Steel and Composite Structures, *Vol. 24, No. 1 (2017) 53-63*
DOI: https://doi.org/10.12989/scs.2017.24.1.053

A simplified matrix stiffness method for analysis of composite and prestressed beams

Biljana Deretić-Stojanović [a] and Svetlana M. Kostić [*]

Faculty of Civil Engineering, University of Belgrade, Bul Kralja Aleksandra 73, Belgrade, Serbia

(*Received October 24, 2016, Revised March 01, 2017, Accepted March 02, 2017*)

Abstract. The paper presents the simplified matrix stiffness method for analysis of composite and prestressed beams. The method is based on the previously developed "exact" analysis method that uses the mathematical theory of linear integral operators to derive all relations without any mathematical simplifications besides inevitable idealizations related to the material rheological properties. However, the method is limited since the closed-form solution can be found only for specific forms of the concrete creep function. In this paper, the authors proposed the simplified analysis method by introducing the assumption that the unknown deformations change linearly with the concrete creep function. Adopting this assumption, the nonhomogeneous integral system of equations of the "exact" method simplifies to the system of algebraic equations that can be easily solved. Therefore, the proposed method is more suitable for practical applications. Its high level of accuracy in comparison to the "exact" method is preserved, which is illustrated on the numerical example. Also, it is more accurate than the well-known EM method.

Keywords: composite beam; viscoelastic analysis; creep; shrinkage; matrix stiffness method; linear integral operators

1. Introduction

Analysis of composite steel and concrete and prestressed structures is more complex in comparison with analysis of pure concrete and steel structures. This firstly comes from different rheological properties of materials that act together in a composite section. There are viscous deformations of concrete, creep and shrinkage, and relaxation of prestressing steel. Over the last several decades, number of studies investigated time-dependent behavior of composite and prestressed structures and proposed different methods for calculating these effects (Amadio *et al.* 2012, Chaudhary *et al.* 2007, Deretić-Stojanović and Kostić 2014, 2015, Dezi and Gara 2001, Dezi *et al.* 1995, 1996, Faella *et al.* 2010, Partov and Kantchev 2009, Ranzi *et al.* 2013). In general, the proposed methods differ in the adopted constitutive stress-strain relation for concrete. This relation is integral and can be converted into an algebraic form applying the specific mathematical transformations. The analysis methods that use the algebraic relations for concrete stresses and strains, i.e., algebraic methods, are approximate methods and effective modulus method (EM) (Fritz 1961, Mirza and Uy 2010), mean stress method (MS) (Mirza and Uy 2010) and age adjusted effective modulus method (AAEM) (Bažant *et al.* 2013, Fan *et al.* 2010, Wu *et al.* 2015) are among these. In the numerical, or step-by-step methods, time is divided into a number of intervals and the integral relations are approximated with finite summations (Fragiacomo *et al.* 2004, Kwak and Seo 2000, Macorini *et al.* 2006, Nguyen *et al.* 2010). On the other side are methods that adopt the integral relation between concrete stresses and strains and do not introduce any additional mathematical approximations, apart from inevitable approximations related to rheological properties of materials. These methods and their solutions can be, therefore, considered as exact (Deretić-Stojanović and Kostić 2014, Deretić-Stojanović and Kostić 2015, Lazic 2003). In this approach, all relations are integral and are derived using the mathematical theory of linear integral operators.

The operators in the aging linear viscoelasticity are firstly used by Mandel (Lazic 2003). In these works, the integral relations are presented using the linear integro-differential operators. These operators are extended to matrix and tensor integro-differential operators in (Bažant and Huet 1999) where beams consisting of several different aging linear viscoelastic materials are analysed. Lazic (2003) was the first who used linear integral operators for analysis of composite and prestressed beams. He derived the force based solution for statically determinate and indeterminate structures. The matrix stiffness method for the analysis of composite and prestressed beams in (Deretić-Stojanović and Kostić 2014, 2015) uses the same operators. In both mentioned methods, the ultimate equations are nonhomogeneous integral equations: in force based method these are equations with the unknown forces, while in the displacement based (matrix stiffness) method the equations are with the unknown displacements. The solutions to these integral equations in the closed form can be obtained only for specific creep functions. For the creep functions of the hereditary theory and the aging theory, with

*Corresponding author, Ph.D., Assistant Professor, E-mail: svetlana@grf.bg.ac.rs
[a] Ph.D., Professor, E-mail: biljads@grf.bg.ac.rs

http://www.techno-press.org/?journal=scs&subpage=6 ISSN: 1229-9367 (Print), 1598-6233 (Online)

the constant concrete modulus of elasticity, solutions may be obtained applying the Laplace transformations.

In order to simplify the analysis, different assumptions can be introduced. Deriving the algebraic relation for concrete stresses and strains in AAEM method, Bazant (1972) assumed that stresses changes linearly with concrete relaxation function if longitudinal dilatation changes linearly with creep function. In the analysis of statically indeterminate composite structures, Lazic assumed that statically indeterminate forces vary linearly with concrete relaxation function. Consequently, the ultimate equations for statically indeterminate forces become algebraic equations and the solution is approximate (Lazic 2003).

The work presented in this paper follows the matrix stiffness method presented in (Deretić-Stojanović and Kostić 2014, 2015), which is denoted as the "exact" method in the following. Starting from this method, the analysis is, further, simplified introducing the assumption that the unknown displacements change linearly over time with the concrete creep function. This way, the system of nonhomogeneous integral equations with the unknown displacements transforms into the system of algebraic equations that can be easily solved. Besides the mentioned assumption, other mathematical simplifications are not used and the obtained solution preserves the high accuracy.

In order to follow the derivations of the proposed simplified method, firstly the basic relations of the "exact" method are summarized. More details are available in (Deretić-Stojanović and Kostić 2014, 2015). Since both of these methods, the "exact" and the proposed simplified method, use the linear integral operators, the basic notations and definitions of the used linear integral operators are summarized in Appendices A and B. More information are available elsewhere (Lazic 2003).

2. Basic relations of the "exact" analysis method

This section contains the short overview of the basic relations of the previously developed "exact" matrix stiffness method for the analysis of composite and prestressed beams (Deretić-Stojanović and Kostić 2014, 2015). The stiffness matrix for a fixed end frame element and a frame element with a moment release at one end, obtained using the basic operator flexibility matrix, are given. In addition, the stiffness matrix for the symmetric frame element (element type "s"), used latter in the numerical example, is derived. In this method, relations between the generalized element deformations and the generalized element forces are integral and presented using the mathematical linear integral operators (Appendices A and B).

Because of the viscoelastic properties of concrete and relaxation of prestressing steel, in composite and prestressed indeterminate structures, deformations and forces change over time. As shown in (Deretić-Stojanović and Kostić 2014, 2015), using the mathematical theory of linear integral operators, it is possible to obtain and represent the element stiffness matrix in the same form as for the elastic homogeneous frame element. Ultimate equations, analogously to the equations for the elastic homogeneous structures, are derived from equilibrium equations for the joints and, in case of composite and prestressed structures, these are system of integral equations with deformations as unknowns.

The "exact" matrix stiffness method of analysis of composite and prestressed beams adopts the following assumptions related to the material properties and element behavior. In general, a cross section consists of concrete, prestressing steel, steel section and reinforcement. The concrete is modeled as linear viscoelastic aging material. Prestressing steel has a relaxation property, and steel and reinforcement behaves as linear-elastic materials. Details about the adopted constitutive equations, written in operator form, are given in Appendix B. In addition, the Bernoulli's hypothesis of plane sections is adopted and it is assumed that there is no slip at the steel section-concrete slab interface.

Fig. 1 shows the fixed-end frame element ik with its nodal forces in the local coordinate system: the bending moments M_i and M_k, the shear forces T_i and T_k and axial forces N_i and N_k. The corresponding element end deformations are rotations φ_i and φ_k and displacements v_i, v_k, u_i and u_k.

As shown in (Deretić-Stojanović and Kostić 2014), the operator stiffness matrix for this element can be written in the following form

$$\left[\widehat{\mathbf{K}}'\right]_{ik} = \begin{bmatrix} \widehat{N}'_{ik} & -\frac{1}{l}(\widehat{S}'_{ik}+\widehat{S}'_{ki}) & -\widehat{S}'_{ik} & -\widehat{N}'_{ik} & \frac{1}{l}(\widehat{S}'_{ik}+\widehat{S}'_{ki}) & -\widehat{S}'_{ki} \\ & \frac{1}{l^2}(\widehat{C}'_{ik}+\widehat{C}'_{ki}) & \frac{1}{l}\widehat{C}'_{ik} & \frac{1}{l}(\widehat{S}'_{ik}+\widehat{S}'_{ki}) & -\frac{1}{l^2}(\widehat{C}'_{ik}+\widehat{C}'_{ki}) & \frac{1}{l}\widehat{C}'_{ki} \\ & & \widehat{A}'_{ik} & \widehat{S}'_{ik} & -\frac{1}{l}\widehat{C}'_{ik} & \widehat{B}'_{ik} \\ & & & \widehat{N}'_{ik} & -\frac{1}{l}(\widehat{S}'_{ik}+\widehat{S}'_{ki}) & \widehat{S}'_{ki} \\ & \textit{symmetric} & & & \frac{1}{l^2}(\widehat{C}'_{ik}+\widehat{C}'_{ki}) & -\frac{1}{l}\widehat{C}'_{ki} \\ & & & & & \widehat{A}'_{ki} \end{bmatrix} \quad (1)$$

where

$$\widehat{C}'_{ik} = \widehat{A}'_{ik} + \widehat{B}'_{ik}\,, \quad \widehat{C}'_{ki} = \widehat{A}'_{ki} + \widehat{B}'_{ki}\,, \quad l = l_{ik} \quad (2)$$

Meaning of the operators $\widehat{A}'_{ik}, \widehat{B}'_{ik}, \widehat{A}'_{ki}$ and $\widehat{B}'_{ki}$ can be found in (Deretić-Stojanović and Kostić 2014). In a special

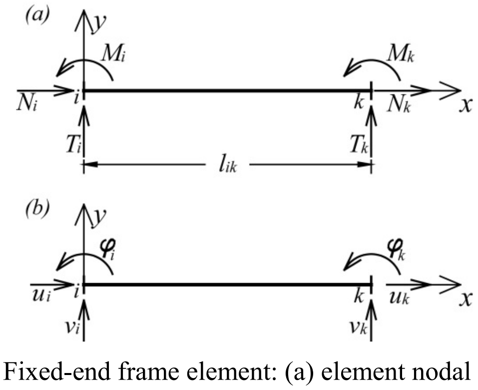

Fig. 1 Fixed-end frame element: (a) element nodal forces in the local coordinate system; (b) element end deformations

case, when the element has a constant cross section, elements of the operator stiffness matrix are proportional to only one operator or are linear combination of two operators

$$
\begin{aligned}
\widehat{N}'_{ik} &= \frac{E_u A_i}{l_{ik}} \widehat{R}'_{11} \\
\widehat{A}'_{ik} &= \frac{E_u J_i}{l_{ik}} \widehat{R}'_{22} + \frac{3E_u J_i}{l_{ik}} \widehat{I}'_{22} = \widehat{A}'_{ki} \\
\widehat{B}'_{ik} &= -\frac{E_u J_i}{l_{ik}} \widehat{R}'_{22} + \frac{3E_u J_i}{l_{ik}} \widehat{I}'_{22} = \widehat{B}'_{ki} \\
\widehat{S}'_{ik} &= -\frac{E_u S_i}{l_{ik}} \widehat{R}'_{12} = -\widehat{S}'_{ki}
\end{aligned}
\tag{3}
$$

where operators $\widehat{R}'_{11}$, $\widehat{R}'_{22}$, $\widehat{R}'_{12}$ and $\widehat{I}'_{22}$ are defined by Eqs. (B17) and (B18); E_u is the relative modulus of elasticity and is equal to the modulus of elasticity of steel section E_a; A_i and J_i are the area and the moment of inertia of a transformed cross section (Appendix C).

Fig. 2 shows the nodal forces in the local coordinate system of a frame element g_k with a moment release at end "g". These are: the bending moment M_k, the shear forces T_g and T_k and axial forces N_g and N_k. The corresponding element end deformations are rotations φ_k and displacements v_g, v_k, u_g and u_k.

For this element, the operator stiffness matrix has the following form (Deretić-Stojanović and Kostić 2015)

$$
\left[\widehat{\mathbf{K}}'\right]_{gk} = \begin{bmatrix}
\widehat{N}'_{gk} & -\frac{1}{l_{gk}}\widehat{S}'_{gk} & -\widehat{N}'_{gk} & \frac{1}{l_{gk}}\widehat{S}'_{gk} & -\widehat{S}'_{gk} \\
 & \frac{1}{l_{gk}^2}\widehat{D}'_{gk} & \frac{1}{l_{gk}}\widehat{S}'_{gk} & -\frac{1}{l_{gk}^2}\widehat{D}'_{gk} & \frac{1}{l_{gk}}\widehat{D}'_{gk} \\
 & & \widehat{N}'_{gk} & -\frac{1}{l_{gk}}\widehat{S}'_{gk} & \widehat{S}'_{gk} \\
 & \text{symmetric} & & \frac{1}{l_{gk}^2}\widehat{D}'_{gk} & -\frac{1}{l_{gk}}\widehat{D}'_{gk} \\
 & & & & \widehat{D}'_{gk}
\end{bmatrix}
\tag{4}
$$

Similarly to the fixed-end frame element, in a case that element has a constant cross section, elements of the operator stiffness matrix become a linear combination of the operators $\widehat{R}'$ and $\widehat{\Phi}'_h$

$$
\begin{aligned}
\widehat{N}'_{gk} &= \frac{E_u A_i}{3l_{ig}} s(n_1 \widehat{1}' + \gamma_{11}\widehat{R}' + n_2\widehat{\Phi}'_1 + n_3\widehat{\Phi}'_2) && (a) \\
\widehat{S}'_{gk} &= \frac{E_u S_i}{2l_{ig}} s(z_1 \widehat{1}' + \gamma_{12}\widehat{R}' + z_2\widehat{\Phi}'_1 + z_3\widehat{\Phi}'_2) && (b) \\
\widehat{D}'_{gk} &= \frac{E_u J_i}{l_{ig}} s(d_1 \widehat{1}' + \gamma_{22}\widehat{R}' + d_2\widehat{\Phi}'_1 + d_3\widehat{\Phi}'_2) && (c)
\end{aligned}
\tag{5}
$$

Operators $\widehat{\Phi}'_h$, (h = 1, 2) and $\widehat{R}'$ are defined latter in Eqs. (25) and (B7), γ_{hl}, (h, l = 1, 2) are the elements of the symmetric scalar matrix of the reduced cross section geometry $[\gamma_{hl}]_{2,2}$, Eq. (C.5), and constants s, n_1-n_3, z_1-z_3 and d_1-d_3 are defined in Appendix C, Eqs. (C.10)-(C.13).

In addition, to take advantage of the analysis of symmetric structures, the operator flexibility matrix for a frame element intersected by the symmetry plane (denoted as element type "s") follows. Its use is illustrated latter in the numerical example.

The element type "s" represents the half of the fixed-end frame element ii' under the symmetric deformation conditions, as depicted in Fig. 3 (s-s is the symmetry plane).

For the element ii', the following relations hold

$$
\begin{aligned}
u_{i'} &= -u_i, v_{i'} = v_i, \varphi_{i'} = -\varphi_i && (a) \\
N_{i'} &= -N_i, T_{i'} = T_i = 0, M_{i'} = -M_i && (b)
\end{aligned}
\tag{6}
$$

The unknowns are displacements u_i, v_i and φ_i at the i end of the element. The stiffness matrix can be derived using Eq. (6) and the stiffness matrix of the fixed end frame element given by Eq. (1) (index k is replaced by index i'). Because of symmetry conditions, it holds

$$
\widehat{A}'_{i'i} = \widehat{A}'_{ii'}, \quad \widehat{B}'_{ii'} = \widehat{B}'_{i'i}, \quad \widehat{S}'_{i'i} = -\widehat{S}'_{ii'} \tag{7}
$$

Introducing the following operator substitutes

$$
2\widehat{N}'_{ii'} = \widehat{N}'_{is}, \quad -2\widehat{S}'_{ii'} = -\widehat{S}'_{is}, \quad \widehat{A}'_{ii'} - \widehat{B}'_{ii'} = \widehat{E}'_{is} \tag{8}
$$

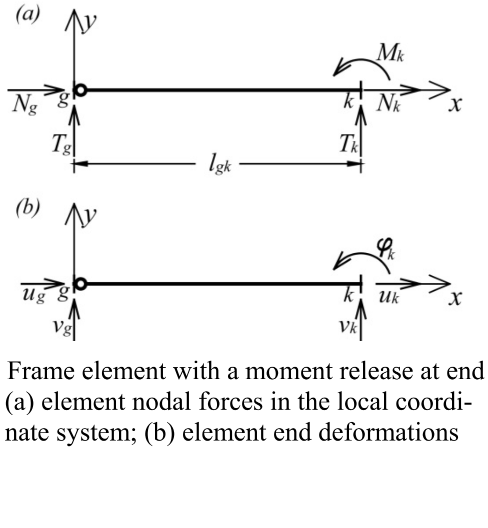

Fig. 2 Frame element with a moment release at end "g": (a) element nodal forces in the local coordinate system; (b) element end deformations

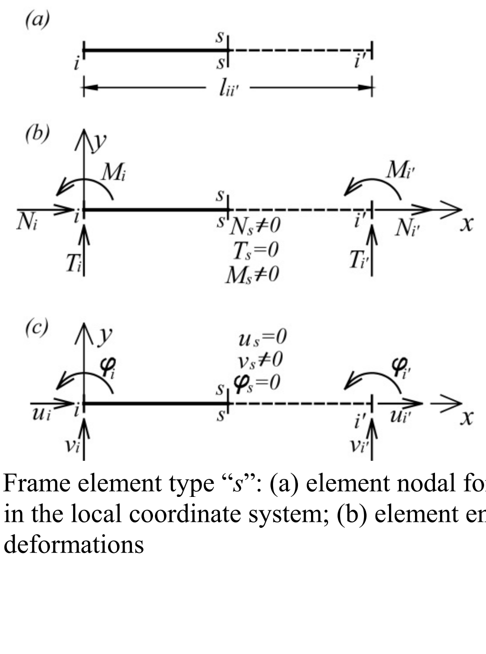

Fig. 3 Frame element type "s": (a) element nodal forces in the local coordinate system; (b) element end deformations

the stiffness matrix of the element type "*s*" can be written in the following form

$$\left[\widehat{K}'\right]_s = \begin{bmatrix} \widehat{N}'_{is} & 0 & \widehat{S}'_{is} \\ 0 & 0 & 0 \\ \widehat{S}'_{is} & 0 & \widehat{E}'_{is} \end{bmatrix} \tag{9}$$

For the element with a constant cross section, the elements of the operator stiffness matrix, Eq. (9), are proportional to one of the operators $\widehat{R}'_{11}$, $\widehat{R}'_{12}$ and $\widehat{R}'_{22}$

$$\begin{aligned} \widehat{N}'_{is} &= \frac{2E_u A_i}{l_{ii'}} \widehat{R}'_{11} \\ \widehat{S}'_{is} &= \frac{2E_u S_i}{l_{ii'}} \widehat{R}'_{12} \\ \widehat{E}'_{is} &= \frac{2E_u J_i}{l_{ii'}} \widehat{R}'_{22} \end{aligned} \tag{10}$$

Vectors of equivalent nodal forces can be determined from relations given in (Deretić-Stojanović and Kostić 2015) and the remaining part of the analysis of composite and prestressed structures follows the standard displacement based finite element method. Therefore, the final system of equations can be written in the form

$$[\widehat{\mathbf{K}}'][\mathbf{q}] = [\mathbf{S}] \tag{11}$$

where $[\widehat{\mathbf{K}}']$ is the stiffness matrix of a structure, $[\mathbf{q}]$ is the vector of displacements and the vector $[\mathbf{S}]$ includes external nodal forces and nodal forces due to the element loads. Decomposing the vector of displacements $[\mathbf{q}]$ into the vector of unknown and known displacements and making the corresponding decompositions on the stiffness matrix and the vector of equivalent nodal forces, the system of ultimate equations for determining the unknown displacements obtains. It should be emphasized here that this ultimate system represent the system of nonhomogeneous integral equations and the stiffness matrix $[\widehat{\mathbf{K}}']$ is the operator matrix. The solution to the integral equations in the closed form can be obtained only for some analytical forms of the concrete creep function (Rate of Creep Method (RCM), Maslov-Arutiunyan's function, Hereditary Theory) (Lazic 2003). In most other cases, the numerical methods are necessary.

3. Simplified analysis method

As explained in the preceding section, the "exact" method of analysis of composite and prestressed beams requires the system of nonhomogeneous integral equations to be solved which complicates the problem. For this reason, the following approximate method is proposed.

We assume that the deformations, i.e. generalized joint displacements (displacements and rotations) q_λ ($\lambda = 1, 2, \ldots n$) change linearly with the concrete creep function F^*, Eq. (B5). If we denote time parameter with t ($t = 0$ is the time of concrete preparing) and with t_0 the age of concrete when first stress and deformation appear (in days), then the introduced assumption implies

$$q_\lambda = q_{\lambda o} 1^* + \Delta q_\lambda (F^* - 1^*), \quad \lambda = 1, 2, \ldots, n \tag{12}$$

where $q_{\lambda o} = q_{\lambda o}(t_o, t_o)$ are known values of deformations, i.e., joint displacements at time $t = t_o$, and Δq_λ are unknowns that should be determined, and which are constant for each pair of time arguments (t, t_o). Bazant in his work (Bazant 1972) introduced the same assumption. With this assumption, the ultimate system of equations modifies. For elements with a constant cross section, in the operator stiffness matrices given by Eqs. (3), (5) and (10) for the fixed end element, element with a moment release at one end and the element type "*s*", respectively, the following integrals appear: $\widehat{R}'_{hl}F^*$, $\widehat{I}'_{22}F^*$, $\widehat{\Phi}'_h F^*$, $h, l = 1, 2$. It will be shown here that, with appropriate transformations, these integrals can be written as a linear combination of functions F^*, B^*_{11} and β^*_h, $h = 1, 2$. Consequently, the ultimate system of nonhomogeneous integral equations transforms into the system of nonhomogeneous algebraic equations with the unknowns Δq_λ ($\lambda = 1, 2, \ldots n$).

Using Eq. (B17), the integral $\widehat{R}'_{hl}F^*$ can be written in the following form

$$\begin{aligned} &\widehat{R}'_{hl}F^* = (\delta_{hl} - \gamma_{hl})\widehat{1}'F^* + \gamma_{hl}\widehat{R}'F^*, \\ &\delta_{hl} = \begin{cases} 1 & \text{for } h = l \\ 0 & \text{for } h \neq l \end{cases} \quad h, l = 1, 2 \end{aligned} \tag{13}$$

According to Eqs. (A3) and (B10), this integral can be written as (F^* is the nondimensional creep function)

$$\begin{aligned} &\widehat{R}'_{hl}F^* = (\delta_{hl} - \gamma_{hl})F^* + \gamma_{hl}1^*, \\ &\delta_{hl} = \begin{cases} 1 & \text{for } h = l \\ 0 & \text{for } h \neq l \end{cases} \quad h, l = 1, 2 \end{aligned} \tag{14}$$

Using relation (B18), the integral $\widehat{I}'_{22}F^*$ has the following form

$$\widehat{I}'_{22}F^* = \frac{\gamma'_1\gamma'_2}{\gamma'_{11}}\widehat{1}'F^* + \frac{\gamma_1\gamma_2}{\gamma_{11}}\widehat{R}'F^* + \left(1 - \frac{\gamma'_1\gamma'_2}{\gamma'_{11}} - \frac{\gamma_1\gamma_2}{\gamma_{11}}\right)\widehat{B}'_{11}F^* \tag{15}$$

In order to determine the value of the integral $\widehat{B}'_{11}F^*$, the following substitutes are useful (Lazic 2003)

$$\widehat{K}'_{11} = \widehat{R}'_{11}F^* \quad \text{(a)}, \qquad \widehat{B}'_{11} = \widehat{R}'\widehat{L}'_{11} \quad (b) \tag{16}$$

where operator $\widehat{L}'_{11}$ is inverse to the operator $\widehat{R}'_{11}$, i.e., the following relation holds true

$$\widehat{R}'_{11}\widehat{L}'_{11} = \widehat{1}' \tag{17}$$

Also, it holds

$$\widehat{K}'_{11}B^*_{11} = 1^* \tag{18}$$

The function K^*_{11} (which is the function corresponding to the operator $\widehat{K}'_{11}$, see Appendix A) depends linearly on the concrete creep function F*:

$$K_{11}^{*} = \gamma_{11} 1^{*} + \gamma_{11}' F^{*} \tag{19}$$

For known creep function F*, and known function K_{11}^{*} (determined from Eq. (19)), function B_{11}^{*} can be determined as a solution to the integral Eq. (18). In its integral form, this equation is

$$K_{11}^{*}(\gamma_{11},t,t)B_{11}^{*}(\gamma_{11},t,\tau) - \int_{\tau}^{t} \frac{\partial K_{11}^{*}(\gamma_{11},t,\theta)}{\partial \theta} B_{h}^{*}(\gamma_{11},t,\theta)d\theta = 1^{*}, \quad \tau \geq t_o. \tag{20}$$

And τ is the age of concrete (in days) when stress, not necessarily first, starts acting, i.e. $t > \tau \geq t_0$.

From Eqs. (B17), (17) and (16b), the following relation may be obtained

$$\widehat{R}_{11}' \widehat{L}_{11}' = \gamma_{11}' \widehat{L}_{11}' + \gamma_{11} \widehat{R}' \widehat{L}_{11}' = \gamma_{11}' \widehat{L}_{11}' + \gamma_{11} \widehat{B}_{11}' = \widehat{1}' \tag{21}$$

Using Eq. (21), operator $\widehat{L}_{11}'$, and its corresponding function L_{11}^{*} are

$$\widehat{L}_{11}' = \frac{1}{\gamma_{11}'} \widehat{1}' - \frac{\gamma_{11}}{\gamma_{11}'} \widehat{B}_{11}', \ (a), \quad L_{11}^{*} = \frac{1}{\gamma_{11}'} 1^{*} - \frac{\gamma_{11}}{\gamma_{11}'} B_{11}^{*} \ \ (b) \tag{22}$$

Multiplying Eq. (16b) by function F* and using Eq. (B10(b)), the following relation obtains

$$\widehat{B}_{11}' F^{*} = \widehat{R}' \widehat{L}_{11}' F^{*} = L_{11}^{*} \tag{23}$$

Also, using relations (A3), (B10), (22b) and (23) integral $\widehat{I}_{22}' F^{*}$ (Eq. (15)), can be written as a linear combination of functions F^{*} and B_{11}^{*}

$$\widehat{I}_{22}' F^{*} = \frac{\gamma_1' \gamma_2'}{\gamma_{11}'} F^{*} + \frac{\gamma_1 \gamma_2}{\gamma_{11}} 1^{*} + \left(1 - \frac{\gamma_1' \gamma_2'}{\gamma_{11}'} - \frac{\gamma_1 \gamma_2}{\gamma_{11}}\right)\left(\frac{1}{\gamma_{11}'} 1^{*} - \frac{\gamma_{11}}{\gamma_{11}'} B_{11}^{*}\right) \tag{24}$$

Operator $\widehat{\Phi}_h'$ (h = 1, 2) in the integral $\widehat{\Phi}_h' F^{*}$ and the corresponding function Φ_h^{*}, are given as (Deretić-Stojanović and Kostić 2014)

$$\widehat{\Phi}_h' = \widehat{1}' - \theta_h \widehat{\beta}_h', \ (a) \quad \Phi_h^{*} = 1^{*} - \theta_h \beta_h^{*}, \ (b) \quad h = 1,2 \tag{25}$$

where θ_h is given by Eq. (C.13), and the operator $\widehat{\beta}_h$ is defined in the following way

$$\widehat{\beta}_h' = \widehat{R}' \widehat{\Phi}_h', \quad h = 1,2 \tag{26}$$

Also, the following relation holds

$$\widehat{\kappa}_h' \beta_h^{*} = \kappa_h^{*} \widehat{\beta}_h' = 1^{*}, \quad h = 1,2 \tag{27}$$

Operator κ_h^{*}, depends linearly on the concrete creep function F^{*}, i.e.

$$\kappa_h^{*} = \kappa_h 1^{*} = \theta_h 1^{*} + F^{*}, \quad h = 1,2 \tag{28}$$

From Eqs. (27) and (28) it follows

$$\theta_h \widehat{\beta}_h' 1^{*} + \widehat{\beta}_h' F^{*} = 1^{*}, \quad h = 1,2 \tag{29}$$

Finaly, multiplying Eq. (25) and using Eq. (29), the integral $\widehat{\Phi}_h' F^{*}$ can be written as a linear combination of functions F^{*} and β_h^{*}, i.e.

$$\widehat{\Phi}_h' F^{*} = F^{*} - \theta_h (1^{*} - \theta_h \beta_h^{*}), \quad h = 1,2 \tag{30}$$

These functions β_h^{*} are solutions to the following parametric nonhomogeneous integral equations (written in the operator form in Eq. (27))

$$\kappa_h^{*}(\lambda,t,t)\beta_h^{*}(\lambda,t,\tau) - \int_{\tau}^{t} \frac{\partial \kappa_h^{*}(\lambda,t,\omega)}{\partial \omega} \beta_h^{*}(\lambda,\omega,\tau)d\omega = 1^{*}, \quad \tau \geq t_o, \ \ h = 1,2 \tag{31}$$

where λ is a parameter ($\lambda = \theta_1$ and $\lambda = \theta_2$), and the integration over the time variable ω goes from time τ to time t.

The application of the presented approximate method is given in the following numerical example. The computations are performed in an own computer program written in FORTRAN.

4. Numerical example

The symmetric composite continuous beam from Fig. 4 is analyzed. Cross sections have two axes of symmetry and the beam is loaded with the uniformly distributed loading q and concentrated forces P (acting at points A, B, A' and B').

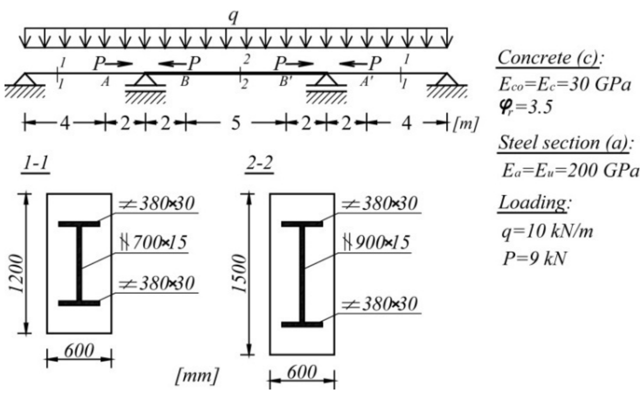

Fig. 4 Continuous composite beam

Table 1 Geometrical properties of cross sections 1-1 and 2-2

	Section 1-1	Section 2-2
A_i (m^2)	0.136305	0.165855
J_i (m^4)	$1.59077915 \cdot 10^{-2}$	$3.02789565 \cdot 10^{-2}$
γ_1	$7.556949488 \cdot 10^{-1}$	$7.811341232 \cdot 10^{-1}$
γ_2	$7.819942511 \cdot 10^{-1}$	$8.070313288 \cdot 10^{-1}$

Fig. 4 contains data about cross sections, loading and materials.

Cross section geometrical properties from Table 1 are calculated according to the expressions given in Appendix C.

Taking advantage of the symmetric characteristics of the continuous beam from Fig. 4, only half of the beam is analyzed. Therefore, there are two unknown generalized displacements: horizontal displacement u and rotation φ (Fig. 5).

The beam from Fig. 5 can be modeled with one fixed end element with a moment release at end 1 (element 1: length $l_1 = l_{gk} = 6$ m), and one element type "s" (element 2: length $l_2 = l_{ii'} = 9$ m). The unknown displacements u and φ are grouped into the vector $\mathbf{q}_n$

$$[\mathbf{q}_n] = \begin{bmatrix} u \\ \varphi \end{bmatrix} \tag{32}$$

Because of the double symmetry of the cross sections 1-1 and 2-2, the following relations hold (according to the Eqs. (C.5)-(C.9))

$$\gamma_{12} = \gamma_{21} = 0,\ \ \gamma_1 = \gamma_{11},\ \ \gamma_2 = \gamma_{22},\ \ \delta\gamma_1 = 0,\ \ \delta\gamma_{21} = \Delta\gamma \tag{33}$$

Therefore, the elements of the operator stiffness matrices for the elements 1 and 2 simplify. According to the Eq. (5), and using Eq. (33), the elements of the operator stiffness matrix for the element 1 with a uniform, double symmetric cross section (Section 1-1) can be found as

$$\begin{aligned} \widehat{N}'_{gk} &= \frac{E_u A_{i,1}}{l_1}(\gamma_1' \widehat{1}' + \gamma_1 \widehat{R}') \\ \widehat{S}'_{gk} &= 0 \\ \widehat{D}'_{gk} &= \frac{3 E_u J_{i,1}}{l_1}(\gamma_2' \widehat{1}' + \gamma_2 \widehat{R}') \end{aligned} \tag{34}$$

The same expressions (34) can, also, be derived directly from the operator flexibility matrix, using Eq. (33) (Deretić-Stojanović and Kostić 2015) and inverting the flexibility matrix to obtain the stiffness matrix.

According to Eqs. (4) and (34), the operator flexibility matrix for the element *1* has the following form

$$\left[\widehat{\mathbf{K}}'\right]_1 = \begin{bmatrix} \widehat{N}'_{gk} & 0 & -\widehat{N}'_{gk} & 0 & 0 \\ & \frac{1}{l_{gk}^2}\widehat{D}'_{gk} & 0 & -\frac{1}{l_{gk}^2}\widehat{D}'_{gk} & \frac{1}{l_{gk}}\widehat{D}'_{gk} \\ & & \widehat{N}'_{gk} & 0 & 0 \\ & \text{symmetric} & & \frac{1}{l_{gk}^2}\widehat{D}'_{gk} & -\frac{1}{l_{gk}}\widehat{D}'_{gk} \\ & & & & \widehat{D}'_{gk} \end{bmatrix}_{l_{gk} = l_1} \tag{35}$$

From Eq. (10), using Eq. (33) and Eq. (B17), the elements of the operator stiffness matrix for the element 2 (element type "s") with a uniform, double symmetric cross section (Section 2-2) can be found as

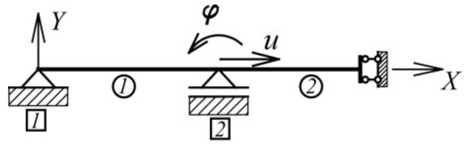

Fig. 5 Symmetric part of the composite beam and unknown displacements

$$\begin{aligned} \widehat{N}'_{is} &= \frac{2 E_u A_{i,2}}{l_2}\widehat{R}'_{11} \\ \widehat{S}'_{is} &= 0 \\ \widehat{E}'_{is} &= \frac{2 E_u J_{i,2}}{l_2}\widehat{R}'_{22} \end{aligned} \tag{36}$$

According to Eqs. (9) and (36), the operator stiffness matrix for the element 2 is in the form

$$\left[\widehat{K}'\right]_2 = \begin{bmatrix} \widehat{N}'_{is} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \widehat{E}'_{is} \end{bmatrix} \tag{37}$$

The vectors of the equivalent nodal forces for the elements 1 and 2 are (Deretić-Stojanović and Kostić 2015)

$$[\mathbf{S}]_1 = \begin{bmatrix} \frac{P}{3}1^* \\ -\frac{ql_1}{2}1^* + \frac{ql_1}{8}1^* \\ \frac{2P}{3}1^* \\ -\frac{ql_1}{2}1^* - \frac{ql_1}{8}1^* \\ \frac{ql_1^2}{8}1^* \end{bmatrix},\ \ [\mathbf{S}]_2 = \begin{bmatrix} -\frac{5P}{9}1^* \\ -\frac{ql_2}{2}1^* \\ -\frac{ql_2^2}{12}1^* \end{bmatrix} \tag{38}$$

Finally, the unknown displacements u and φ can be obtained from the following uncoupled system of two integral equations

$$\begin{aligned} (\widehat{N}'_{gk} + \widehat{N}'_{is})u &= \frac{P}{9}1^* & (a) \\ (\widehat{D}'_{gk} + \widehat{E}'_{is})\varphi &= \frac{ql_1^2}{8}1^* - \frac{ql_2^2}{12}1^* & (b) \end{aligned} \tag{39}$$

In this example, the concrete creep function is adopted in accordance with the creep function of the aging theory with constant concrete modulus of elasticity since, in this case, the system of equations (39) can be solved applying the Laplace transformations. Therefore, the concrete creep function F^* and the corresponding concrete relaxation function R^* (that is solution to the integral Eq. (B11)) are

$$F^* = 1^* + \varphi_r \quad (a), \qquad R^* = e^{-\varphi_r} \quad (b) \tag{40}$$

where φ_r is the reduced concrete creep coefficient defined in Eq. (B2).

Using the assumption of the approximate method (Eq.

(12)), i.e. that displacements u and φ linearly depend on the concrete creep function F^* in the system of Eq. (39), we have

$$\begin{aligned} &(\widehat{N}'_{gk} + \widehat{N}'_{is})(u_o 1^* + \Delta u(F^* - 1^*)) = \frac{P}{9} 1 && (a) \\ &(\widehat{D}'_{gk} + \widehat{E}'_{is})(\varphi_o 1^* + \Delta\varphi(F^* - 1^*)) = \frac{ql_1^2}{8} 1^* - \frac{ql_2^2}{12} 1^* && (b) \end{aligned} \tag{41}$$

Using Eqs. (34), (36) and (B17) for operators $\widehat{N}'_{gk}$, $\widehat{N}'_{is}$, $\widehat{D}'_{gk}$ and $\widehat{E}'_{is}$, the product of the operator $\widehat{R}'$ and the function F^*, i.e., integral $\widehat{R}'F^*$, appears in Eq. (41). According to the relation (B10(b)) this integral is equal to the function 1*, and the Eq. (41) linearly depend on the functions R^* and F^*, i.e., Eq. (41) becomes system of algebraic (not integral) equations with unknowns Δu and $\Delta\varphi$.

For the considered continuous beam, solutions at time $t = t_0$ and time $t \to \infty$ are given in Table 2. In addition, for comparison, the solutions obtained applying the "exact" analysis method from Section 2, and the widely used EM method are also given. As known, the EM method is based on the following algebraic relation between stress σ_c and strain ε_c for concrete

$$\sigma_c(t) = E_{c,eff}(\varepsilon_c - \varepsilon_{cs}), \quad E_{c,eff} = \frac{E_{co}}{1+\varphi_r} \tag{42}$$

where ε_{cs} is the shrinkage strain, and $E_{c,eff}$ is the effective elastic modulus of concrete (Fritz 1961). The EM method is accurate only for the creep function of the Hereditary Theory (HT) in time $t \to \infty$. At $t = t_0$ all three methods give the same solution.

In addition, it should be noted that the creep function of the RCM used in the "exact" method and the creep function of the HT in the EM method give, respectively, the upper and lower bounds for the displacements. Results obtained using other theories, should take place in-between, as solutions of the simplified method satisfy.

Results from the Table 2 show that the simplified method solutions are closer to the solutions of the "exact" analysis method than the solutions of the EM method. This confirms that the introduced assumption about the linear dependency of the displacements on the concrete creep function F^* is reasonable. On the other side, introducing this assumption, the analysis simplify significantly since instead of a system of integral equations, the system of algebraic equations need to be solved. This is a major advantage of the presented simplified method which allows its use in practical engineering applications.

Table 2 Solution for generalized displacements u and φ at time $t = t_0$ and time $t \to \infty$ according to the simplified, EM and the "exact" analysis methods

Time	Method	u (m)	φ (rad)
$t = t_0$	all	8.39E-08	-7.66E-06
	simplified	2.26E-07	-2.15E-05
$t \to \infty$	"exact"	2.39E-07	-2.28E-05
	EM	2.10E-07	-2.00E-05

5. Conclusions

This paper presents the simplified method for the analysis of composite and prestressed structures. The proposed method is based on the previously developed "exact" matrix stiffness method, but introduces the assumption that unknown deformations change linearly with the concrete creep function F^*. With this assumption, the ultimate equations with the unknown deformations that are nonhomogeneous integral equations in the "exact" method, transform into the system of algebraic equations which can be easily solved. Besides this assumption, no other mathematical simplifications are applied. Consequently, as illustrated on the numerical example, the results obtained with the proposed method are very close to the "exact" solution, while the solution algorithm is much simpler and faster, and therefore, more suitable for practical applications.

Acknowledgments

The second author thanks the Ministry of Science of the Republic of Serbia for financial support under project number TR 36046.

References

Amadio, C., Fragiacomo, M. and Macorini, L. (2012), "Evaluation of the deflection of steel-concrete composite beams at serviceability limit state", *J. Constr. Steel Res.*, **73**, 95-104.

Bažant, Z.P. (1972), "Prediction of concrete creep effects using age-adjusted effective modulus method", *J. Am. Concrete Inst.*, **69**, 212-217.

Bažant, Z.P. and Huet, C. (1999), "Thermodynamic functions for ageing viscoelasticity: Integral form without internal variables", *Int. J. Solids Struct.*, **36**(26), 3993-4016.

Bažant, Z.P., Hubler, M.H. and Jirásek, M. (2013), "Improved estimation of long-term relaxation function from compliance function of aging concrete", *J. Eng. Mech.*, **139**(2), 146-152.

Chaudhary, S., Pendharkar, U. and Nagpal, A.K. (2007), "Hybrid procedure for cracking and yime-dependent effects in composite frames at service load", *J. Struct. Eng.*, **133**(2), 166-175.

Deretić-Stojanović, B. and Kostić, S.M. (2014), "Time-dependent analysis of composite and prestressed beams using the slope deflection method", *Arch. Appl. Mech.*, **85**(2), 257-272.

Deretić-Stojanović, B. and Kostić, S.M. (2015), "Matrix stiffness method for composite and prestressed beam analysis using linear integral operators", *Arch. Appl. Mech.*, **85**(12), 1961-1981.

Dezi, L. and Gara, F. (2001), "Time-dependent analysis of shear-lag effect in composite beams", *J. Eng. Mech.*, **127**(1), 71-79.

Dezi, L., Leoni, G. and Tarantino, A.M. (1995), "Time-dependent analysis of prestressed composite beams", *J. Struct. Eng.*, **121**(4), 621-633.

Dezi, L., Leoni, G. and Tarantino, A.M. (1996), "Algebraic methods for creep analysis of continuous composite beams", *J. Struct. Eng.*, **122**(4), 423.

Faella, C., Martinelli, E. and Nigro, E. (2010), "Steel–concrete composite beams in partial interaction: Closed-form "exact" expression of the stiffness matrix and the vector of equivalent nodal forces", *Eng. Struct.*, **32**(9), 2744-2754.

Fan, J., Nie, X., Li, Q. and Li, Q. (2010), "Long-term behavior of

composite beams under positive and negative bending. II: Analytical atudy", *J. Struct. Eng.*, **136**(7), 858-865.
Fragiacomo, M., Amadio, C. and Macorini, L. (2004), "Finite-element model for collapse and long-term analysis of steel–concrete composite beams", *J. Struct. Eng.*, **130**(3), 489-497.
Fritz, B. (1961), *Verbundträger*, Springer-Verlag.
Kwak, H.-G. and Seo, Y.-J. (2000), "Long-term behavior of composite girder bridges", *Comput. Struct.*, **74**(5), 583-599.
Lazic, V.B. (2003), *Mathematical Theory of Composite and Prestressed Structures*, Mathematical Institute SANU, Belgrade, Serbia.
Macorini, L., Fragiacomo, M., Amadio, C. and Izzuddin, B.A. (2006), "Long-term analysis of steel–concrete composite beams: FE modelling for effective width evaluation", *Eng. Struct.*, **28**(8), 1110-1121.
Mirza, O. and Uy, B. (2010), "Finite element model for the long-term behaviour of composite steel-concrete push tests", *Steel Compos. Struct., Int. J.*, **10**(1), 45-67.
Nguyen, Q.-H., Hjiaj, M. and Uy, B. (2010), "Time-dependent analysis of composite beams with continuous shear connection based on a space-exact stiffness matrix", *Eng. Struct.*, **32**(9), 2902-2911.
Partov, D. and Kantchev, V. (2009), "Time-dependent analysis of composite steel-concrete beams using integral equation of volterra, according to Eurocode 4", *Eng. Mech.*, **16**(5), 367-392.
Ranzi, G., Leoni, G. and Zandonini, R. (2013), "State of the art on the time-dependent behaviour of composite steel–concrete structures", *J. Constr. Steel Res.*, **80**, 252-263.
Wu, J., Frangopol, D.M. and Soliman, M. (2015), "Simulating the construction process of steel-concrete composite bridges", *Steel Compos. Struct., Int. J.*, **18**(5), 1239-1258.

BU

Appendix A
Linear integral operators

The linear integral operator $\widehat{G}$ is associated with a function of two variables $G\,(t,\,\tau)$ which satisfies the condition $G\,(t,\tau) \equiv 0$ for $t < \tau$. The operator $\widehat{G}$ is defined for any function $U = U\,(t,\,\tau)$, $\tau \geq t_o$ as

$$I = I(t,\tau) = \widehat{G}U = \int_{\tau}^{t} G(t,\theta)U(\theta,\tau)d\theta \tag{A1}$$

In functions $I\,(t,\,\tau)$ and $U\,(t,\,\tau)$, the second variable τ is a parameter if $\tau = t_o$.

The laws of algebra of ordinary numbers are valid for linear integral operators with the exception of the commutative law, which, generally, does not hold. However, for the rheological properties of materials which are adopted in this paper, all defined operators have the commutative property.

The following notations are adopted

$$\begin{aligned} &G' = G'(t,\tau) = \partial G(t,\tau)/\partial\tau, \\ &1' = 1'(t,\tau) = \delta(t-\tau) = \delta(\tau-t) \\ &1^* = 1^*(t,\tau) = H(t-\tau) = \begin{cases} 1 & \text{for} \quad t > \tau \\ 0 & \text{for} \quad t \leq \tau \end{cases} \qquad \tau \geq t_o \end{aligned} \tag{A2}$$

where $\delta\,(t\text{-}\tau)$ is the Dirac function and $H\,(t\text{-}\tau)$ is the Heaviside step function.

The operator $\widehat{1}'$, associated with the Dirac function, is the unity operator. It can be shown that the following relations hold

$$\begin{aligned} &\widehat{1}'U = U, \quad \widehat{1}'1^* = 1^* \\ &G^* = G^*(t,\tau) = \widehat{G}'1^* \\ &\quad = \int_{\tau}^{t} \frac{\partial G(t,\theta)}{\partial\theta} H(\theta-\tau)d\theta \\ &\quad = G(t,t) - G(t,\tau) \qquad\qquad \tau \geq t_o \end{aligned} \tag{A3}$$

The function G^* is denoted as the integral of the function G'.

Operators $\widehat{G}'$ and $\widehat{L}'$ are inverse when they satisfy relations

$$\widehat{G}'\widehat{L}' = \widehat{1}', \;(a) \qquad \widehat{L}'\widehat{G}' = \widehat{1}' \;\;(b) \tag{A4}$$

Multiplying Eq. (A4) with function 1^* and using Eq. (A3), obtains

$$\widehat{G}'L^* = 1^*, \;(a) \qquad \widehat{L}'G^* = 1^* \;\;(b) \tag{A5}$$

The above relations are the nonhomogeneous integral equations. These equations can be transformed to the Volterra integral equations of the second kind. In that form, the Eq. (A5(a)) is

$$G^*(t,t)L^*(t,\tau) - \int_{\tau}^{t} \frac{\partial G(t,\theta)}{\partial\theta} L^*(\tau,\theta)d\theta = 1^*, \quad \tau \geq t_o \tag{A6}$$

Appendix B
Linear integral operators in linear viscoelastic aging creep theory

The analyzed cross section of a composite and prestressed beam consists of concrete, prestressing steel, steel section and reinforcement. Concrete is modeled as linear viscoelastic aging material, prestressing steel has a relaxation property, while other materials (reinforcement and steel section) behave as linear-elastic.

The following concrete constitute relation is adopted. Using the Boltzmann-Volterra's superposition principle, the uniaxial creep law for concrete is

$$\varepsilon(t,\tau)-\varepsilon_{cs}(t,\tau) = \frac{1}{E_{co}}\left[\frac{1}{e(t)}\sigma_c(t,\tau)+\int_\tau^t \frac{\partial f(t,\theta)}{\partial\theta}\sigma_c(\theta,\tau)d\theta\right], \quad \tau \ge t_o \tag{B1}$$

where $\varepsilon(t,\tau)$ is the total strain, $\varepsilon_{cs}(t,\tau)$ is the shrinkage strain; $\sigma_c(\theta,\tau)$ is the concrete stress. Also

$$\begin{aligned} &e=e(t)=\frac{E_c(t)}{E_{co}},\quad E_{co}=E_c(t_o),\quad e(t_o)=1,\\ &f^*=f^*(t,\tau)=\frac{1}{e(\tau)}1^*-\frac{1}{e(t)}1^*+\frac{E_{co}}{E_{c28}}\varphi(t,\tau),\\ &\frac{E_{co}}{E_{c28}}\varphi(t,\tau)=\varphi_r(t,\tau)=\varphi_r \end{aligned} \tag{B2}$$

where $\varphi(t,\tau)$ is the creep coefficient, $E_c(t)$ is the concrete modulus of elasticity; E_{co} is the elastic modulus of concrete at time t_o, E_{c28} is the elastic modulus of concrete at time t = 28 days, $\varphi_r(t,\tau)=\varphi_r$ is the reduced concrete creep coefficient used in the numerical example and $1^* = 1^*(t, \tau)$ is the Heaviside step function (see Appendix A, Eq. (A2)).

The integral Eq. (B1) may be written symbolically in the operator form (see Appendix A) as

$$\varepsilon-\varepsilon_{cs}=\frac{1}{E_{co}}\widehat{F}'\sigma_c,\quad (\tau=t_o) \tag{B3}$$

where $\widehat{F}'$ is the linear integral operator, with the meaning

$$\widehat{F}'=\frac{1}{e}\widehat{1}'+\widehat{f}' \tag{B4}$$

and the operator $\widehat{1}'$ is the unity operator associated with the Dirac function (see Eq. (A2)).

The non-dimensional concrete creep function F^* is defined as the integral of the function F'

$$\widehat{F}'1^*=\left(\frac{1}{e}\widehat{1}'+\widehat{f}'\right)1^*=F^*(t,\tau)=F^* \tag{B5}$$

The solution to the integral Eq. (B3) is

$$\sigma_c=E_{co}\widehat{R}'(\varepsilon-\varepsilon_{cs}),\quad (\tau=t_o) \tag{B6}$$

where

$$\widehat{R}'=e\widehat{1}'-\widehat{\psi}' \tag{B7}$$

and e is as defined in Eq. (B2); and the operator $\widehat{\psi}'$ is associated with the function ψ' defined below.

The non-dimensional concrete relaxation function R^* is the integral of the function R'

$$R^*=R^*(t,\tau)=\widehat{R}'1^*=e1^*-\psi^*,\quad \tau\ge t_o \tag{B8}$$

where

$$\psi'=\psi'(t,\theta)=-\frac{\partial R^*(t,\theta)}{\partial\theta},\quad t\ge\theta>t_o \tag{B9}$$

It can be shown that the operators $\widehat{F}'$ and $\widehat{R}'$ are inverse, i.e., the following relations hold

$$\widehat{R}'\widehat{F}'=\widehat{F}'\widehat{R}'=\widehat{1}',\ (a)\qquad \widehat{R}'F^*=\widehat{F}'R^*=1^*\ \ (b) \tag{B10}$$

When concrete creep function F^* is known, the concrete relaxation function R^* obtains as a solution to the integral Eq. (B10(b)). Using Eqs. (A1) and (A5) this equation can be written as

$$F^*(t,t)R^*(t,\tau)-\int_\tau^t\frac{\partial F^*(t,\theta)}{\partial\theta}R^*(\tau,\theta)d\theta=1^*,\quad \tau\ge t_o \tag{B11}$$

In the theory of composite structures, the concrete shrinkage function ε_{cs} usually is defined in two ways: as an arbitrary function which describes experimentally obtained curves and as a function with the similar time variation as the concrete creep function F^*, i.e.

$$\varepsilon_{cs}=r(F^*-1^*) \tag{B12}$$

where a constant r is determined for each pair of time arguments (t, t_o).

The non-dimensional relaxation function, $R_p^*=R_p^*(t,\tau)$, is assumed to depends linearly on the concrete non-dimensional relaxation function $R^* = R^*(t, \tau)$ over the interval (τ, t)

$$\begin{aligned} &R_p^*=\widehat{R}_p'1^*=(1-\rho)1^*+\rho R^*\quad (a),\\ &\rho=\frac{\zeta_p}{1^*-R^*}\quad (b)\qquad (\tau\ge t_o) \end{aligned} \tag{B13}$$

In the preceding relations, $\widehat{R}_p'$ is the linear integral operator; ρ is constant for each pair (τ, t) and for the initial stress σ_{po} in prestressing steel; ζ_p is relaxation of prestressing steel that corresponds to the interval (τ, t) and to the initial stress σ_{po}. Finally, for prestressing steel the adopted stress-strain relation can be written in the following operator form

$$\sigma_p=E_p\widehat{R}_p'\varepsilon \tag{B14}$$

E_p is the elastic modulus of prestressing steel.

As mentioned, steel section (a) and reinforcing steel (s), follow the Hook's law

$$\sigma_k = E_k \varepsilon, \quad k = a, s \tag{B15}$$

Referring to the finite element formulation derived in the paper, the following assumptions are adopted: the Bernoulli's hypothesis is valid, i.e., a plane cross section remains plane and there is no slip at the steel section-concrete slab interface. Consequently, the axial strain $\eta = \eta(x,t,t_o)$ and the curvature change of the element axis $\kappa = \kappa(x,t,t_o)$ are solutions of the following system of two integral equations (Deretić-Stojanović and Kostić 2014)

$$\begin{aligned} E_u A_i \widehat{R}'_{11}\eta + E_u S_i \widehat{R}'_{12}\kappa = N, \\ E_u S_i \widehat{R}'_{21}\eta + E_u J_i \widehat{R}'_{22}\kappa = M \end{aligned} \tag{B16}$$

where $N = N(x, t, t_o)$ is the axial force, and $M = M(x, t, t_o)$ is the bending moment. The elements of the symmetric operator matrix $\left[\widehat{R}'_{hl}\right]_{2,2}$ are

$$\begin{aligned} \widehat{R}'_{hl} &= (\delta_{hl} - \gamma_{hl})\widehat{1}' + \gamma_{hl}\widehat{R}', \\ \delta_{hl} &= \begin{cases} 1 \text{ za } h = l \\ 0 \text{ za } h \neq l \end{cases} \quad h,l = 1,2 \end{aligned} \tag{B17}$$

Elements of the symmetric matrix $[\gamma_{hl}]_{2,2}$ and its principal values γ_h (h = 1, 2) are defined in Appendix C.

New operator $\widehat{I}'_{22}$ is defined as

$$\widehat{I}'_{22} = \frac{\gamma'_1\gamma'_2}{\gamma'_{11}}\widehat{1}' + \frac{\gamma_1\gamma_2}{\gamma_{11}}\widehat{R}' + \left(1 - \frac{\gamma'_1\gamma'_2}{\gamma'_{11}} - \frac{\gamma_1\gamma_2}{\gamma_{11}}\right)\widehat{B}'_{11} \tag{B18}$$

Operators $\widehat{R}'$ and $\widehat{B}'_{11}$ are defined in Eqs. (B7) and (16b).

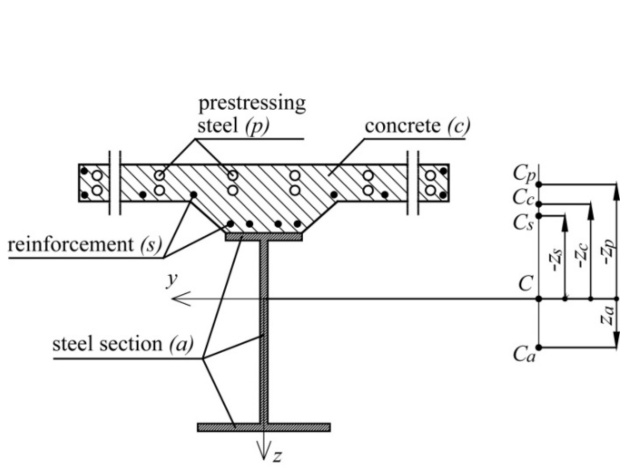

Fig. C1 Composite cross section

Appendix C
Reduced cross section geometrical properties

The reduced geometrical properties of the cross section from Fig. C1 are defined as follows. The area of each part of a composite section k ($k = c, p, a, s$) is denoted with A_k. The reduced areas, A_{kr}, and the area of the transformed cross section (with centroid denoted with C in Fig. C1) A_i are

$$A_{kr} = \nu_k A_k, \quad (a) \qquad A_i = \sum_k A_{kr}, \quad k = c, p, a, s \quad (b) \tag{C1}$$

where ν_k are reducing factors

$$\nu_c = \frac{E_{co}}{E_u}, \quad \nu_p = \frac{E_p}{E_u}, \quad \nu_k = \frac{E_k}{E_u}, \quad k = a, s \tag{C2}$$

and E_p, E_a and E_s are the modulus of elasticity of prestressing steel, steel section and reinforcement, respectively; E_u is the relative modulus of elasticity.

The first moment of area of a reduced area A_{kr} with respect to the y axis is

$$S_{kr} = z_k A_{kr}, \quad k = c, p, a, s \tag{C3}$$

z_k is the ordinate of the centroid C_k ($k = c, p, a, s$) of an area A_k (Fig. C1).

The moment of inertia of a reduced area A_{kr} about the axes y passing through the centroid C, J_{kr}, and the moment of inertia of a transformed cross section, J_i, are

$$\begin{aligned} J_{kr} &= I_{kr} + z_k^2 A_{kr}, \quad \text{(a)} \\ J_i &= \sum_k J_{kr}, \quad k = c, p, a, s \quad \text{(b)} \end{aligned} \tag{C4}$$

where I_{kr} is the moment of inertia of a reduced area A_{kr} about the axes passing through the centroid C_k, parallel to the y axis.

The elements γ_{hl} of the symmetric scalar matrix of the reduced cross section geometry $[\gamma_{hl}]_{2,2}$ are defined as

$$\begin{aligned} \gamma_{11} = \frac{A_{cr}}{A_i} + \rho\frac{A_{pr}}{A_i}, \quad \gamma_{22} = \frac{J_{cr}}{J_i} + \rho\frac{J_{pr}}{J_i}, \\ \gamma_{12} = \gamma_{21} = \frac{S_{cr}}{S_i} + \rho\frac{S_{pr}}{S_i} \end{aligned} \tag{C5}$$

where ρ is defined in Eq. (B13(b)) and S_i is defined as $S_i = \sqrt{A_i J_i}$. The elements of the symmetric matrix $[\gamma'_{hl}]_{2,2}$ are

$$\gamma'_{11} = 1 - \gamma_{11}, \quad \gamma'_{22} = 1 - \gamma_{22}, \quad \gamma'_{12} = \gamma'_{21} = -\gamma_{12} \tag{C6}$$

The principal values of the matrix $[\gamma_{hl}]_{2,2}$ are denoted with γ_h and of matrix $[\gamma'_{hl}]_{2,2}$ are denoted with γ'_h. The following relations hold

$$\gamma_h + \gamma'_h = 1, \quad h = 1, 2 \tag{C7}$$

The principal values are ordered in the following way

$$1 > \gamma_1 > \gamma_2 > 0, \ (a) \quad 1 > \gamma_2' > \gamma_1' > 0, \ (b) \tag{C8}$$

In addition, the following substitutes are used in the paper

$$\delta\gamma_1 = \gamma_1 - \gamma_{11}, \quad \delta\gamma_2 = \gamma_{11} - \gamma_2, \quad \Delta\gamma = \gamma_1 - \gamma_2 \tag{C9}$$

Finally, the constants with the following meaning are used in Eq. (5)

$$\begin{aligned}
&n_1 = \gamma_{11}' + \gamma_{11}\left(\frac{t}{\theta_1} + \frac{g}{\theta_2}\right), \ n_2 = t\left(\gamma_{11}' - \frac{\gamma_{11}}{\theta_1}\right), \\
&n_3 = g\left(\gamma_{11}' - \frac{\gamma_{11}}{\theta_2}\right), \\
&d_1 = \gamma_{22}' + \gamma_{22}\left(\frac{t}{\theta_1} + \frac{g}{\theta_2}\right), \ d_2 = t\left(\gamma_{22}' - \frac{\gamma_{22}}{\theta_1}\right), \\
&d_3 = g\left(\gamma_{22}' - \frac{\gamma_{22}}{\theta_2}\right), \\
&z_1 = \gamma_{12}\left(-1 + \frac{t}{\theta_1} + \frac{g}{\theta_2}\right), \ z_2 = -\gamma_{12} t\left(1 + \frac{1}{\theta_1}\right), \\
&z_3 = -\gamma_{12} g\left(1 + \frac{1}{\theta_2}\right)
\end{aligned} \tag{C10}$$

$$\begin{aligned}
&s = \frac{\gamma_1\gamma_2}{p\theta_1\theta_2}, \quad g = \frac{-\theta_1(\gamma_1 - \theta_1\gamma_1')(\gamma_2 - \theta_2\gamma_2')}{\Delta\theta\gamma_1\gamma_2}, \\
&t = \frac{\theta_2(\gamma_1 - \theta_1\gamma_1')(\gamma_2 - \theta_2\gamma_2')}{\Delta\theta\gamma_1\gamma_2}
\end{aligned} \tag{C11}$$

$$\begin{aligned}
&p = \frac{4\gamma_1'\gamma_2' + \gamma_{12}^2}{12}, \quad r = \frac{4\gamma_1\gamma_2 + \gamma_{12}^2}{2}, \\
&q = \frac{4(\gamma_1'\gamma_2 + \gamma_1\gamma_2') - 2\gamma_{12}^2}{2}
\end{aligned} \tag{C12}$$

$$\theta_1 + \theta_2 = \frac{q}{p}, \quad \theta_1\theta_2 = \frac{r}{p}, \quad \Delta\theta = \theta_1 - \theta_2 \tag{C13}$$