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A novel four variable refined plate theory for laminated composite plates

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Abstract. A novel four variable refined plate theory is proposed in this work for laminated composite plates. The theory considers a parabolic distribution of the transverse shear strains, and respects the zero traction boundary conditions on the surfaces of the plate without employing shear correction coefficient. The displacement field is based on a novel kinematic in which the undetermined integral terms are used, and only four unknowns are involved. The analytical solutions of antisymmetric cross-ply and angle-ply laminates are determined via Navier technique. The obtained results from the present model are compared with three-dimensional elasticity solutions and results of the first-order and the other higher-order theories reported in the literature. It can be concluded that the developed theory is accurate and simple in investigating the bending and buckling responses of laminated composite plates.

Keywords: laminated composite plates; refined plate theory; navier solution

1. Introduction

The composite materials are widely utilized in civil, aerospace, automobile and other engineering industries because of their advantage of high stiffness and strength to weight ratio. With the ever-increasing use of laminated composites in engineering structures, a variety of laminated models have been proposed to predict its behavior. A critical review of more recent studies on the development of laminated models can be found in Refs. (Ghugal and Shimpi 2002, Khandan *et al*. 2012). These theories can be classified as equivalent single layer (ESL), layer-wise and zig–zag theories. The ESL theories can be divided into three main categories: classical laminated plate theory (CLPT), first-order shear deformation theory (FSDT), and higher-order shear deformation theories (HSDTs).

The classical laminated plate theory (CLPT), which ignores the transverse normal and shear stresses, predicts acceptable results for thin plates. However, it underestimates deflections and overestimates frequencies as well as buckling loads with moderately thick laminates (Reddy 1997). The first-order shear deformation theory (FSDT) based on Reissner (1945) and Mindlin (1951) is simple to implement and applied for moderately thick plates and provides acceptable results but

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depends on a shear correction factor which is hard to compute as it depends on many parameters (Whitney and Pagano 1970, Noor and Burton 1989, Khdeir 1989, Chakraborty *et al*. 2003, Li 2008, Sina *et al*. 2009, Wei *et al*. 2012, Bellifa *et al*. 2016, Bouderba *et al*. 2016). However, there is no requiring of shear correction coefficients when employing higher-order shear deformation theories (HSDTs). Among these models we can cite the third-order theory of Reddy (Reddy 1984, Yesilce 2010, Yesilce and Catal 2009 and 2011, Zidi *et al*. 2014, Ait Atmane *et al*. 2015, Ait Yahia *et al*. 2015, Boukhari *et al*. 2016, Bounouara *et al*. 2016, Bourada *et al*. 2016), the sinusoidal theories (Touratier 1991, Tounsi *et al*. 2013, Bouderba *et al*. 2013, Ait Amar Meziane *et al*. 2014, Draiche *et al*. 2014, Al-Basyouni *et al*. 2015, Hamidi *et al*. 2015, Beldjelili *et al*. 2016, Houari *et al*. 2016), the hyperbolic models (Soldatos 1992, Belabed *et al*. 2014, Akavci 2014, Bousahla *et al*. 2014, 2016, Hebali *et al*. 2014, Mahi *et al*. 2015, Bourada *et al*. 2015, Attia *et al*. 2015, Bouchafa *et al*. 2015, Belkorissat *et al*. 2015, Bennoun *et al*. 2016, Tounsi *et al*. 2016), the inverse hyperbolic theories (Sahoo and Singh 2013, Grover *et al*. 2013), and the exponential theory of Karama *et al*. (2003). Xiang *et al*. (2011) presented a *n*-order shear deformation theory in which Reddy's theory comes out as special case. Kant and Pandya (1988), Mallikarjuna and Kant (1989) and Kant and Khare (1997) proposed also polynomial HSDTs with cubic variations for axial displacements as in the article by Reddy (1984). To consider the thickness stretching effect (i.e., $\varepsilon_z \neq 0$), Lo *et al*. (1977) and Kant *et al*. (1988) proposed HSDTs in which axial and transverse displacements are supposed as cubic and parabolic distributions within the thickness, respectively. A review of various shear deformation models for the investigation of laminated composite plates is available in references (Reddy 1990, Mallikarjuna and Kant 1993).

It is worth indicating that some of the above cited HSDTs are computational costly due to additional unknowns introduced to the theory (e.g., theories by Kant and Pandya (1988) and Mallikarjuna and Kant (1989) with seven unknowns, Kant and Khare (1997) with nine unknowns, Lo *et al*. (1977) and Kant *et al*. (1988) with 11 unknowns). Although some well-known HSDTs contain five unknowns as in the case of FSDT (e.g., theories by Reddy (1984), Xiang *et al*. (2009, 2011), Touratier (1991), Ferreira *et al*. (2005), Soldatos (1992), Akavci (2010), Grover *et al*. (2013) and Karama *et al*. (2003)), their equations of motion are much more complicated than those of FSDT. Thus, needs exist for the development of shear deformation theory which is simple to use.

Recently, a new FSDT with four variables is proposed by Mantari and Ore (2015). The aim of this work is to improve the novel FSDT developed by Mantari and Ore (2015) by considering higher-order variations of axial displacements across the plate thickness and studying the bending and buckling behavior of laminated composite plates. Navier solution is employed to determine the analytical solutions for simply supported antisymmetric cross-ply and angle-ply laminates. To demonstrate the accuracy of the present formulation, the computed results are compared with three-dimensional elasticity solutions and results of the FSDT and HSDTs.

2. Theory and formulation

Consider a rectangular plate of total thickness *h* composed of *n* orthotropic layers with the coordinate system as indicated in Fig. 1.

2.1 Kinematics

In this work, further simplifying assumptions are considered to the existing HSDT so that the number of variables is diminished. The displacement field of the existing HSDT is given by

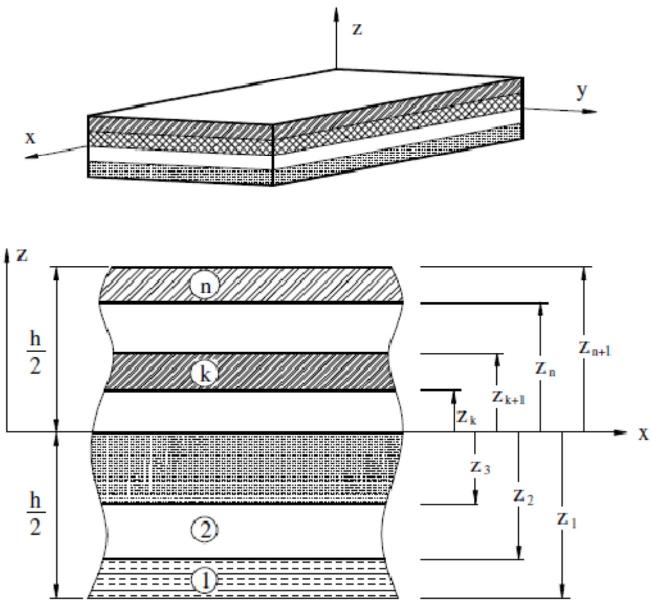

Fig. 1 Coordinate system and layer numbering used for a typical laminated plate

$$u(x,y,z) = u_0(x,y) - z\frac{\partial w_0}{\partial x} + f(z)\varphi_x(x,y) \tag{1a}$$

$$v(x,y,z) = v_0(x,y) - z\frac{\partial w_0}{\partial y} + f(z)\varphi_y(x,y) \tag{1b}$$

$$w(x,y,z) = w_0(x,y) \tag{1c}$$

where u_0, v_0, w_0, φ_x and φ_y are five generalized displacements, $f(z)$ is the shape function representing the distribution of the transverse shear strains and stresses across the thickness. By supposing that $\varphi_x = \int \theta(x,y)dx$ and $\varphi_y = \int \theta(x,y)dy$, the kinematic of the present theory can be written in a simpler form as

$$u(x,y,z) = u_0(x,y) - z\frac{\partial w_0}{\partial x} + k_1 f(z)\int \theta(x,y)\,dx \tag{2a}$$

$$v(x,y,z) = v_0(x,y) - z\frac{\partial w_0}{\partial y} + k_2 f(z)\int \theta(x,y)\,dy \tag{2b}$$

$$w(x,y,z) = w_0(x,y) \tag{2c}$$

where $u_0(x, y)$, $v_0(x, y)$, $w_0(x, y)$ and $\theta(x, y)$ are the four unknown displacement functions of middle surface of the plate. The constants k_1 and k_2 depends on the geometry. The integrals employed are undetermined.

In this article, the current higher-order shear deformation plate theory is obtained by putting

$$f(z) = z\left(\frac{5}{4} - \frac{5z^2}{3h^2}\right) \tag{3}$$

The strains associated with the displacements in Eq. (2) are

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} k_x^b \\ k_y^b \\ k_{xy}^b \end{Bmatrix} + f(z) \begin{Bmatrix} k_x^s \\ k_y^s \\ k_{xy}^s \end{Bmatrix}, \qquad \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = g(z) \begin{Bmatrix} \gamma_{yz}^0 \\ \gamma_{xz}^0 \end{Bmatrix}, \tag{4}$$

where

$$\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial x} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{Bmatrix}, \qquad \begin{Bmatrix} k_x^b \\ k_y^b \\ k_{xy}^b \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\partial^2 w_0}{\partial x^2} \\ -\dfrac{\partial^2 w_0}{\partial y^2} \\ -2\dfrac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix},$$

$$\begin{Bmatrix} k_x^s \\ k_y^s \\ k_{xy}^s \end{Bmatrix} = \begin{Bmatrix} k_1 \theta \\ k_2 \theta \\ k_1 \dfrac{\partial}{\partial y} \int \theta \, dx + k_2 \dfrac{\partial}{\partial x} \int \theta \, dy \end{Bmatrix}, \qquad \begin{Bmatrix} \gamma_{yz}^0 \\ \gamma_{xz}^0 \end{Bmatrix} = \begin{Bmatrix} k_1 \int \theta \, dy \\ k_2 \int \theta \, dx \end{Bmatrix}, \tag{5a}$$

and

$$g(z) = \frac{df(z)}{dz} \tag{5b}$$

The integrals used in the above relations shall be resolved by a Navier solution and can be expressed by

$$\frac{\partial}{\partial y} \int \theta \, dx = A' \frac{\partial^2 \theta}{\partial x \partial y}, \qquad \frac{\partial}{\partial x} \int \theta \, dy = B' \frac{\partial^2 \theta}{\partial x \partial y}, \qquad \int \theta \, dx = A' \frac{\partial \theta}{\partial x}, \qquad \int \theta \, dy = B' \frac{\partial \theta}{\partial y} \tag{6}$$

where the parameters A' and B' are defined according to the type of solution employed, in this case via Navier. Hence, A' and B' are expressed by

$$A' = -\frac{1}{\alpha^2}, \quad B' = -\frac{1}{\beta^2}, \quad k_1 = \alpha^2, \; k_2 = \beta^2 \tag{7}$$

where α and β are defined in expression (22).

2.2 Constitutive equations

Under the supposition that each layer contains a plane of elastic symmetry parallel to the x–y plane, the constitutive equations for a layer can be expressed as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 \\ 0 & 0 & 0 & 0 & Q_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \tag{8}$$

where Q_{ij} are the plane stress-reduced stiffnesses, and are expressed in terms of the engineering constants in the material axes of the layer

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}};\; Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}};\; Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}};\; Q_{66} = G_{12};\; Q_{44} = G_{23};\; Q_{55} = G_{13} \tag{9}$$

The constitutive equations of each lamina must be transformed to the laminate coordinates (x, y, z). The stress-strain relations in the laminate coordinates of the kth layer are expressed as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix}^{(k)} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} & 0 & 0 \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} & 0 & 0 \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} & 0 & 0 \\ 0 & 0 & 0 & \overline{Q}_{44} & \overline{Q}_{45} \\ 0 & 0 & 0 & \overline{Q}_{45} & \overline{Q}_{55} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}^{(k)} \tag{10}$$

where $\overline{Q}_{ij}$ are the transformed material constants given in Reddy (1997).

2.3 Governing equations

The governing equations will be determined by employing principle of the minimum total potential energy as follows

$$\delta\Pi = \delta(U - V) = 0 \tag{11}$$

where Π is the total potential energy. δU is the virtual variation of the strain energy; and δV is the variation of work done by external forces. The first variation of the strain energy is given as

$$\begin{aligned} \delta U &= \int_V \left[\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \tau_{xy}\delta\gamma_{xy} + \tau_{yz}\delta\gamma_{yz} + \tau_{xz}\delta\gamma_{xz}\right] dV \\ &= \int_A \Big[N_x \delta\varepsilon_x^0 + N_y \delta\varepsilon_y^0 + N_{xy}\delta\gamma_{xy}^0 + M_x^b \delta k_x^b + M_y^b \delta k_y^b + M_{xy}^b \delta k_{xy}^b \\ &\quad + M_x^s \delta k_x^s + M_y^s \delta k_y^s + M_{xy}^s \delta k_{xy}^s + S_{yz}^s \delta\gamma_{yz}^s + S_{xz}^s \delta\gamma_{xz}^0\Big] dA = 0 \end{aligned} \tag{12}$$

where A is the top surface and the stress resultants N, M, and S are expressed by

$$\left(N_i, M_i^b, M_i^s\right) = \int_{-h/2}^{h/2} (1, z, f)\sigma_i dz, \quad (i = x, y, xy) \quad \text{and} \quad \left(S_{xz}^s, S_{yz}^s\right) = \int_{-h/2}^{h/2} g\left(\tau_{xz}, \tau_{yz}\right) dz \tag{13}$$

The variation of the external work can be expressed as

$$\delta V = -\int_A q\delta\, w_0 dA - \int_A \left(N_x^0 \frac{\partial\, w_0}{\partial x}\frac{\partial \delta\, w_0}{\partial x} + 2N_{xy}^0 \frac{\partial\, w_0}{\partial x}\frac{\partial \delta\, w_0}{\partial y} + N_y^0 \frac{\partial\, w_0}{\partial y}\frac{\partial \delta\, w_0}{\partial y} \right) dA \tag{14}$$

where q and (N_x^0, N_y^0, N_{xy}^0) are transverse and in-plane applied loads, respectively.

Substituting the relations for δU, and δV from Eqs. (12) and (14) into Eq. (11) and integrating by parts, and collecting the coefficients of δu_0, δv_0, δw_0, and $\delta\theta$, the following governing equations for the laminate plate are obtained as follows

$$\begin{aligned}
&\delta\, u_0 :\ \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \\
&\delta\, v_0 :\ \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \\
&\delta\, w_0 :\ \frac{\partial^2 M_x^b}{\partial x^2} + 2\frac{\partial^2 M_{xy}^b}{\partial x \partial y} + \frac{\partial^2 M_y^b}{\partial y^2} + q + N_x^0 \frac{\partial^2 w_0}{\partial x^2} + 2\, N_{xy}^0 \frac{\partial^2 w_0}{\partial x \partial y} + N_y^0 \frac{\partial^2 w_0}{\partial y^2} = 0 \\
&\delta\, \theta :\ -\mathrm{k}_1 M_x^s - \mathrm{k}_2 M_y^s - \left(k_1 A' + k_2 B'\right)\frac{\partial^2 M_{xy}^s}{\partial x \partial y} + k_1 A' \frac{\partial S_{xz}^s}{\partial x} + k_2 B' \frac{\partial S_{yz}^s}{\partial y} = 0
\end{aligned} \tag{15}$$

Substituting Eq. (10) into Eq. (13) and integrating within the thickness of the plate, the stress resultants are written as

$$\begin{Bmatrix} \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} \\ \begin{Bmatrix} M_x^b \\ M_y^b \\ M_{xy}^b \end{Bmatrix} \\ \begin{Bmatrix} M_x^s \\ M_y^s \\ M_{xy}^s \end{Bmatrix} \end{Bmatrix} = \begin{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} & \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} & \begin{bmatrix} B_{11}^s & B_{12}^s & B_{16}^s \\ B_{12}^s & B_{22}^s & B_{26}^s \\ B_{16}^s & B_{26}^s & B_{66}^s \end{bmatrix} \\ \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} & \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} & \begin{bmatrix} D_{11}^s & D_{12}^s & D_{16}^s \\ D_{12}^s & D_{22}^s & D_{26}^s \\ D_{16}^s & D_{26}^s & D_{66}^s \end{bmatrix} \\ \begin{bmatrix} B_{11}^s & B_{12}^s & B_{16}^s \\ B_{12}^s & B_{22}^s & B_{26}^s \\ B_{16}^s & B_{26}^s & B_{66}^s \end{bmatrix} & \begin{bmatrix} D_{11}^s & D_{12}^s & D_{16}^s \\ D_{12}^s & D_{22}^s & D_{26}^s \\ D_{16}^s & D_{26}^s & D_{66}^s \end{bmatrix} & \begin{bmatrix} H_{11}^s & H_{12}^s & H_{16}^s \\ H_{12}^s & H_{22}^s & H_{26}^s \\ H_{16}^s & H_{26}^s & H_{66}^s \end{bmatrix} \end{bmatrix} \begin{Bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} \\ \begin{Bmatrix} k_x^b \\ k_y^b \\ k_{xy}^b \end{Bmatrix} \\ \begin{Bmatrix} k_x^s \\ k_y^s \\ k_{xy}^s \end{Bmatrix} \end{Bmatrix} \tag{16a}$$

$$\begin{Bmatrix} S_{yz}^s \\ S_{xz}^s \end{Bmatrix} = \begin{bmatrix} A_{44}^s & A_{45}^s \\ A_{45}^s & A_{55}^s \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^0 \\ \gamma_{xz}^0 \end{Bmatrix} \tag{16b}$$

and stiffness components are given as

$$(A_{ij}, B_{ij}, B_{ij}^s, D_{ij}, D_{ij}^s, H_{ij}^s) = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} \overline{Q}_{ij}^{(k)} (1, z, f(z), z^2, z\,f(z), f(z)^2)\; dz \;, \;(i,j = 1,2,6), \tag{17a}$$

$$A_{ij}^s = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} \overline{Q}_{ij}^{(k)} \,(g(z))^2\, dz\,,\; (i,j = 4,5) \tag{17b}$$

Eq. (15) can be written in terms of displacements (u_0, v_0, w_0, θ) by substituting for the stress resultants from Eq. (16). For homogeneous laminates, the governing Eq. (15) take the form

$$\begin{aligned}
&A_{11} d_{11} u_0 + 2A_{16}\, d_{12} u_0 + A_{66}\, d_{22} u_0 + (A_{12} + A_{66}) d_{12} v_0 + A_{26}\, d_{22} v_0 + A_{16}\, d_{11} v_0 \\
&- (B_{11} d_{111} w_0 + 3B_{16} d_{112} w_0 + (B_{12} + 2B_{66}) d_{122} w_0 + B_{26} d_{222} w_0) \\
&+ (k_1 A' + k_2 B') B_{66}^s\, d_{122}\theta + (k_1 B_{11}^s + k_2 B_{12}^s)\, d_1\theta + (k_1 A' + k_2 B') B_{16}^s\, d_{112}\theta - (k_1 B_{16}^s + k_2 B_{26}^s) d_2\theta = 0,
\end{aligned} \tag{18a}$$

$$\begin{aligned}
&A_{11} d_{16} u_0 + (A_{12} + A_{66}) d_{12} u_0 + A_{26}\, d_{22} u_0 + A_{66}\, d_{22} v_0 + 2A_{26}\, d_{12} v_0 + A_{22}\, d_{22} v_0 \\
&- (B_{16} d_{111} w_0 + 3B_{26} d_{122} w_0 + (B_{12} + 2B_{66}) d_{112} w_0 + B_{22} d_{222} w_0) \\
&+ (k_1 A' + k_2 B') B_{66}^s\, d_{112}\theta + (k_2 B_{22}^s + k_1 B_{12}^s)\, d_2\theta + (k_1 A' + k_2 B') B_{26}^s\, d_{122}\theta + (k_1 B_{16}^s + k_2 B_{26}^s) d_1\theta = 0,
\end{aligned} \tag{18b}$$

$$\begin{aligned}
&(B_{11} d_{111} u_0 + 3B_{16} d_{112} u_0 + (B_{12} + 2B_{66}) d_{122} u_0 + B_{26} d_{222} u_0) \\
&+ (B_{16} d_{111} v_0 + 3B_{26} d_{122} v_0 + (B_{12} + 2B_{66}) d_{112} v_0 + B_{22} d_{222} v_0) \\
&- D_{11} d_{1111} w_0 - 2(D_{12} + 2D_{66})\, d_{1122} w_0 - D_{22}\, d_{2222} w_0 - 4D_{16}\, d_{1112} w_0 - 4D_{26}\, d_{1222} w_0 \\
&+ (k_1 D_{11}^s + k_2 D_{12}^s)\, d_{11}\theta + 2(k_1 A' + k_2 B') D_{66}^s\, d_{1122}\theta + (k_1 D_{12}^s + k_2 D_{22}^s)\, d_{22}\theta + 2(k_1 D_{16}^s + k_2 D_{26}^s)\, d_{12}\theta \\
&+ (k_1 A' + k_2 B') D_{16}^s\, d_{1112}\theta + (k_1 A' + k_2 B') D_{26}^s\, d_{1222}\theta + N_x^0\, d_{11} w_0 + 2N_{xy}^0\, d_{12} w_0 + N_y^0\, d_{22} w_0 + q = 0
\end{aligned} \tag{18c}$$

$$\begin{aligned}
&-\begin{pmatrix} (k_1 A' + k_2 B') B_{66}^s\, d_{122} u_0 + (k_1 B_{11}^s + k_2 B_{12}^s)\, d_1 u_0 \\ + (k_1 A' + k_2 B') B'\, B_{16}^s\, d_{112} u_0 - (k_1 B_{16}^s + k_2 B_{26}^s) d_2 u_0 \end{pmatrix} \\
&-\begin{pmatrix} (k_1 A' + k_2 B') B_{66}^s\, d_{112} v_0 + (k_2 B_{22}^s + k_1 B_{12}^s)\, d_2 v_0 \\ + (k_1 A' + k_2 B') B_{26}^s\, d_{122} v_0 + (k_1 B_{16}^s + k_2 B_{26}^s) d_1 v_0 \end{pmatrix} \\
&+ (k_1 D_{11}^s + k_2 D_{12}^s)\, d_{11} w_0 + 2(k_1 A' + k_2 B') D_{66}^s\, d_{1122} w_0 + (k_1 D_{12}^s + k_2 D_{22}^s)\, d_{22} w_0 \\
&+ 2(k_1 D_{16}^s + k_2 D_{26}^s)\, d_{12} w_0 + (k_1 A' + k_2 B') D_{16}^s\, d_{1112} w_0 + (k_1 A' + k_2 B') D_{26}^s\, d_{1222} w_0 \\
&- H_{11}^s k_1\theta - H_{22}^s\, k_2\theta - 2H_{12}^s k_1 k_2\theta - (k_1 A' + k_2 B')^2 H_{66}^s\, d_{1122}\theta - 2(k_1 A' + k_2 B')(H_{16}^s + H_{26}^s) d_{12}\theta \\
&+ A_{44}^s (k_2 B')^2\, d_{22}\theta + A_{55}^s\, (k_1 A')^2\, d_{11}\theta + 2A_{45}^s\, k_1 k_2\, A' B'\, d_{12}\,\theta = 0
\end{aligned} \tag{18d}$$

where d_{ij}, d_{ijl} and d_{ijlm} are the following differential operators

$$d_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}, \quad d_{ijl} = \frac{\partial^3}{\partial x_i \partial x_j \partial x_l}, \quad d_{ijlm} = \frac{\partial^4}{\partial x_i \partial x_j \partial x_l \partial x_m}, \quad d_i = \frac{\partial}{\partial x_i}, \quad (i, j, l, m = 1,2). \tag{19}$$

3. Analytical solutions for anti-symmetric cross-ply laminates

The Navier procedure is utilized to determine the closed-form solutions of the partial differential equations in Eq. (18) for simply supported rectangular plates. For anti-symmetric cross-ply laminated plates, the following stiffness components are identically zero

$$\begin{aligned} &A_{16} = A_{26} = D_{16} = D_{26} = B_{16}^s = B_{26}^s = D_{16} = D_{26} = D_{16}^s = D_{26}^s = H_{16}^s = H_{26}^s = A_{45}^s = 0 \\ &B_{12} = B_{66} = B_{12}^s = B_{66}^s = 0 \end{aligned} \tag{20}$$

Based on the Navier method, the following expansions of displacements are adopted to automatically respect the simply supported boundary conditions of plate

$$\begin{Bmatrix} u_0 \\ v_0 \\ w_0 \\ \theta \end{Bmatrix} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \begin{Bmatrix} U_{mn}\cos(\alpha\, x)\sin(\beta\, y) \\ V_{mn}\sin(\alpha\, x)\cos(\beta\, y) \\ W_{mn}\sin(\alpha\, x)\sin(\beta\, y) \\ X_{mn}\sin(\alpha\, x)\sin(\beta\, y) \end{Bmatrix} \tag{21}$$

where U_{mn}, V_{mn}, W_{mn} and X_{mn} are coefficients, and α and β are expressed as

$$\alpha = m\pi / a \qquad \text{and} \qquad \beta = n\pi / b \tag{22}$$

The transverse load q is also expanded in the double-Fourier sine series as

$$q(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Q_{mn}\sin(\alpha\, x)\sin(\beta\, y), \tag{23}$$

Substituting Eqs. (23), (21) and (20) into Eq. (18), the Navier solution of anti-symmetric cross-ply laminates can be deduced from equations

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{12} & S_{22} & S_{23} & S_{24} \\ S_{13} & S_{23} & S_{33}+k & S_{34} \\ S_{14} & S_{24} & S_{34} & S_{44} \end{bmatrix} \begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \\ X_{mn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ q_{mn} \\ 0 \end{Bmatrix} \tag{24}$$

where

$$S_{11} = -\left(A_{11}\alpha^2 + A_{66}\beta^2\right), \quad S_{12} = -\alpha\beta\left(A_{12} + A_{66}\right), \quad S_{13} = \alpha\left(B_{11}\alpha^2\right), \quad S_{14} = \alpha\left(k_1 B_{11}^s\right), \tag{25}$$

$$
\begin{aligned}
&S_{22} = -\left(A_{66}\,\alpha^2 + A_{22}\,\beta^2\right), \quad S_{23} = \beta\left(B_{22}\beta^2\right), \quad S_{24} = \beta\left(k_2 B_{22}^s\right),\\
&S_{33} = -\left(D_{11}\alpha^4 + 2(D_{12} + 2D_{66})\alpha^2\beta^2 + D_{22}\beta^4\right),\\
&S_{34} = -\left(k_1 D_{11}^s\alpha^2 + k_2 D_{12}^s\beta^2\right) + 2(k_1 A' + k_2 B')\,D_{66}^s\alpha^2\beta^2 - \left(k_2 D_{22}^s\beta^2 + k_1 D_{12}^s\alpha^2\right)\\
&S_{44} = -k_1\left(k_1 H_{11}^s + k_2 H_{12}^s\right) - (k_1 A' + k_2 B')\left((k_1 A' + k_2 B')H_{66}^s\alpha^2\beta^2\right) - k_2\left(k_1 H_{12}^s + k_2 H_{22}^s\right)\\
&\qquad - \left(k_1 A'\right)^2 A_{55}^s\alpha^2 - \left(k_2 B'\right)^2 A_{44}^s\beta^2\\
&k = N_x^0\alpha^2 + N_y^0\beta^2
\end{aligned}
\tag{25}
$$

4. Analytical solutions for anti-symmetric angle-ply laminates

For anti-symmetric angle-ply laminated plates, the following stiffness components are identically zero

$$
\begin{aligned}
&A_{16} = A_{26} = D_{16} = D_{26} = B_{16}^s = B_{26}^s = D_{16} = D_{26} = D_{16}^s = D_{26}^s = H_{16}^s = H_{26}^s = H_{45}^s = 0\\
&B_{11} = B_{12} = B_{22} = B_{66} = B_{11}^s = B_{12}^s = B_{22}^s = B_{66}^s = 0
\end{aligned}
\tag{26}
$$

The following expansions of displacements are adopted to automatically respect the simply supported boundary conditions of anti-symmetric angle-ply laminated plate

$$
\begin{Bmatrix} u_0 \\ v_0 \\ w_0 \\ \theta \end{Bmatrix} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\begin{Bmatrix} U_{mn}\,\sin(\alpha\,x)\cos(\beta\,y) \\ V_{mn}\cos(\alpha\,x)\sin(\beta\,y) \\ W_{mn}\sin(\alpha\,x)\sin(\beta\,y) \\ X_{mn}\sin(\alpha\,x)\sin(\beta\,y) \end{Bmatrix}
\tag{27}
$$

Substituting Eqs. (23), (26) and (27) into Eq. (18), the equations of the form in Eq. (24) are obtained with the following coefficients

$$
\begin{aligned}
&S_{11} = -\left(A_{11}\alpha^2 + A_{66}\beta^2\right), \quad S_{12} = -\alpha\beta\left(A_{12} + A_{66}\right), \quad S_{13} = 3B_{16}\alpha^2\beta + B_{26}\,\beta^3,\\
&S_{14} = -\left(k_1 A' + k_2 B'\right)B_{16}^s\alpha^2\beta + \beta\left(k_1 B_{16}^s + k_2 B_{26}^s\right),\\
&S_{22} = -\left(A_{66}\,\alpha^2 + A_{22}\,\beta^2\right), \quad S_{23} = 3B_{26}\,\alpha\beta^2 + B_{16}\alpha^3,\\
&S_{24} = -\left(k_1 A' + k_2 B'\right)B_{26}^s\alpha\beta^2 + \alpha\left(k_1 B_{16}^s + k_2 B_{26}^s\right),\\
&S_{33} = -\left(D_{11}\alpha^4 + 2(D_{12} + 2D_{66})\alpha^2\beta^2 + D_{22}\beta^4\right),\\
&S_{34} = -\left(k_1 D_{11}^s\alpha^2 + k_2 D_{12}^s\beta^2\right) + 2(k_1 A' + k_2 B')\,D_{66}^s\alpha^2\beta^2 - \left(k_2 D_{22}^s\beta^2 + k_1 D_{12}^s\alpha^2\right)\\
&S_{44} = -k_1\left(k_1 H_{11}^s + k_2 H_{12}^s\right) - (k_1 A' + k_2 B')\left((k_1 A' + k_2 B')H_{66}^s\alpha^2\beta^2\right)\\
&\qquad - k_2\left(k_1 H_{12}^s + k_2 H_{22}^s\right) - \left(k_1 A'\right)^2 A_{55}^s\alpha^2 - \left(k_2 B'\right)^2 A_{44}^s\beta^2\\
&k = N_x^0\alpha^2 + N_y^0\beta^2
\end{aligned}
\tag{28}
$$

5. Numerical results and discussion

In this section, various numerical examples are proposed for checking the exactitude and efficiency of the present model in predicting the bending and buckling responses of simply supported antisymmetric cross-ply and angle-ply laminated plates. For the checking purpose, the results computed using the present four variable refined plate theory are compared with those of the CLPT, FSDT, HSDT, RPT and exact solution of 3D elasticity. The presentation of various displacement models is shown in Table 1. In all examined examples, a shear correction factor of 5/6 is utilized for FSDT. The following lamina characteristics are employed:

- Material 1 (Pagano 1970)

$$E_1 = 25\,E_2,\quad G_{12} = G_{13} = 0.5\,E_2,\quad G_{23} = 0.2\,E_2,\quad \nu_{12} = 0.25 \tag{29a}$$

- Material 2 (Ren 1990)

$$E_1 = 40\,E_2,\quad G_{12} = G_{13} = 0.5\,E_2,\quad G_{23} = 0.6\,E_2,\quad \nu_{12} = 0.25 \tag{29b}$$

- Material 2 (Ren 1990)

$$E_1 = 40\,E_2,\quad G_{12} = G_{13} = 0.6\,E_2,\quad G_{23} = 0.5\,E_2,\quad \nu_{12} = 0.25 \tag{29c}$$

For convenience, the following dimensionless quantities are utilized in investigating the numerical results

$$\begin{gathered}\overline{w} = \frac{100E_c h^3}{q_0\, a^4} w\left(\frac{a}{2},\frac{b}{2}\right),\quad \left(\overline{\sigma}_x,\overline{\sigma}_y\right)\left(\frac{a}{2},\frac{b}{2},\frac{z}{h}\right) = \frac{h^2}{qa^2}\left(\sigma_x,\sigma_y\right),\quad \overline{\tau}_{xy}\left(\frac{a}{2},\frac{b}{2},\frac{z}{h}\right) = \frac{h^2}{qa^2}\tau_{xy},\\ \overline{\tau}_{xz}\left(0,\frac{b}{2},\frac{z}{h}\right) = \frac{h}{qa}\tau_{xz},\quad \overline{\tau}_{yz}\left(\frac{b}{2},0,\frac{z}{h}\right) = \frac{h}{qa}\tau_{yz},\quad \overline{N} = \frac{N_{cr}\, a^2}{E_m h^3}\end{gathered} \tag{30}$$

Table 1 Displacement models

Model	Theory	Unknown variables
CLPT	Classical laminate plate theory	3
FSDT	First-order shear deformation theory [(a)]	5
HSDT	Higher-order shear deformation theory [(b)]	5
Ren	Higher-order shear deformation theory [(c)]	5
RPT	Refined plate theory [(d)]	4
Present	Present higher-order shear deformation theory	4

[(a)] Whitney and Pagano (1970)
[(b)] Reddy (1980)
[(c)] Ren (1990)
[(d)] Kim *et al*. (2009)

5.1 Bending problem

The bending solution can be deduced from Eq. (24) by setting the in-plane loads to zero

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{12} & S_{22} & S_{23} & S_{24} \\ S_{13} & S_{23} & S_{33} & S_{34} \\ S_{14} & S_{24} & S_{34} & S_{44} \end{bmatrix} \begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \\ X_{mn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ q_{mn} \\ 0 \end{Bmatrix} \tag{31}$$

Example 1: A simply supported two-layer anti-symmetric cross-ply (0/90) square plate subjected to sinusoidal transverse force is examined using Material 1. The obtained numerical values of the non-dimensional transverse displacement are presented in Table 2. From this example, it can be observed that the results determined by employing HSDT, RPT and the present theory are identical. Compared to the elasticity solution (Pagano 1970), the present model underpredicts deflection by 3.57% for a / h ratio equal to 5. This small difference is due to the thickness stretching effect which is neglected in the present model (ε_z = 0). Table 3 shows comparison of displacement and stresses for the (0/90) laminated plate subjected to single sine load. The deflection and stresses predicted by the present model are identical with those of HSDT of Reddy. The maximum deflections computed by present theory are also in good agreement with those of exact solution (Pagano 1970) and other solutions of Sayyad and Ghugal (2014) whereas CLPT underestimates the results for all slenderness ratios. The axial normal stress $\bar{\sigma}_x$ calculated by the present model is in good agreement with that of Sayyad and Ghugal (2014) and in tune with exact solution whereas FSDT and CLPT underestimate this stress for all slenderness ratios when compared with the values of other refined theories. Both the present theory and the theory proposed by Sayyad and Ghugal (2014), give the same values of the axial normal stress $\bar{\sigma}_y$ and shear stress $\bar{\tau}_{xy}$. These results are also in good agreement with those of exact solution (Pagano 1970). The proposed theory predicts more accurate transverse shear stresses than those reported by other refined theories as compared to exact values. Fig. 2 demonstrates the variation of non-dimensional transverse displacement versus thickness ratios a / h by employing all displacement models. It can be confirmed that the curves of present theory and HSDT are identical, and the CLPT underestimates the deflection of composite plate. Since the transverse shear deformation effects are not considered in CLPT, the non-dimensional transverse displacement $\bar{w}$ calculated by CLPT are not influenced by the variation of thickness ratio a / h. Thus, in general, the proposed theory is successfully verified.

Table 2 Non-dimensional deflections of simply supported two-layer (0/90) square laminates under sinusoidal transverse load

a / h	Source	$\bar{w}$
2	Pagano (1970)	4.9362
	HSDT	4.5619
	FSDT	5.4103
	RPT	4.5619
	Present	4.5619

Table 2 Continued

a / h	Source	$\overline{w}$
5	Pagano (1970)	1.7287
	HSDT	1.6670
	FSDT	1.7627
	RPT	1.6670
	Present	1.6670
10	Pagano (1970)	1.2318
	HSDT	1.2161
	FSDT	1.2416
	RPT	1.2161
	Present	1.2161
20	Pagano (1970)	1.1060
	HSDT	1.1018
	FSDT	1.1113
	RPT	1.1018
	Present	1.1018
100	Pagano (1970)	1.0742
	HSDT	1.0651
	FSDT	1.0653
	RPT	1.0651
	Present	1.0651
	CLPT	1.0636

Table 3 Comparison of transverse displacement and stresses for simply supported two-layer (0/90) square laminated plate subjected to single sine load

a / h	Theory	Model	$\overline{w}$ (z = 0)	$\overline{\sigma}_x$ ($z = -h$ / 2)	$\overline{\sigma}_y$ ($z = -h$ / 2)	$\overline{\tau}_{xy}$ ($z = -h$ / 2)	$\overline{\tau}_{xz}$ (z = 0)	$\overline{\tau}_{yz}$ (z = 0)
4	Present	/	1.9985	0.9060	0.0891	0.0577	0.3128	0.3128
	Ref[(a)]	TSDT	1.9424	0.9063	0.0964	0.0562	0.3189	0.3189
	Reddy	HSDT	1.9985	0.9060	0.0891	0.0577	0.3128	0.3128
	Mindlin	FSDT	1.9682	0.7157	0.0843	0.0525	0.2274	0.2274
	Kirchhoff	CLPT	1.0636	0.7157	0.0843	0.0525	—	—
	Pagano	Elasticity	2.0670	0.8410	0.1090	0.0591	0.3210	0.3130
10	Present	/	1.2161	0.7468	0.0851	0.0533	0.3190	0.3190
	Ref[(a)]	TSDT	1.2089	0.7471	0.0876	0.0530	0.3261	0.3261
	Reddy	HSDT	1.2161	0.7468	0.0851	0.0533	0.3190	0.3190
	Mindlin	FSDT	1.2083	0.7157	0.0843	0.0525	0.2274	0.2274
	Kirchhoff	CLPT	1.0636	0.7157	0.0843	0.0525	—	—
	Pagano	Elasticity	1.2250	0.7302	0.0886	0.0535	0.3310	0.3310

[(a)] Results taken from reference Sayyad and Ghugal (2014)

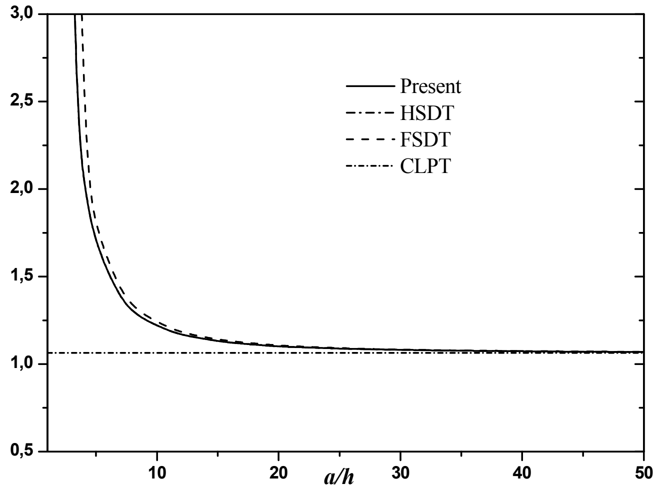

Fig. 2 The effect of thickness ratio on non-dimensional deflection of simply supported two-layer (0/90) square laminates under sinusoidal transverse load

Example 2: A simply supported two-layer anti-symmetric angle-ply (45/–45) laminate subjected to sinusoidal transverse force is investigated by considering Material 2. Dimensionless deflections for the square and rectangular plates are presented in Table 4 for various values of thickness ratio a / h. The computed results are compared with those generated via HSDT, RPT, and FSDT. It can be confirmed that a good agreement is demonstrated for all values of thickness ratio a / h. The dimensionless deflections of two-layer (45/–45) square laminates under sinusoidal transverse force

Table 4 Non-dimensional deflections of simply supported two-layer (45/– 45) laminates under sinusoidal transverse load

a / h	Source	$\overline{w}$	
		Square plate ($a = b$)	Rectangular plate ($a = 3b$)
4	HSDT	1.0203	3.1560
	FSDT	1.1576	3.3814
	RPT	1.0203	3.0971
	Present	1.0203	3.0971
10	HSDT	0.5581	2.2439
	FSDT	0.5773	2.2784
	RPT	0.5581	2.2325
	Present	0.5581	2.2325
100	HSDT	0.4676	2.0671
	FSDT	0.4678	2.0674
	RPT	0.4676	2.0670
	CLPT	0.4667	2.0653
	Present	0.4676	2.0670

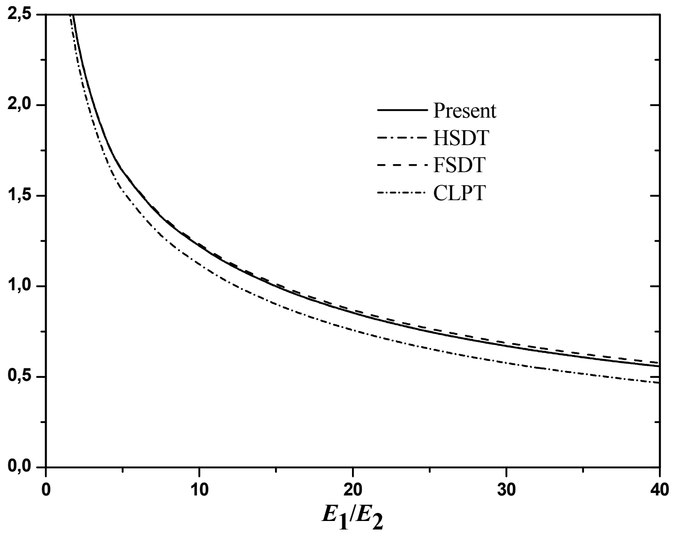

Fig. 3 The effect of modulus ratio on non-dimensional deflection of simply supported two-layer (45/-45) square laminates under sinusoidal transverse load (a / h = 10)

are illustrated in Fig. 3 for various ratio of modulus E_1 / E_2. Again, it can be concluded that the results predicted by the present theory, HSDT and RPT are in excellent agreement for a wide range of values of modulus ratio.

5.2 Buckling problem

For buckling investigation, the applied loads are supposed to be in-plane forces

$$N_x^0 = -N_0, \quad N_x^0 = \gamma\, N_0, \quad \gamma = \frac{N_x^0}{N_y^0}, \quad N_{xy}^0 = 0 \tag{32}$$

The buckling solution can be determined from Eq. (24) by setting the transverse loads to zero

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{12} & S_{22} & S_{23} & S_{24} \\ S_{13} & S_{23} & S_{33}-N_0(\alpha^2+\gamma\beta^2) & S_{34} \\ S_{14} & S_{24} & S_{34} & S_{44} \end{bmatrix} \begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \\ X_{mn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{33}$$

Example 3: A simply supported anti-symmetric cross-ply $(0/90)_n$ (n = 2, 3,5) square plate subjected under uniaxial compressive force on sides x = 0, a is examined by considering Material 3. Table 5 demonstrates a comparison between the results computed by employing the various models and the 3D elasticity solutions reported by Noor (1975). The examination of presented results clearly shows that both the present theory and HSDT predict identical values of the buckling load and these results are in good agreement also with those given by 3D elasticity solutions (Noor and Burton 1990). Compared to the 3 D elasticity solution, the buckling loads predicted by the present model, HSDT, and FSDT are 6.11%, and 7.17%, respectively, for four-layer antisymmetric cross-ply (0/90/0/90) square laminates. The observed difference is due to the thickness stretching effect which is neglected in the present model (ε_z = 0). The influence of thickness ratio a / h on buckling force of simply supported four-layer (0/90/0/90) square plates is also shown in Fig. 4. Again, it can be confirmed that the curves of present theory and HSDT are identical, and the CLPT overestimates the buckling load of composite plate.

Table 5 Non-dimensional uniaxial buckling load of simply supported anti-symmetric cross-ply (0/90/…) square laminates (a / h = 10)

a / h	Source	$\overline{N}$
4	Exact [(a)]	21.2796
	HSDT	22.5790
	FSDT	22.8060
	Present	22.5790
	CLPT	30.3591
10	Exact [(a)]	23.6689
	HSDT	24.4596
	FSDT	24.5777
	Present	24.4596
	CLPT	33.5817
100	Exact [(a)]	24.9636
	HSDT	25.4225
	FSDT	25.4500
	Present	25.4225
	CLPT	35.4225

[(a)] Noor (1975)

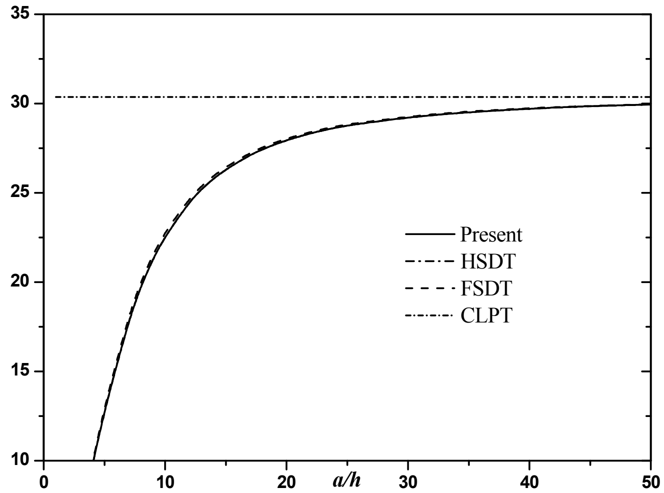

Fig. 4 The effect of thickness ratio on non-dimensional uniaxial buckling load of simply supported four-layer (0/90/0/90) square laminates

Table 6 Non-dimensional uniaxial buckling load of simply supported two-layer (θ / $-\theta$) square laminates

a / h	Source	$\overline{N}$	
		$\theta = 30°$	$\theta = 45°$
4	Ren (1990)	9.5368	9.8200
	HSDT	9.3391	8.2377
	FSDT	7.5450	6.7858
	RPT	9.3518	8.3963
	Present	9.3518	8. 3963
10	Ren (1990)	15.7517	16.4558
	HSDT	17.1269	18.1544
	FSDT	16.6132	17.5522
	RPT	17.2795	18.1544
	Present	17.2795	18.1544
100	Ren (1990)	20.4793	21.6384
	HSDT	20.5017	21.6663
	FSDT	20.4944	21.6576
	RPT	20.5040	21.6663
	Present	20.5040	21.6663
	CLPT	20.5026	21.6643

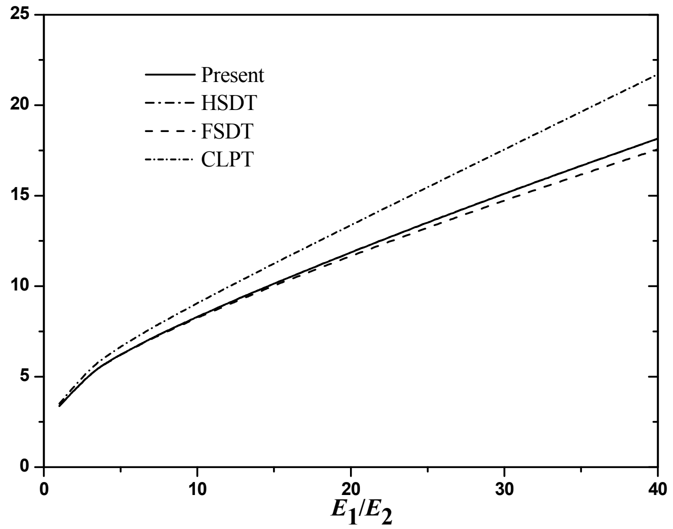

Fig. 5 The effect of modulus ratio on non-dimensional uniaxial buckling load of simply supported two-layer (45/-45) square laminates (a/h = 10)

Example 4: A simply supported two-layer anti-symmetric angle-ply (θ/-θ) square plate under an uniaxial compressive force on sides x = 0, a, a is investigated by considering Material 3. The numerical values of dimensionless buckling load are reported in Table 6. The computed results are compared with those predicted by Ren (1990). For all values of thickness ratio and fiber orientation, the buckling forces determined by the present theory, RPT and HSDT are almost identical. For *a* / *h* ratio equal to 4 and the fiber orientation equal to 30°, the buckling force values computed by FSDT, HSDT, and the present theory are 20.88 %, 2.07 %, and 1.94 % lower as compared to the results determined by Ren (1990). The buckling force values calculated by employing the five models are in an excellent agreement with those given by Ren (1990) for thin plates (a/h = 100). The influence of modulus ratio on dimensionless uniaxial buckling force of simply supported two-layer (45/-45) square plate is demonstrated in Fig. 5. It can be concluded that increasing the modular ratio makes the plate stiffer because the buckling load is reduced.

6. Conclusions

A simplified HSDT is developed for laminated composite plates. By proposing some additional simplifying assumptions to the existing HSDT, with the consideration of an undetermined integral term, the number of variables and governing equations of the proposed HSDT are reduced by one, and thus, make this formulation simple and efficient to utilize. The theory provides parabolic variable of the transverse shear strains, and respects the zero traction boundary conditions on the surfaces of the plate without employing shear correction coefficients. Verification investigations demonstrate that the predictions by the proposed HSDT and existing HSDT for anti-symmetric cross-ply and angle- ply laminates are close to each other. In conclusion, the present model can improve the numerical computational cost because of their diminished degrees of freedom

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