+(A^cB^c)\left[-\frac{t^tt^b}{16}I^c_0+\frac{t^tt^b}{4{t^c}^2}I^c_2\right]\ddot{\theta}^t_\beta$$

$$
\begin{aligned}
&+(A^cB^c)\left[-\frac{t^b t^{c^2}}{16} I_0^c + \frac{t^b t^c}{8} I_1^c + \frac{t^b}{4} I_2^c - \frac{t^b}{2t^c} I_3^c\right]\ddot{v}_2^c \\
&+(A^cB^c)\left[-\frac{t^b t^{c^2}}{16} I_1^c + \frac{t^b t^c}{8} I_2^c + \frac{t^b}{4} I_3^c - \frac{t^b}{2t^c} I_4^c\right]\ddot{v}_3^c \\
&+\left[(A^cB^c)\left(\frac{t^b}{8} I_0^c - \frac{t^b}{2t^c} I_1^c + \frac{t^b}{2t^{c^2}} I_2^c\right)\right. \\
&\quad \left. + (A^bB^b) I_1^b\right]\ddot{v}_0^b \\
&+\left[(A^cB^c)\left(\frac{t^{b^2}}{16} I_0^c - \frac{t^{b^2}}{4t^c} I_1^c + \frac{t^{b^2} I_2^c}{4t^{c^2}}\right)\right. \\
&\quad \left. + (A^bB^b) I_2^b\right]\ddot{\theta}_\beta^b
\end{aligned}
\tag{35p}
$$

Boundary conditions are

at $\alpha_1 = 0$ and $\alpha_2 = \alpha$	**at $\beta_1 = 0$ and $\beta_2 = \beta$**
$\delta u_0^t = 0$ or $B^t N_\alpha^t = 0$	$\delta u_0^t = 0$ or $A^t N_{\beta\alpha}^t = 0$
$\delta v_0^t = 0$ or $B^t N_{\alpha\beta}^t = 0$	$\delta v_0^t = 0$ or $A^t N_\beta^t = 0$
$\delta w^t = 0$ or $B^t Q_{\alpha z}^t = 0$	$\delta w^t = 0$ or $A^t Q_{\beta z}^t = 0$
$\delta \theta_\alpha^t = 0$ or $B^t M_\alpha^t = 0$	$\delta \theta_\alpha^t = 0$ or $A^t M_{\beta\alpha}^t = 0$
$\delta \theta_\beta^t = 0$ or $B^t M_{\alpha\beta}^t = 0$	$\delta \theta_\beta^t = 0$ or $A^t M_\beta^t = 0$
$\delta u_2^c = 0$ or $h_\alpha B^c N_\alpha^{*c} = 0$	$\delta u_2^c = 0$ or $h_\alpha A^c N_{\beta\alpha}^{*c} = 0$
$\delta v_2^c = 0$ or $h_{\alpha\beta} B^c N_{\alpha\beta}^{*c} = 0$	$\delta v_2^c = 0$ or $h_{\alpha\beta} A^c N_\beta^{*c} = 0$
$\delta w_2^c = 0$ or $B^c Q_{\alpha z}^{*c} = 0$	$\delta w_2^c = 0$ or $A^c Q_{\beta z}^{*c} = 0$ (36)
$\delta u_3^c = 0$ or $h_\alpha B^c M_\alpha^{*c} = 0$	$\delta u_3^c = 0$ or $h_\alpha A^c M_{\beta\alpha}^{*c} = 0$
$\delta v_3^c = 0$ or $h_{\alpha\beta} B^c M_{\alpha\beta}^{*c} = 0$	$\delta v_3^c = 0$ or $h_{\alpha\beta} A^c M_\beta^{*c} = 0$
$\delta w_3^c = 0$ or $B^c S_{\alpha z}^{*c} = 0$	$\delta w_3^c = 0$ or $A^c S_{\beta z}^{*c} = 0$
$\delta u_0^b = 0$ or $B^b N_\alpha^b = 0$	$\delta u_0^b = 0$ or $A^b N_{\beta\alpha}^b = 0$
$\delta v_0^b = 0$ or $B^b N_{\alpha\beta}^b = 0$	$\delta v_0^b = 0$ or $A^b N_\beta^b = 0$
$\delta w^b = 0$ or $B^b Q_{\alpha z}^b = 0$	$\delta w^b = 0$ or $A^b Q_{\beta z}^b = 0$
$\delta \theta_\alpha^b = 0$ or $B^b M_\alpha^b = 0$	$\delta \theta_\alpha^b = 0$ or $A^b M_{\beta\alpha}^b = 0$
$\delta \theta_\beta^b = 0$ or $B^b M_{\alpha\beta}^b = 0$	$\delta \theta_\beta^b = 0$ or $A^b M_\beta^b = 0$

By substituting components of the face sheets and the core resultant in Eq. (35) and considering the strain components (Eqs. (21), (11) and (12)), the equations of motion are expressed as follows

$$[L]_{(16,16)}\{d\}_{(16*1)} = 0 \tag{37}$$

In Eq. (37), L_{ij} are differential operators and the matrix is (Liew and Lim 1996)

$$\{d\} = \begin{Bmatrix} u_0^t, v_0^t, w^t, \theta_\alpha^t, \theta_\beta^t, u_2^c, v_2^c, w_2^c, u_3^c, v_3^c \\ , w_3^c, u_0^b, v_0^b, w^b, \theta_\alpha^b, \theta_\beta^b \end{Bmatrix}^T \tag{38}$$

2.9 Free vibration analysis

In this section, the Galerkin method based on the double Fourier series is used for free vibration analysis of simply-supported thick orthotropic DCSP. Simply-supported B.C., implies the following conditions (Qato 2004)

$$
\begin{gathered}
v^i = w^i = N_\alpha^i = M_\alpha^i = N_\alpha^{*c} = M_\alpha^{*c} = 0, \\
\text{on an edge } \alpha = 0, a \\
u^i = w^i = N_\beta^i = M_\beta^i = N_\beta^{*c} = M_\beta^{*c} = 0, \\
\text{on an edge } \beta = 0, b \quad \text{that} \quad i = t, b, c
\end{gathered}
\tag{39}
$$

The component of generalized displacement field is considered as follow

$$\{d_{i,1}\} = \Delta.\ T_{mn}(t), \quad i = 1, \dots, 16 \tag{40}$$

where $T_{mn}(t) = e^{i\omega_{mn} t}$, $i = \sqrt{-1}$ and ω_{mn} is the natural frequency; $\{\Delta\}$ is the weighting functions vector which is

$$\{\Delta\} = \sum_m \sum_n (d_{mn}[1, i].\psi[\mathrm{i}, 1]) \quad \text{that} \quad i = 1, \dots, 16, \tag{41}$$

where $\{d_{mn}\}$ and $\{\psi\}$ are the natural mode shape constants and natural mode shape vector, respectively, which are

$$\{d_{mn}\} = \begin{Bmatrix} u_{0mn}^t, v_{0mn}^t, w_{mn}^t, \theta_{\alpha mn}^t, \theta_{\beta mn}^t, \\ u_{2mn}^c, v_{2mn}^c, w_{2mn}^c, u_{3mn}^c, v_{3mn}^c, w_{3mn}^c, \\ u_{0mn}^b, v_{0mn}^b, w_{mn}^b, \theta_{\alpha mn}^b, \theta_{\beta mn}^b \end{Bmatrix}^T \tag{42a}$$

$$\{\psi\} = \begin{Bmatrix} C.S \quad S.C \quad C.S \\ C.S \quad S.C \quad S.S \\ S.C \quad S.S \quad C.S \\ S.C \quad S.S \quad C.S \\ S.C \quad S.S \quad C.S \quad S.C \end{Bmatrix}^T \tag{42b}$$

$$\text{that} \quad S = \sin(p\alpha), \quad C = \cos(p\alpha)$$

where $p = \frac{m\pi\alpha}{a}$, $q = \frac{n\pi\beta}{b}$; m and n are the numbers of longitudinal half wave and circumferential wave, respectively. Then, by substituting Eq. (41) into equations of motions and applying Galerkin method yields

$$\{\psi\} = \begin{Bmatrix} C.S \quad S.C \quad C.S \\ C.S \quad S.C \quad S.S \\ S.C \quad S.S \quad C.S \\ S.C \quad S.S \quad C.S \\ S.C \quad S.S \quad C.S \quad S.C \end{Bmatrix}^T \tag{43}$$

$$\text{that} \quad S = \sin(p\alpha), \quad C = \cos(p\alpha)$$

By integrating Eq. (43) and collecting coefficients, the eigenvalue equations are obtained as follow

$$\{[K] - \lambda_{mn}[M]\}\{d\} = 0 \tag{44}$$

Where $[K]$ and $[M]$ are the stiffness and the mass matrices, respectively. Also, $\lambda_{mn} = \omega_{mn}^2$ is the mode shape vector coefficients for any value of m and n. The eigenvalue of Eq. (44) can be solved for various eigenvalues and is associated to eigenvectors. Fundamental frequency of vibration is the lowest eigen value λ_{mn}.

3. Results and discussion

3.1 Validation

This section deals with the analysis of free vibration of thick DCSP based on an efficient computer program developed for numerical analysis of DCSP equations of motion obtained by Galerkin method. The purpose is to compute the natural frequency based on High-order sandwich panel theory (HSAPT) by considering all of the stress components in the core and the face sheets. First, the results of the present model in this paper are compared with the results in the literature by considering various geometries such as sandwich plate, cylindrical sandwich panel and spherical sandwich panel (compressible and incompressible) well as various radii curvature and thicknesses. It is important to note that the core is compressible when in-plane stress is not considered and is incompressible when in-plane stress is considered.

A three-layer laminated sandwich panel with fiber reinforced polymer (FRP) face sheet made of glass fiber reinforced polyester and HerexC70.130 PVC foam core are considered. The mechanical properties for the core and the face sheets are given in Table 1 and are used for the validation process and the free vibration analysis.

3.1.1 Example 1.Consider an antisymmetric cross-ply (0/90/core/0/90) laminated square flat composite sandwich panel

Table 2 shows the six non-dimensional natural frequencies (NDNF) $\Omega = \omega a^2 \sqrt{\rho_c / E_2^c} / H$ of antisymmetric cross-ply laminated sandwich panel composite with plane form laminates ($a/b = 1$, square plate), the side-to-thickness ratio ($a/H = 10$) and the core thickness to face thickness ratio $t_c/t_t = 10$.

The results are compared with those available in the existed literature as follows:

(1) The results by Biglari and Jafari (2010) who used an analytical displacement method as the High-order Sandwich Panel Theory (HSAPT) (see Frostig and Thomson 2004) and Mixed Layer-Wise Theory (MLWT) (see Rao and Desai 2004). In their method, the order of core displacement for u and v is 3 and w is 2 and the face sheets are based on FSDT with neglected in-plane stress in the core.
(2) The results by Rao and Desai (2004) based on mixed layerwise theory (MLWT).
(3) The results by Ćetković and Vuksanović (2009) using finite element method (FEM).
(4) The results reported by Rahmani *et al*. (2010)

Table 1 Materials properties used for the analysis (Garg *et al*. 2006)

Material properties	Face sheets	Core
	E_1 = 131 GPa, $E_2 = E_3$ = 10.34 GPa	$E_1 = E_2 = E_3$ = 0.00689 GPa
(0/90/core/0/90)	$G_{12} = G_{13}$ = 6.895 GPa, G_{13} = 6.205 GPa	$G_{12} = G_{13} = G_{23}$ = 3.45 GPa
	$\nu_{12} = \nu_{13}$ = 0.22, ν_{23} = 0.49, ρ = 1627 kg/m^3	ν = 0, ρ = 94.195 kg/m^3

Table 2 Comparison of the first of six NDNF Ω of simply supported antisymmetric (0/90/core/0/90) sandwich plate with $a/b = 1$ and $t_c/t_t = 10$

m, *n*	Present results	HSART	MLWT	FEM	ANSYS	ESL
1,1	1.8577 (12.21%)	1.8627 (12.51%)	1.848 (11.62%)	1.8627 (12.69%)	1.6556	4.8 (193%)
1,2	3.2667 (15.64%)	3.2799 (16.12%)	3.2196 (13.98%)	3.2882 (16.41%)	2.8247	8.0 (183%)
2,2	4.3493 (9.72%)	4.3843 (10.60%)	4.2894 (8.21%)	4.3981 (15.02%)	3.9641	10.3 (159%)
1,3	5.3594 (14.08%)	5.3902 (14.73%)	5.2234 (11.18%)	5.4040 (15.02%)	4.6981	11.7 (149%)
2,3	6.1786 (9.83%)	6.2840 (11.71%)	6.0942 (8.33%)	6.3024 (12.03%)	5.6254	13.5 (139%)
3,3	7.7663 (6.70%)	7.9414 (9.11%)	7.6762 (5.47%)	7.9629 (9.41%)	7.2783	16.1 (121%)

*Numbers in parentheses are the discrepancies with repect to ANSYS results (Rahani *et al*. 2010)

Table 3 Comparison of the first mode of NDNF Ω of simply supported antisymmetric (0/90/core/0/90) sandwich plate with $a/b = 1$ and $a/H = 10$

t_c / t_t	Present results	HOST11	ESL	Reddy	ESL/FSDT
4	1.9458	9.1427	8.9948	10.7409	13.919
10	1.8574	4.9586	4.8594	7.0473	13.8694
20	2.134	3.1824	3.1435	4.3734	12.8946
30	2.3345	2.8646	2.8481	3.4815	11.976
40	2.4712	2.8348	2.8266	3.1664	11.2036
50	2.568	2.8669	2.8625	3.0561	10.5557
100	2.7898	3.0293	3.029	3.05	8.4349

obtained by parametric design language (APDL) of ANSYS commercial FE code using a shell/solid/shell layered model.

(5) The results by Kant and Swaminathan's model (2001) based on ESL model with 12 degrees of freedom.

For further validation of the analytical method presented in the present article, the variation of fundamental frequency ($m, n = 1$) of the present model based on different t_c/t_t and $a/H = 10$ is compared with those results reported in the literature by Khare (higher-order shear deformation theory with 11 displacement components (HOST11) (Garg *et al*. 2006), Kant and Swaminathan (ESL) (2001), Reddy (2003) and Pagano (ESL/FSDT) (1970). The comparisons are tabulated in Table 3. Considering the results obtained by ANSYS (Rahmani *et al*. 2010) in Table 2, the results achieved by the model presented in the current study show more accuracy as compared with those obtained by HSAPT (2010), FEM (2009) and ESL (2001). The reason behind is consideration of the in-plane stresses for the current model. Moreover, taking into account the results in Table 3 indicates that considering the in-plane stresses by the present model culminates in much more accuracy as compared with other LW models.

3.1.2 Example 2. Antisymmetric cross-ply (0/90/core/0/90) cylindrical sandwich panel

Table 4 shows the variations of NDNF with respect to radius-to-side ratio (R/a) and the thickness-to-side ratio a/H for a five-layer simply supported cylindrical sandwich shell which has square plane form ($a/b = 1$) with antisymmetric cross-ply face sheets. The core-to-face sheet thickness ratio (t_c/t_t) is considered to be 10. The mechanical properties for the core and the face sheet are similar to those considered for cross-ply sandwich plate in the Example 1. The results are presented with two assumptions (**a**: in-plane stress in the core and **b**: Z/R) and compared with the analytical ESL theory results reported by Garg *et al*. (2006). It is to be noted that both FSDT (2006) and HSDT (2006) methods mentioned in Table 4 employed ESL method. Analytical HSAPT1 and ANSYS results reported by Rahmani *et al*. (2010) and analytical HSAPT2 results reported by Biglari and Jafari (2010) are considered. The results of the present analysis by considering two assumptions (Z/R and in-plane stress) are in good agreement with the numerical ANSYS results reported by Rahmani *et al*. (2010). The results obtained by ESL models (both FSDT and HSDT) have less accuracy for thick shells ($a/H = 10$) in comparison with those obtained by other models in table 4, however higher accuracy is observed for thin shells ($a/H = 100$). Also, by considering the parameter Z/R in the present study, higher accuracy is obtained.

3.1.3 Example 3. Antisymmetric cross-ply (0/90/core/0/90) spherical sandwich panel

Table 5 shows a comparison for the first mode of dimensionless fundamental frequency with respect to

Table 4 Comparison of first mode NDNF Ω of simply supported antisymmetric (0/90/core/0/90) cylindrical sandwich shells with $a/b = 1$ and $t_c/t_t = 10$

a/H	*R/a*	Present results a	Present results b	HASPT1	HASPT2	ANSYS	ESL/HSDT	ESL/FSDT
100	1	68.284 (0.59%)	64.17 (0.695%)	63.27 (2.09%)	64.23 (0.6%)	64.62	64.64 (0.03%)	64.801 (0.3%)
	2	34.746 (0.70%)	34.71 (0.606%)	33.87 (1.83%)	34.71 (0.6%)	34.5	35.9 (4.06%)	36.214 (5.0%)
	3	24.977 (7.46%)	26.64 (7.389%)	24.17 (2.58%)	24.95 (0.56%)	24.81	26.7 (7.62%)	27.119 (9.3%)
10	1	6.536 (1.16%)	6.528 (1.054%)	5.65 (12.54%)	6.57 (1.7%)	6.46	7.71 (19.35%)	14.164 (119%)
	2	3.733 (0.83%)	3.737 (0.742%)	2.96 (20.22%)	3.74 (0.81%)	3.71	5.82 (56.87%)	14.026 (278%)
	3	2.86 (1.01%)	2.8 (1.076%)	2.19 (22.61%)	2.86 (1.06%)	2.83	5.36 (89.4%)	14.004 (4.0%)

*Numbers in parentheses are the discrepancies with respect to ANSYS results (Rahmani *et al*. 2010)

Table 5 Comparison of first mode NDNF Ω of simply supported antisymmetric (0/90/core/0/90) spherical sandwich shells with $a/b = 1$ and $t_c/t_t = 10$

a/H	*R/a*	Present results (a)	Present results (b)	HASPT2	ESL/FSDT
100	1	125.26	125.27	123.56	123.57
	2	62.53	562.53	65.86	66.33
	3	41.66	41.66	45.24	46.11
	5	24.99	24.99	28.95	30.45
	10	12.49	12.49	17.9	20.34
10	1	12.46	12.44	12.29	12.94
	2	6.27	6.27	6.71	8
	3	4.18	4.18	4.73	6.52
	5	2.51	2.51	3.22	5.58
	10	1.25	1.25	2.28	5.12

radius-to-side ratio (R/a) and the side-to-thickness ratio (H/a) of the five-layer simply supported spherical sandwich panels which have square plane form (a/b = 1) with antisymmetric cross-ply face sheets. The core-to-face sheets thickness (t_c/t_t) is equal to 10. The mechanical properties for the core and the face sheet are similar to those considered for antisymmetric cross-ply cylindrical sandwich shell in Example 1. The results presented on NDNF by considering $\frac{Z}{R}$ effect and the correct Lame's parameters for spherical shells ($A = R$, $B = R\sin\beta$) are compared with ESL model based on higher-order shear deformation theories (HOST11) by Garg *et al.* (2006) and analytical HSAPT2 results reported by Biglari and Jafari (2010). It is worth noting that the mentioned research works (2010), (Garg *et al*. 2006) assume the structure as a shallow shell and Lame's parameters to be $A = B = 1$. This is whilst, these Lame's parameters aren't suitable for thick shells.

3.2 Results

All of the formulations for free vibration analysis of different sandwich panels such as sandwich plate, cylindrical and spherical were validated using the above examples. In this section, the examples in the previous Section 3.1 are considered and the obtained results are presented and discussed. Also in these examples, the mechanical properties of the sandwich panels are given in table 1. In this section, the effect of different parameter such as core-to-face sheet stiffness ratio, plane stress of core, side-to-thickness ratio, the numbers of longitudinal half wave and circumferential wave, curvature of face sheets and radius to length are shown on the dimensionless frequency of structure.

3.2.1 Example 1

In this example, the free vibration of antisymmetric cross-ply (0/90/core/0/90) laminated square flat composite sandwich panel is investigated. Fig. 3 shows the variation of NDNF (Ω) with respect to the core-to-face sheet stiffness ratio with and without in-plane stress.

The following points can be elicited from the diagram:

(1) The NDNF is considerably affected for the core-to-face sheet stiffness ratio $E_c/E_t < 0.0005$, however it reaches a plateau for $E_c/E_t > 0.0005$, when the in-plane stress in the core is neglected. While the NDNF experiences a mild increase for $E_c/E_t > 0.0005$, when the in-plane stress in the core is considered.

(2) If the core-to-face sheet stiffness ratio becomes more than 0.02 ($E_c/E_t > 0.02$), the difference created on NDNF for the two conditions (with and without in-

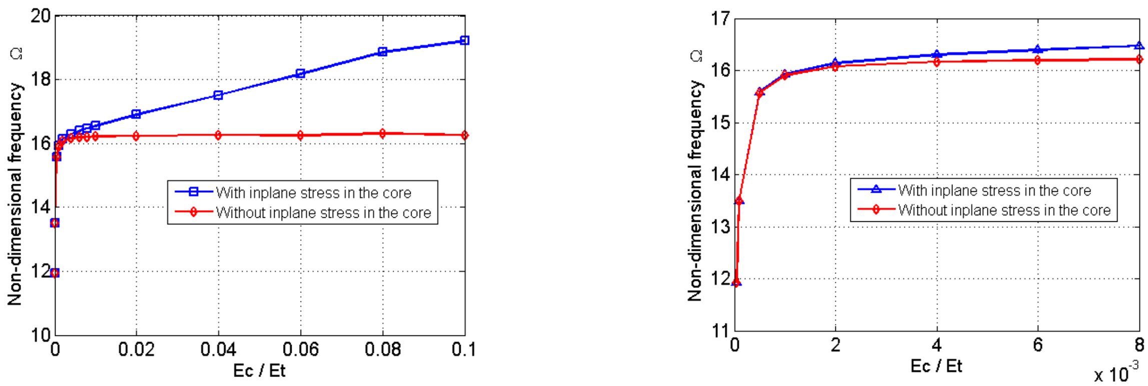

Fig. 3 The effect of in-plane stress of the core on variations of NDNF of the sandwich panel with respect to core to face sheets stiffness with $a/H = 100$, $t_c/t_t = 10$ and wave number in the first mode

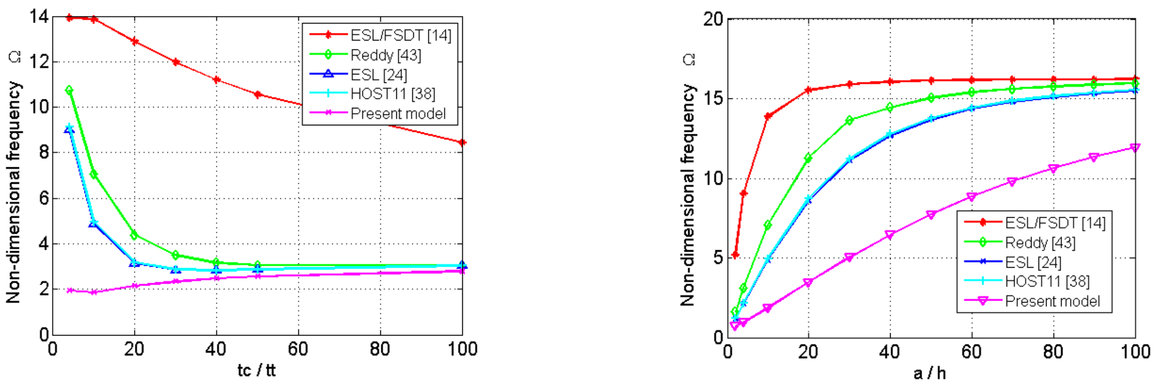

Fig. 4 The comparison of the variations of NDNF of the sandwich panel to: (a) the core-to-face sheet thickness ratio in different models with a / H = 10 in the first mode; (b) the side-to- thickness ratio in different models with $t_c/t_t = 10$ in the first mode

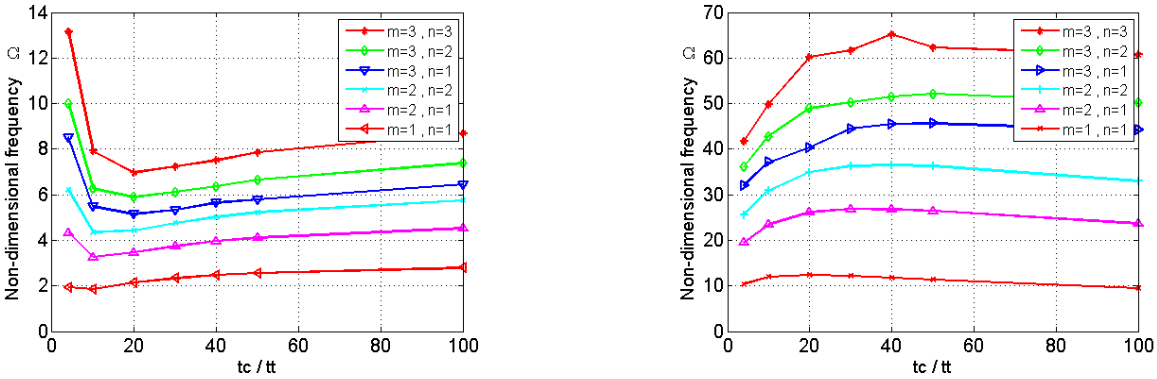

Fig. 5 The comparison of the variations of NDNF of the sandwich panel with respect to the core-to-face sheet thickness ratio in different wave number: (a) with $a/H = 10$; and (b) with $a/H = 100$

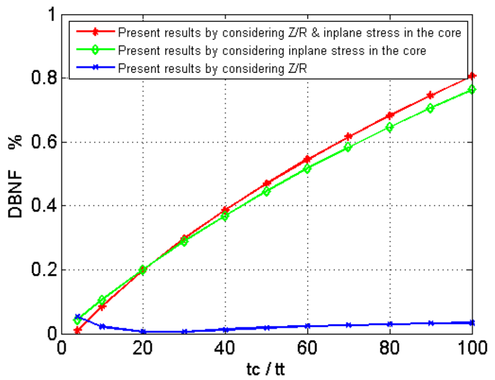

Fig. 6 The discrepancy between the non-dimensional frequencies (DBNF) of the cylindrical sandwich shell to the variations of the core-to-face sheet thickness ratio with $H/a = 0.1$ and $R/a = 1$ in the first mode

plane stress) becomes considerable (about 5%). It is an important factor in analyzing sandwich panels and therefore, can be used as a criterion for choosing flexibility or inflexibility of sandwich panel based on the amount of the core-to-face sheet stiffness ratio.

The variations of NDNF with respect to variation of core-to-face sheet thickness ratio and side-to-thickness ratio (a/H) based on various theories are presented in Fig. 4(a) and (b), respectively. It is observed that higher-order ESL model (HOST11) by Kant and Swaminathan (2010) has more accuracy in comparison with first-order ESL model (FSDT) by Pagano (1970), while the present LW model has more accuracy when it is compared with HOST11 (Garg *et al*. 2006). Fig. 4(a) depicts that the NDNF obtained by LW higher-order models and ESL models (Kant and Swaminathan 2001) becomes nearly equal when $t_c/t_t >$ 30, i.e., for t_c/t_t higher than 30, simple models of ESL with less calculations can be used as compared with LW models with costly computation. Fig. 5 shows the NDNF with respect to core-to-face sheet stiffness ratio in different modes. As can be seen, increasing (m, n) results in a general increase in NDNF. In addition, NDNF for all modes becomes nearly stable, when $t_c/t_t > 30$ for both $a/H =$ 10 and $a/H = 100$.

3.2.2 Example 2

In this example, the free vibration of cross-ply (0/90/core/0/90) cylindrical sandwich panel is investigated. Fig. 6 shows the effect of parameter Z/R and in-plane stress on discrepancy between NDNF $\left(\frac{f-f_0}{f_0} \times 100\right)$ in which f is NDNF by considering Z/R or in-plane stress in the core or both, and f_0 is NDNF without Z/R and in-plane stress in the core. As can be seen, considering the Z/R only causes no considerable effect on NDNF, however the in-plane

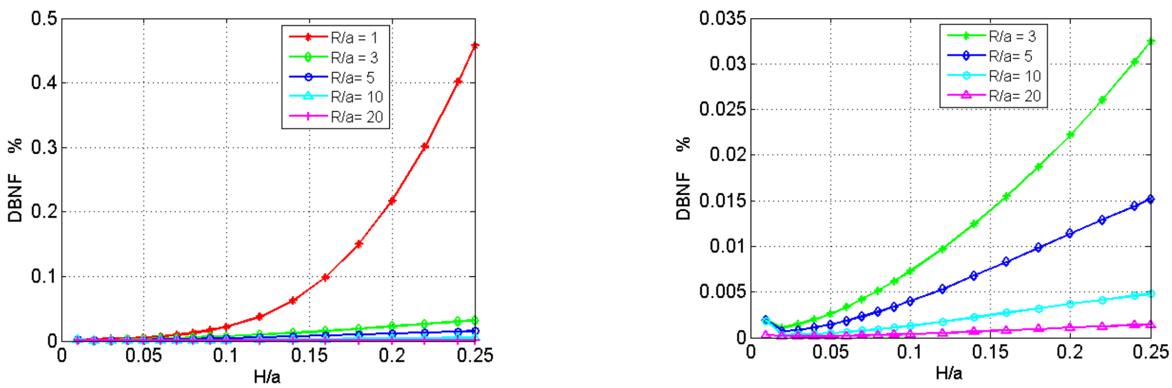

Fig. 7 The DBNF for the cylindrical sandwich shell by considering Z/R versus side-to-thickness ratio for different R/a with $t_c/t_t = 10$ in the first mode

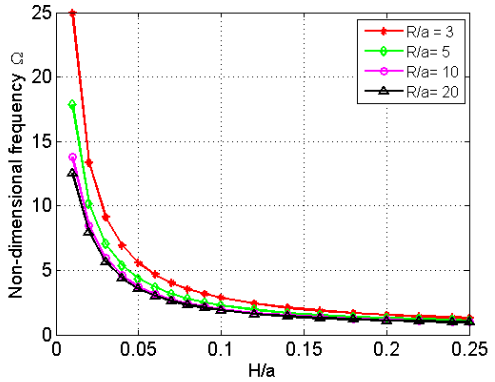

Fig. 8 The NDNF for the cylindrical sandwich shell by considering Z/R versus side-to-thickness ratios for different R/a when $t_c/t_t = 10$ in the first mode

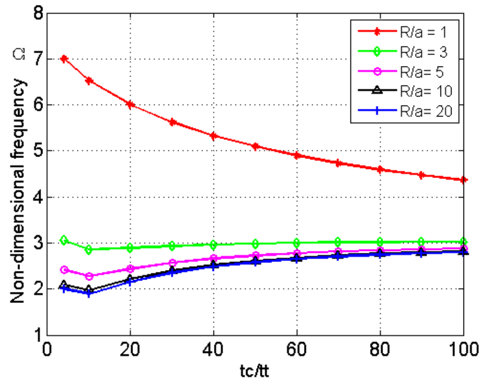

Fig. 10 The comparison of the variations of NDNF of the cylindrical sandwich shell by considering Z/R to the variations of the core-to-face sheet thickness ratio with $H/a = 0.1$ in the first mode

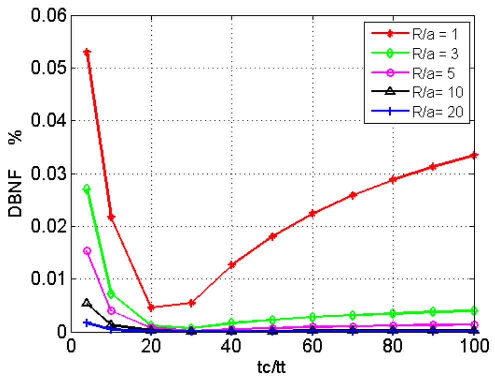

Fig. 9 The DBNF for the cylindrical sandwich shell by considering Z/R versus the core-to-face sheet thickness ratio with different R/a with $H/a = 0.1$ in the first mode

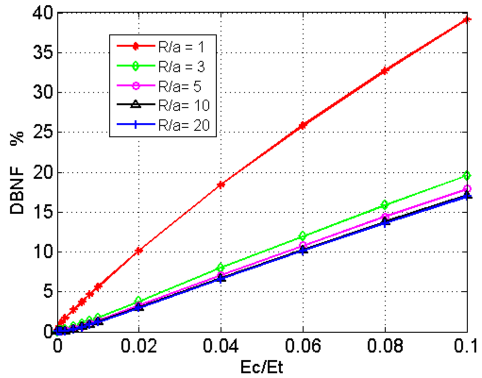

Fig. 11 The DBNF for the cylindrical sandwich shell by considering in-plane stress with respect to the core-to-face sheet stiffness ratio for different R/a with $H/a = 0.1$, $t_c/t_t = 10$ in the first mode

stress in the core plays a key role in changing this parameter.

The variations of DBNF with respect to thickness-to-side ratio of sandwich panel (a/H) for various R/a by considering Z/R are shown in Fig. 7. As can be seen, the DBNF is considerable when $R/a = 1$, however DBNF palpably decreases when R/a reaches 3 and higher. This situation is intensified by increasing the H/a.

Fig. 8 shows the variations of NDNF with respect to thickness-to-side ratio of panel for various R/a by considering Z/R. It is seen that the amount of NDNF decreases as H/a increases. Moreover, the natural frequency significantly decreases by increasing the thickness-to-side ratio of panel for $H/a < 0.05$, however no palpable effect on NDNF occurs when $H/a > 0.05$.

The variations of DBNF with respect to core-to-face sheet thickness ratio (t_c/t_t) for various curvature-to-side ratios (R/a) of sandwich panel by considering Z/R is presented in Fig. 9. As can be seen, a dramatic decrease occurs for DBNF when $t_c/t_t < 20$ for all R/a, however a considerable increase occurs when $t_c/t_t > 20$. It is worth noting that this phenomenon is intensified when $R/a = 1$ and is diminished when $R/a = 10$ or higher.

The NDNF with respect to core-to-face sheet thickness ratio (t_c/t_t) for various ratios of R/a by considering Z/R is depicted by Fig. 10. As can be seen, R/a =1 experiences a visible decrease in NDNF by increasing t_c/t_t, however the NDNF for $R/a = 3$ and higher marginally increase to reach a plateau and nearly converge in higher core-to-face sheet thickness ratios.

Fig. 11, shows the DBNF with respect to core-to-face sheet stiffness ratio (E_c/E_t) by considering in-plane stress for various R/a when $t_c/t_t = 10$, $a/H = 10$. It can be observed that considering the in-plane stress and increasing the core stiffness results in increasing the DBNF. The DBNF reaches 40% when the core-to-face sheet stiffness ratio is $R/a = 1$, $E_c/E_t = 0.1$, and reaches 20% when $R/a > 1$, $E_c/E_t = 0.1$.

Fig. 12 shows the DBNF for the cylindrical sandwich shell by considering in-plane stress to the core-to-face sheet stiffness ratio for: (a) various H/a with $R/a = 1$, $t_c/t_t = 10$ in the first mode; and (b) various t_c/t_t with $R/a = 1$, $H/a = 10$ in the first mode. As can be seen, the assumption for core compressibility based on core-to-face sheet stiffness ratio is reasonable when $E_c/E_t < 0.01$, because in this range, the maximum value of DBNF with

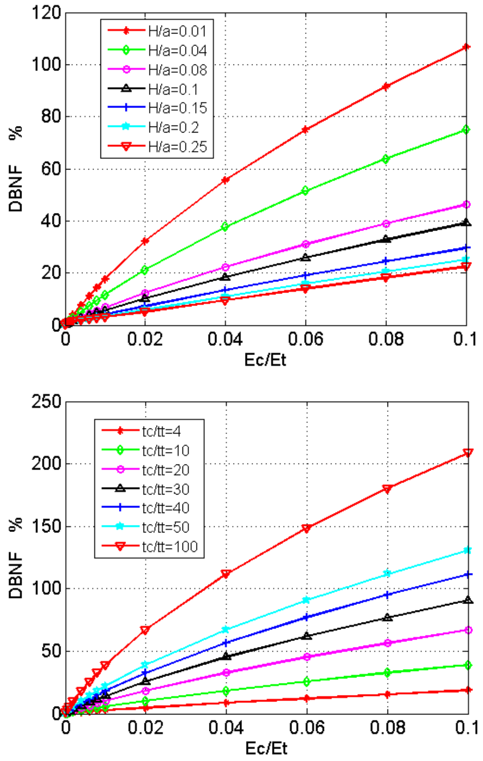

Fig. 12 The DBNF of the cylindrical sandwich shell by considering in-plane stress to the core-to-face sheet stiffness ratio for: (a) various H/a with $R/a=1$, $t_c/t_t=10$ in the first mode; (b) various t_c/t_t with $R/a=1$, $H/a=0.1$ in the first mode

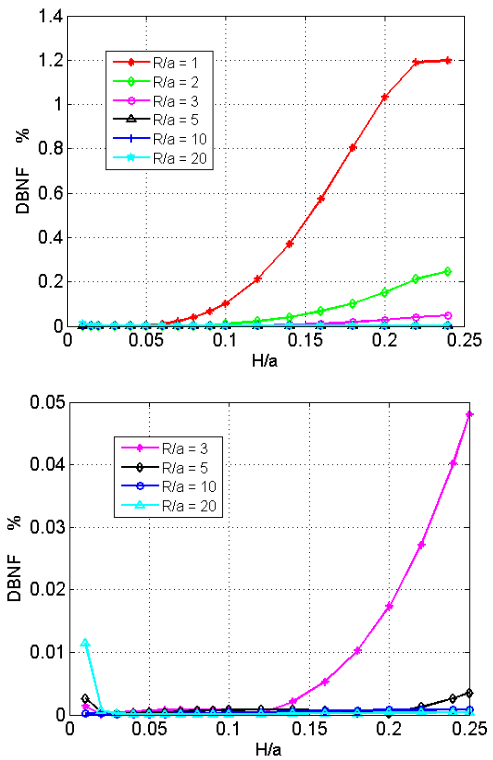

Fig. 13 The DBNF for the spherical sandwich shell by considering Z/R and in-plane stress in the core versus various H/a with different R/a with $t_c/t_t=10$ in the first mode

considering in-plane stress reaches 5%. DBNF dramatically increases by increasing the H/a and E_c/E_t. Also, DBNF vividly escalates by increasing E_c/E_t and t_c/t_t, exceeding 200% when $E_c/E_t=0.1$ and $t_c/t_t=100$.

3.2.3 Example 3

In this example, the free vibration of antisymmetric cross-ply (0/90/core/0/90) spherical sandwich panel is investigated. The effect of parameter Z/R on NDNF is shown by Fig. 13. As can be seen, the highest discrepancy between NDNF by considering Z/R and in-plane stress in the core occurs for $R/a=1$, when the side-to-thickness ratio H/a increases. Nevertheless, when $R/a>1$, the discrepancy experience less increase even at higher H/a values.

4. Conclusions

In this article, an analytical approach was developed for free vibration analysis of simply supported thick doubly curved sandwich panels with compressible/incompressible core using high-order shear deformation theory and Hamilton's principle. The face sheets are considered as laminated composite which follow first-order shear deformation theory and the core is considered compressible (with transverse stress only) and incompressible (with in-plane and transverse stresses) based on high-order shear deformation theory of sandwich structure. The present results are compared with those for the exact 3D elasticity and numerical results available in the literature. A good agreement is found between the results. The present validated model is used to carry out several parametric studies on the effects of radii of curvature, trapezoidal shape factor (the $\left(1\pm\frac{z}{R}\right)$ terms), thickness and flexibility of the core on the free vibration of thick DCSP in detail. The output of the present model and its numerical results yield the following conclusions:

(1) A general formulation is presented for a wide range of geometries such as sandwich plates by taking both the radii of curvature as infinity and cylindrical sandwich shells/panels by taking one radius of curvature as infinity and spherical sandwich shells/ panels by taking $R_\alpha=R_\beta=R$. Despite the previous research works based on shallow shell which assume the Lame' parameters to be 1, the current model considers the DCSP with a general view (deep and shallow) in such a way that the real values of Lame' parameters for different geometries are taken into account.

(2) One of the novelties in this work in comparison with the previous research works dealing with the free vibration analysis of doubly curved sandwich panels is considering different radii for three layers of top, bottom and core in the equations of motions.

(3) Sixteen displacement parameters are unknown in the equation of motions for DCSP. These are not dependent on the number of composite sheet layers and so are always constant. Considering the

continuity of conditions between the top and the bottom layers as well as the core, 22 displacement components are existed out of which 6 components are dependent on the other 16 components.

(4) The results suggest that the present high-order model is applicable to determine the natural frequencies of sandwich panels/shells with compressible and incompressible core for a wide range of the core to face sheet thickness ratios and various radii of curvatures.

(5) The parameter Z/R plays a pivotal role in free vibration analysis of curved structures such as cylinder, sphere, etc particularly where the radius of curvature is small. Therefore, an optimum range of DCSP thickness by considering Z/R is presented in which the influence ability of Z/R is highly considerable.

(6) The effect of in-plane stress is also very important in analyzing free vibration of DCSP. This study presents an optimum range for the core to face sheet stiffness ratio in which considering the existence of the in-plane stress, significantly affects the natural frequencies of DCSP.

References

Amabili, M. (2008), *Nonlinear Vibrations and Stability of Shells and Plates*, Cambridge University Press.

Biglari, H. and Jafari, A.A. (2010), "High-order free vibrations of doubly-curved sandwich panels with flexible core based on a refined three-layered theory", *Compos. Struct.*, **92**(11), 2685-2694.

Bhimaraddi, A. (1984), "A higher order theory for free vibration analysis of circular cylindrical shells", *Int. J. Solids Struct.*, **20**(7), 623-630.

Ćetković, M. and Vuksanović, D. (2009), "Bending, free vibrations and buckling of laminated composite and sandwich plates using a layerwise displacement model", *Compos. Struct.*, **88**(2), 219-227.

Chang, J.D. (1992), *Theory of Thick, Laminated Composite Shallow Shells*, Ohio State University.

Fares, M.E. and Youssif, Y.G. (2001), "A refined equivalent single-layer model of geometrically non-linear doubly curved layered shells using mixed variational approach", *Int. J. Non-Linear Mech.*, **36**(1), 117-124.

Ferreira, A.J.M. (2005), "Analysis of composite plates using a Layerwise Theory and multiquadrics discretization", *Mech. Adv. Mater. Struct.*, **12**(2), 99-112.

Frostig, Y. (1992), "Behavior of delaminated sandwich beam with transversely flexible core — high order theory", *Compos. Struct.*, **20**(1), 1-16.

Frostig, Y. and Thomsen, O.T. (2004), "High-order free vibration of sandwich panels with a flexible core", *Int. J. Solids Struct.*, **41**(5-6), 1697-1724.

Frostig, Y., Thomsen, O.T. and Vinson, J.R. (2004), "Bending analysis of unidirectional curved "Soft" sandwich panels with disbonds and slipping layers", *J. Sandw. Struct. Mater.*, **6**(2), 167-194.

Garg, A.K., Khare, R.K. and Kant, T. (2006), "Higher-order closed-form solutions for free vibration of laminated composite and sandwich shells", *J. Sandw. Struct. Mater.*, **8**(3), 205-235.

Hause, T. and Librescu, L. (2006), "Flexural free vibration of sandwich flat panels with laminated anisotropic face sheets", *J. Sound Vib.*, **297**(3-5), 823-841.

Hohe, J., Librescu, L. and Oh, S.Y. (2006), "Dynamic buckling of flat and curved sandwich panels with transversely compressible core", *Compos. Struct.*, **74**(1), 10-24.

Jedari Salami, S. (2016), "Free vibration analysis of sandwich beams with carbon nanotube reinforced face sheets based on extended high-order sandwich panel theory", *J. Sandw. Struct. Mater.* DOI: https://doi.org/10.1177/1099636216649788

Kant, T. and Swaminathan, K. (2001), "Free vibration of isotropic, orthotropic, and multilayer plates based on higher order refined theories", *J. Sound Vib.*, **241**(2), 319-327.

Kapoor, H. and Kapania, R.K. (2012), "Geometrically nonlinear NURBS isogeometric finite element analysis of laminated composite plates", *Compos. Struct.*, **94**(12), 3434-3447.

Kardomateas, G. (2005), "Wrinkling of wide sandwich panels/beams with orthotropic phases by an elasticity approach", *J. Appl. Mech.*, **72**(6), 818-825.

Khalili, S.M.R., Davar, A. and Malekzadeh Fard, K. (2012), "Free vibration analysis of homogeneous isotropic circular cylindrical shells based on a new three-dimensional refined higher-order theory", *Int. J. Mech. Sci.*, **56**(1), 1-25.

Khdeir, A.A. and Reddy, J.N. (1997), "Free and forced vibration of cross-ply laminated composite shallow arches", *Int. J. Solids Struct.*, **34**(10), 1217-1234.

Kheirikhah, M.M., Khalili, S.M.R. and Malekzadeh Fard, K. (2012), "Biaxial buckling analysis of soft-core composite sandwich plates using improved high-order theory", *Eur. J. Mech. - A/Solids*, **31**(1), 54-66.

Leissa, A.W. and Chang, J.D. (1996), "Elastic deformation of thick, laminated composite shells", *Compos. Struct.*, **35**(2), 153-170.

Leissa, A.W. and Qatu, M.S. (2011), *Vibration of Continuous Systems*, McGraw-Hill.

Librescu, L., Khdeir, A.A. and Frederick, D. (1989), "A shear deformable theory of laminated composite shallow shell-type panels and their response analysis I: Free vibration and buckling", *Acta Mech.*, **76**(1-2), 1-33.

Liew, K. and Lim, C. (1996), "A higher-order theory for vibration of doubly curved shallow shells", *J. Appl. Mech.*, **63**(3), 587-593.

Malekzadeh Fard, K.M., Livani, M. and Gholami, M. (2014), "Improved high-order bending analysis of double curved sandwich panels subjected to multiple loading conditions", *Latin Am. J. Solids Struct.*, **11**(9), 1591-1614.

Noor, A.K., Burton, W.S. and Bert, C.W. (1996), "Computational models for sandwich panels and shells", *Appl. Mech. Rev.*, **49**(3), 155-199.

Pagano, N.J. (1970), "Exact solutions for rectangular bidirectional composites and sandwich plates", *Mech. Compos. Mater.*, 86-101.

Qatu, M.S. (2004), *Vibration of Laminated Shells and Plates*, Elsevier.

Qatu, M.S. and Asadi, E. (2012), "Vibration of doubly curved shallow shells with arbitrary boundaries", *Appl. Acoust.*, **73**(1), 21-27.

Rabinovitch, O. and Frostig, Y. (2001), "High-order analysis of unidirectional sandwich panels with flat and generally curved faces and a soft core", *J. Sandw. Struct. Mater.*, **3**(2), 89-116.

Rahmani, O., Khalili, S.M.R. and Malekzadeh Fard, K. (2010), "Free vibration response of composite sandwich cylindrical shell with flexible core", *Compos. Struct.*, **92**(5), 1269-1281.

Rao, M.K. and Desai, Y.M. (2004), "Analytical solutions for vibrations of laminated and sandwich plates using mixed theory", *Compos. Struct.*, **63**(3-4), 361-373.

Reddy, J. (1984), *Energy and Variational Methods*, John Wiley, New York, NY, USA.

Reddy, J. (2003), *Mechanics of Laminated Composite Plates and*

Shells, *Theory and Application*, CRC Press, Boca Raton, FL, USA.

Reissner, E. (1953), "On a variational theorem for finite elastic deformations", *J*. *Math*. *Phys*., **32**(2-3), 129-135.

Reissner, E. and Wan, F.Y.M. (1982), "A note on the linear theory of shallow shear-deformable shells", *ZAMP*, **33**(3), 425-427.

Singh, A.V. (1999), "Free vibration analysis of deep doubly curved sandwich panels", *Comput*. *Struct*., **73**(1-5), 385-394.

Thai, C.H., Nguyen-Xuan, H., Nguyen-Thanh, N., Le, T.H., Nguyen-Thoi, T. and Rabczuk, T. (2012), "Static, free vibration, and buckling analysis of laminated composite Reissner–Mindlin plates using NURBS-based isogeometric approach", *Int*. *J*. *Numer*. *Meth*. *Eng*., **91**(6), 571-603.

Toorani, M. and Lakis, A. (2000), "General equations of anisotropic plates and shells including transverse shear deformations, rotary inertia and initial curvature effects", *J*. *Sound Vib*., **237**(4), 561-615.

Valizadeh, N., Natarajan, S., Gonzalez-Estrada, O.A., Rabczuk, T., Bui, T.Q. and Bordas, S.P. (2013), "S.P.A. NURBS-based finite element analysis of functionally graded plates: Static bending, vibration, buckling and flutter", *Compos*. *Struct*., **99**, 309-326.

Vinson, J.R. and Sierakowski, R.L. (2006), *The Behavior of Structures Composed of Composite Materials*, Springer Science & Business Media.

Viola, E., Tornabene, F. and Fantuzzi, N. (2013), "General higher-order shear deformation theories for the free vibration analysis of completely doubly-curved laminated shells and panels", *Compos*. *Struct*., **95**, 639-666.

Wu, Z., Cheung, Y.K., Lo, S.H. and Chen, W. (2008), "Effects of higher-order global–local shear deformations on bending, vibration and buckling of multilayered plates", *Compos*. *Struct*., **82**(2), 277-289.

CC

Appendix A

Membrane, Flexure, Coupling, and Shear Rigidity Matrices of core (Garg *et al*. 2006)

$$[A^c] = \begin{bmatrix} Q_{11}H_0^5 & Q_{12}H_0^0 & Q_{14}H_0^0 & Q_{14}H_0^5 & Q_{11}H_2^5 & Q_{12}H_2^0 & Q_{14}H_2^0 & Q_{14}H_2^5 & Q_{13}H_0^2 & Q_{13}H_2^2 \\ & Q_{22}H_0^4 & Q_{24}H_0^4 & Q_{24}H_0^0 & Q_{21}H_2^0 & Q_{22}H_2^4 & Q_{24}H_2^4 & Q_{24}H_2^0 & Q_{23}H_0^1 & Q_{23}H_2^1 \\ & & Q_{44}H_0^4 & Q_{44}H_0^0 & Q_{41}H_2^0 & Q_{42}H_2^4 & Q_{44}H_2^4 & Q_{44}H_2^0 & Q_{43}H_0^1 & Q_{43}H_2^1 \\ & & & Q_{44}H_0^5 & Q_{41}H_2^5 & Q_{42}H_2^0 & Q_{44}H_2^0 & Q_{44}H_2^5 & Q_{43}H_0^2 & Q_{43}H_2^2 \\ & & & & Q_{11}H_4^5 & Q_{12}H_4^0 & Q_{14}H_4^0 & Q_{14}H_4^5 & Q_{13}H_2^2 & Q_{13}H_4^2 \\ & & & & & Q_{22}H_4^4 & Q_{24}H_4^4 & Q_{24}H_4^0 & Q_{23}H_2^1 & Q_{23}H_4^1 \\ & Sym. & & & & & Q_{44}H_4^4 & Q_{44}H_4^0 & Q_{43}H_2^1 & Q_{43}H_4^1 \\ & & & & & & & Q_{44}H_4^5 & Q_{43}H_2^2 & Q_{43}H_4^2 \\ & & & & & & & & Q_{33}H_0^3 & Q_{33}H_2^3 \\ & & & & & & & & & Q_{33}H_4^3 \end{bmatrix}^c \tag{A1}$$

$$[D_s^c] = \begin{bmatrix} Q_{55}H_0^3 & Q_{55}H_0^2 & Q_{56}H_0^3 & Q_{56}H_0^1 & Q_{55}H_2^3 & Q_{55}H_2^2 & Q_{56}H_2^3 & Q_{56}H_2^1 & Q_{55}H_1^3 & Q_{55}H_1^2 & Q_{56}H_1^3 & Q_{56}H_1^1 & Q_{55}H_3^2 & Q_{56}H_3^1 \\ & Q_{55}H_0^5 & Q_{56}H_0^2 & Q_{56}H_0^0 & Q_{55}H_2^2 & Q_{55}H_2^5 & Q_{56}H_2^2 & Q_{56}H_2^0 & Q_{55}H_1^2 & Q_{55}H_1^5 & Q_{56}H_1^2 & Q_{56}H_1^0 & Q_{55}H_3^5 & Q_{56}H_3^0 \\ & & Q_{66}H_0^3 & Q_{66}H_0^1 & Q_{65}H_2^3 & Q_{65}H_2^2 & Q_{66}H_2^3 & Q_{66}H_2^1 & Q_{65}H_1^3 & Q_{65}H_1^2 & Q_{66}H_1^3 & Q_{66}H_1^1 & Q_{65}H_3^2 & Q_{66}H_3^1 \\ & & & Q_{66}H_0^4 & Q_{65}H_2^1 & Q_{65}H_2^0 & Q_{66}H_2^1 & Q_{66}H_2^4 & Q_{65}H_1^1 & Q_{65}H_1^0 & Q_{66}H_1^1 & Q_{66}H_1^4 & Q_{65}H_3^0 & Q_{66}H_3^4 \\ & & & & Q_{55}H_4^3 & Q_{55}H_4^2 & Q_{56}H_4^3 & Q_{56}H_4^1 & Q_{55}H_3^3 & Q_{55}H_3^2 & Q_{56}H_3^3 & Q_{56}H_3^1 & Q_{55}H_5^2 & Q_{56}H_5^1 \\ & & & & & Q_{55}H_4^5 & Q_{56}H_4^2 & Q_{56}H_4^0 & Q_{55}H_3^2 & Q_{55}H_3^5 & Q_{56}H_3^2 & Q_{56}H_3^0 & Q_{55}H_5^5 & Q_{56}H_5^0 \\ & & & & & & Q_{66}H_4^3 & Q_{66}H_4^1 & Q_{65}H_3^3 & Q_{65}H_3^2 & Q_{66}H_3^3 & Q_{66}H_3^1 & Q_{65}H_5^2 & Q_{66}H_5^1 \\ & & & & & & & Q_{66}H_4^4 & Q_{65}H_3^1 & Q_{65}H_3^0 & Q_{66}H_3^1 & Q_{66}H_3^4 & Q_{65}H_5^0 & Q_{66}H_5^4 \\ & & & & & & & & Q_{55}H_2^3 & Q_{55}H_2^2 & Q_{56}H_2^3 & Q_{56}H_2^1 & Q_{55}H_4^2 & Q_{56}H_4^1 \\ & & & & & & & & & Q_{55}H_2^5 & Q_{56}H_2^2 & Q_{56}H_2^0 & Q_{55}H_4^5 & Q_{56}H_4^0 \\ & & & Sym. & & & & & & & Q_{66}H_2^3 & Q_{66}H_2^1 & Q_{65}H_4^2 & Q_{66}H_4^1 \\ & & & & & & & & & & & Q_{66}H_2^4 & Q_{65}H_4^0 & Q_{66}H_4^4 \\ & & & & & & & & & & & & Q_{55}H_6^5 & Q_{56}H_6^0 \\ & & & & & & & & & & & & & Q_{66}H_6^4 \end{bmatrix}^c \tag{A2}$$

It is worth noting that $[B^c]$matrices is similar to $[E^c]$ matrices and the difference between $[A^c]$, $[B^c], [E^c]$ and $[D^c]$ matrices are subscript "*j*" in "H_j^i" parameter in $[A^c]$ matrices is "*j*", in $[B^c]$ and $[E^c]$ matrices is equal to "*j*+1" and in $[D^c]$ matrices is equal to "*j*+2" .

where

$$H_i^o = \int_{h^k}^{h^{k+1}} z^i dz, \quad H_i^1 = \int_{h^k}^{h^{k+1}} k_1 z^i dz, \quad H_i^2 = \int_{h^k}^{h^{k+1}} k_2 z^i dz,$$
$$H_i^3 = \int_{h^k}^{h^{k+1}} k_1 k_2 z^i dz, \quad H_i^4 = \int_{h^k}^{h^{k+1}} \frac{k_1}{k_2} z^i dz, \quad H_i^5 = \int_{h^k}^{h^{k+1}} \frac{k_2}{k_1} z^i dz \tag{A3}$$
$$\text{That} \quad i = 1, 2, 3, 4, 5, 6, \quad k_1 = \left(1 + \frac{z}{R_\alpha}\right), \quad k_2 = \left(1 + \frac{z}{R_\beta}\right)$$

and $[Q]$ matrix refers to elastic stiffness in principle material axes (Reddy 2003).

And membrane, flexure, coupling, and shear rigidity matrices of face sheet are

$$[A^i] = \sum_L^{NL} \begin{bmatrix} Q_{11}H_0^5 & Q_{12}H_0^0 & Q_{14}H_0^0 & Q_{14}H_0^5 \\ & Q_{22}H_0^4 & Q_{24}H_0^4 & Q_{24}H_0^0 \\ & Sym. & Q_{44}H_0^4 & Q_{44}H_0^0 \\ & & & Q_{442}H_0^5 \end{bmatrix}^i \tag{A4}$$

$$[D_s^i]=\sum_{L}^{NL}\begin{bmatrix} Q_{55}H_0^3 & Q_{55}H_0^2 & Q_{56}H_0^3 & Q_{56}H_0^1 & Q_{55}H_1^2 & Q_{56}H_1^1 \\ & Q_{55}H_0^5 & Q_{56}H_0^2 & Q_{56}H_0^0 & Q_{55}H_1^5 & Q_{56}H_1^0 \\ & & Q_{66}H_0^3 & Q_{66}H_0^1 & Q_{65}H_1^2 & Q_{66}H_1^1 \\ & & & Q_{66}H_0^4 & Q_{65}H_1^0 & Q_{66}H_1^4 \\ & Sym. & & & Q_{55}H_2^5 & Q_{56}H_2^0 \\ & & & & & Q_{66}H_2^4 \end{bmatrix}^i \tag{A5}$$

that $i=t$ (top face sheet), b (buttom face sheet)

It is worth noting that $[B^i]$ matrices is similar to $[E^i]$ matrices and difference of $[A^i]$, $[B^i]$, $[E^i]$ and $[D^i]$ matrices are subscript "*j*" in "H_j^i" parameter in $[A^i]$ matrices is "*j*", in $[B^i]$ and $[E^i]$ matrices is equal to "*j*+1" and in $[D^i]$ matrices is equal to "*j*+2".

Nomenclature

Membrane, Flexure, Coupling, and Shear Rigidity Matrices of core (Garg *et al*. 2006)

I (= c, t, b)	indices for core, top and bottom facesheets
$p, q\ (= \alpha, \beta, z)$	indices for curvilinear coordinate axes
$\alpha\ , \beta, z$	curvilinear coordinate axes
a, b	length and width of sandwich panel
H	Total thickness of the sandwich panel
t^i	Thickness of layers
$R_\alpha^i,\ R_\beta^i$	the radii of curvature to mid surface of the layers in the α and β directions
$R_{\alpha\beta}^i$	the radii of twist of the surface
$A_1,\ A_2,\ A_3$	geometrical scale factor quantities
A, B	Lame' parameters
$u^i,\ v^i,\ w^i$	displacements in α, β and z directions
$u_0^i,\ v_0^i,\ w^i$	displacements at the mid surface of the face sheets
$\theta_\alpha^i,\ \theta_\beta^i$	slopes in α–z and β-z planes in the face sheets
$u_j^c,\ v_j^c,\ w_j^c,$ $j = 0, 1, 2, 3$	displacement components of core
$C_0,\ C_1$	Trapezoidal effect coefficient
$\varepsilon_p^i,\ \varepsilon_{pq}^i,\ \gamma_{pq}^i$	Engineering strain components
$\varepsilon_{0p}^i,\ \kappa_p^i,\ \varepsilon_{0pq}^i,$ $\chi_{pq}^i,\ \gamma_{opq}^i$	mid-plane strains and curvatures of face sheets
$\varepsilon_{0p}^c,\ \kappa_p^c,\ \varepsilon_{0p}^{*c},\ \kappa_p^{*c},$ $\varepsilon_{0pq}^c,\ \chi_{pq}^c,\ \varepsilon_{0pq}^{*c},\ \chi_{pq}^{*c}$	mid-plane strains and curvatures of core
$E_{11},\ E_{22},\ E_{33}$	Young's modulus in principle directions
$G_{12},\ G_{13},\ G_{23}$	shear modulus
the principle axes 1, 2 and 3	parallel and perpendicular of fiber
$\sigma_1,\ \sigma_2,\ \sigma_3$	normal stresses in the principle axes
$\tau_{12},\ \tau_{13},\ \tau_{23}$	shear stresses in the principle axes
$\sigma_\alpha,\ \sigma_\beta,\ \sigma_z$	normal stress components referred to the laminate coordinate
$\tau_{\alpha\beta},\ \tau_{\alpha z},\ \tau_{\beta z}$	shear stress components referred to the laminate coordinate
$\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3$	normal strains in the principle axes
$\gamma_{12},\ \gamma_{13},\ \gamma_{23}$	Shear strains in the principle axes
Q_{ij}	transformed elastic constant with respect to the laminateaxes
C_{ij}	elastic constant of layers with reference to thefiber axes
$\nu_{21},\ \nu_{23},\ \nu_{13}$	Poisson's ratio
$[T]$	transformation matrix
$[D]$	rigidity matrix
$[A]^i$	membrane matrix
$[B]^i\ , [E]^i$	shear matrix
$[D]^i$	bending matrix
$[D_s^i]$	membrane matrix
k_o	shear correction factor
$\{\bar{\varepsilon}_i\}$	midsurface strain vector of top, bottom and core
$\{\bar{\sigma}_i\}$	stress-resultant vector of top, bottom and core
$N_{pq}^i,\ N_{pq}^{*i},\ M_{pq}^i,\ M_{pq}^{*i},$ $Q_{pq}^i,\ Q_{pq}^{*i},\ S_{pq}^i,\ S_{pq}^{*i}$	stress resultants
k_1^i and $k_2^i = 0$ or 1	Factor trapezoidal curvature parameter
E	kinetic energy
U	potential energy
W	potential of the external loads
ρ^i	density of the top, bottom face sheet and the core
$I_n^i n = (1$ to $6)$	moment of inertia
m, n	longitudinal half and circumferential wave numbers
$\{\Delta\}$	natural mode shape vector
$T_{mn}(t)$	generalized coordinates
$\{\psi\}$	weighting functions vector
$[K]$	stiffness matrix
$[M]$	mass matrix
λ_{mn}	the lowest eigenvalue
$\{d\}$	displacement vector
ω_{mn}	natural frequency
Ω (NDNF)	non-dimensional natural frequencies
(DBNF)	The discrepancy between the non-dimensional frequencies