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Free vibration analysis of composite cylindrical shells with non-uniform thickness walls

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Abstract. The paper proposes to characterize the free vibration behaviour of non-uniform cylindrical shells using spline approximation under first order shear deformation theory. The system of coupled differential equations in terms of displacement and rotational functions are obtained. These functions are approximated by cubic splines. A generalized eigenvalue problem is obtained and solved numerically for an eigenfrequency parameter and an associated eigenvector which are spline coefficients. Four and two layered cylindrical shells consisting of two different lamination materials and plies comprising of same as well as different materials under two different boundary conditions are analyzed. The effect of length parameter, circumferential node number, material properties, ply orientation, number of lay ups, and coefficients of thickness variations on the frequency parameter is investigated.

Keywords: free vibration; anti-symmetric; non-uniform thickness; shear deformation; spline approximation; frequency parameter

1. Introduction

Structural components comprising of composite materials facilitate engineers to manufacture light weight, better temperature resistant, damp and shock absorbing structures. Composites offer high specific strength (strength-to-weight) and specific stiffness (stiffness-to-weight ratio) which in return provide fuel economic structures. Cylinders of variable thickness have been used in many engineering structures which in return help to construct low cost and light weight structures. Due to such superior characteristics, number of researchers analyzed the cylindrical shells such as Edalat *et al*. (2014) analyzed free vibration of cylindrical shells having variable radii of curvature by using Galerkin method. Selahi *et al*. (2014) studied the functionally graded truncated conical shells with variable thickness walls under asymmetric pressure using differential quadrature method. Zerin (2013) analyzed the free vibration of laminated homogeneous and non-homogeneous cylindrical and conical shells using Galerkin method. Ritz method was used to analyse joined thick conical-cylindrical shells having variable thickness by Kang (2012). Khalifa (2012) used transfer matrix approach and Romberg intergration method for analyzing free

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vibration and buckling of cylindrical shells with variable thickness under axially compressive loads. Alibeigloo (2009) and Alibeigloo and Shakeri (2007) suggested that the frequencies were significantly affected by lamination scheme, number of layers, end conditions and ideal design and desired structural properties can be achieved by adjusting orientation angle, geometric and material parameters. These factors also help to modify the frequency of a structure (Alibeigloo and Shakeri 2007, Alibeigloo *et al*. 2012).

Recently, Khan *et al*. (2015) analysed the free and forced vibration of laminated cylindrical shells. Moreover, free vibration of functionaly graded viscoelastic cylindrical shells was examined for various bouandry conditions by Hosseini-Hashemi *et al*. (2015). In addition to that, anti-symmetric angle-ply cylindrical shells were investigated by Shao and Ma (2007) using fourier series expansion. Chaudhuri and Abu-Arja (1991) studied the static analysis of moderatly-thick anti-symmetric angle-ply cylindrical panels and shells using Sander's first order shear deformation theory. A shear deformable cylindrical shell model was analyzed using couple stress theory by Zeighampour and Beni (2015). Lopatin and Morozov (2015) investigated the fundamental frequency of laminated cylindrical shells under clamped edges. Asghari (2015) analyzed 2D heterogeneous cylinders under thermomechanical loading. Four variable sinusoidal theory was used by Chattibi *et al*. (2015) to analyse the thermomechanical and bending effect of antisymmetric cross-ply composite plates. Sofiyev and Kuruoglu (2015) studied the buckling of orthotropic conical shells subjected to loads. Cross-ply laminated shells were analysed under FSDT using different formulation by Sahan (2015). Nonlinear flexural vibration of shear deformable spherical shells was examined by Kar and Panda (2015). Functionally graded sandwich plates were analyzed for bending, vibration and buckling under higher order shear deformation theory by Nguyen *et al*. (2015). Radial basis function was used to analyse the thickness deformations of laminated doubly curved shells by using layerwise theory (Ferreira *et al*. 2011). Three dimensional vibration of cylindrical shell was investigated using continuum and discrete approaches by Liew *et al*. (2000). Free vibration of microtubules was studied as elastic shell model by Beni and Zeverdejani (2014). Chorfi and Houmat (2010) studied non-linear free vibration of a functionally graded doubly-curved shallow shell. Laminated composite curved shell panels were analysed by Katariya *et al*. (2015). Mahapatra *et al*. (2014) investigated the large amplitude of laminated composite spherical panels. FEM method was used by Mahapatra and Panda (2015) to analyze the free vibration of shallow panels. Nonlinear free vibration of laminated composite shell panels were investigated by Mahapatra *et al*. (2015). Vibration of spherical shell was studied by Panda and Mahapatra (2014) using FEM. Panda and Singh (2009) analysed the free vibration of spherical shell using FEM under higher order shear deformation theory. Experimental and numerical investigation of static and free vibration responses of glass/epoxy laminated composite plate was done by Sahoo *et al*. (2015). Zeighampour *et al*. (2015) studied the shear deformable conical shell using couple stress theory. The spline method was used by Viswanathan *et al*. (Viswanathan *et al*. 2015a, b, c, Viswanathan and Javed 2016) to solve the free vibrational problems of conical shells, cylindrical shells and annular circular plates using first order shear deformation theory. Further, Javed *et al*. (2016) analyze the free vibration of antisymmetric angle-ply plates using first order shear deformation theory.

The main objective of this study is to analyze the effect of variable thickness on the vibration of cylindrical shells. The thickness variation can considerably affect the performance of equipment (Featherston and Barabasz 2000). Further, slight thickness variation may affect the frequency significantly, which in return will affect the performance of any structure. The effect of variable thickness on the frequency of the shells may be significantly useful for designers in creating desired structure.

The knowledge gap to the already exisiting literature is that the displacement and rotational functions are approximated by splines. The spline function has very attractive characteristics for computational work in terms of convergence and accuracy as compare to FEM and FDM. Moreover Bickely (1968) concluded through his work that splines chain of low-order approximations may yield better accuracy than a global high-order approximation. Since to the authors knowledge for the sake comparison of current results with others, authors are unable to find published data on free vibration of antisymmetric angle-ply composite cylindrical shells with non-uniform thickness using spline approximation.

The present study aim to analyze the free vibration of anti-symmetric angle-ply composite cylindrical shells with non-uniform thickness along the axial direction using spline approximation under first order shear deformation theory. The displacement and rotational functions are approximated by Bickley-type splines. Collocation with these splines yields a set of field equations which along with the equations of boundary conditions, in return reduce to system of homogeneous simultaneous algebraic equations on the assumed spline coefficients. Then the problem is solved using eigensolution technique to obtain the frequency parameter. The eigenvector are the spline coefficients from which the mode shapes can be constructed. The stability of the cylindrical structure is analyzed with respect to the ply angles, length parameter, coefficients of thickness variations, circumferential node number, number of lay-ups, lamination materials and two different boundary conditions. Results are presented in terms of graphs and tables.

2. Formulation

Consider a composite laminated circular cylindrical shell of having length ℓ, thickness h, radius r. The x coordinates of the shell is taken along the meridional direction, θ coordinate along the circumferential direction and z along the thickness direction having the origin at the mid plane of the shell.

The equilibrium equations of circular cylindrical shells including shear deformation which are as follows.

$$
\begin{aligned}
&\frac{\partial N_x}{\partial x}+\frac{1}{r}\frac{\partial N_{\theta x}}{\partial \theta}=I_1\frac{\partial^2 u}{\partial t^2}\\
&\frac{\partial N_{x\theta}}{\partial x}+\frac{1}{r}\frac{\partial N_{\theta}}{\partial \theta}+\frac{1}{r}Q_{\theta z}=I_1\frac{\partial^2 v}{\partial t^2}\\
&\frac{\partial Q_{xz}}{\partial x}+\frac{1}{r}\frac{\partial Q_{\theta z}}{\partial \theta}-\frac{1}{r}N_{\theta}=I_1\frac{\partial^2 w}{\partial t^2}\\
&\frac{\partial M_x}{\partial x}+\frac{1}{r}\frac{\partial M_{\theta x}}{\partial \theta}-Q_{xz}=I_3\frac{\partial^2 \psi_x}{\partial t^2}\\
&\frac{\partial M_{x\theta}}{\partial x}+\frac{1}{r}\frac{\partial M_{\theta}}{\partial \theta}-Q_{\theta z}=I_3\frac{\partial^2 \psi_\theta}{\partial t^2}
\end{aligned}
\tag{1}
$$

where N_x, N_θ, $N_{x\theta}$ are the stress resultants in the respective directions, M_x, M_θ, $M_{x\theta}$ are the moment

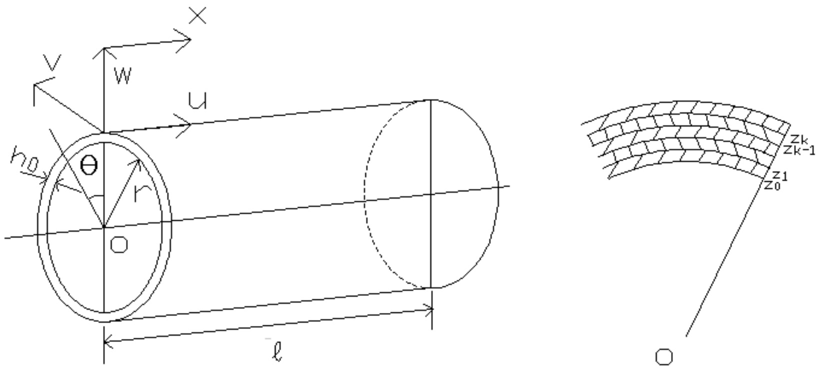


Fig. 1 Layered circular cylindrical shell of constant thickness: geometry

resultants in the respective directions Q_{xz}, $Q_{\theta z}$ are the transverse shear resultants in the respective directions.

According to the first order shear deformation theory, the displacements components u, v, w can be written as

$$\begin{aligned} u(x,\theta,z,t) &= u_{\circ}(x,\theta,t) + z\psi_x(x,\theta,t) \\ v(x,\theta,z,t) &= v_{\circ}(x,\theta,t) + z\psi_\theta(x,\theta,t) \\ w(x,\theta,z,t) &= w(x,\theta,t) \end{aligned} \tag{2}$$

where u_0, v_0, w_0 are the mid plane displacements, and ψ_x, ψ_θ are the shear rotations of any point on the mid surface normal to the xz and θz plane respectively and t is the time. Using the strain-displacement relations and stress-strain relations of the k-th layer by neglecting the transverse normal strain and stress for cylindrical shells Viswanathan *et al*. (2008).

In this study, the thickness of the k-th layer is assumed in the form

$$h_k(x) = h_{0k}\, g(x) \tag{3}$$

where h_{0k} is a constant thickness.

In general, the thickness variation of each layer is assumed in the form

$$h(x) = h_0\, g(x)$$

and

$$g(x) = 1 + C_\ell \frac{x}{\ell} + C_e \exp\left(\frac{x}{\ell}\right) + C_s \sin\left(\frac{\pi x}{\ell}\right) \tag{4}$$

The thickness becomes uniform if $g(x) = 1$. Therefore the elastic coefficients A_{ij}, B_{ij} and D_{ij} corresponding to layers of uniform thickness with superscript 'c', one easily finds

$$A_{ij} = A_{ij}^c g(x), \quad B_{ij} = B_{ij}^c g(x),\ D_{ij} = D_{ij}^c g(x) \tag{5}$$

where

$$A^c_{ij} = \sum_k \overline{Q}^{(k)}_{ij}\,(z_k - z_{k-1}), \qquad B^c_{ij} = \frac{1}{2}\sum_k \overline{Q}^{(k)}_{ij}\,(z^2{}_k - z^2{}_{k-1}),$$
$$D^c_{ij} = \frac{1}{3}\sum_k \overline{Q}^{(k)}_{ij}\,(z^3{}_k - z^3{}_{k-1}) \qquad \text{for} \qquad i,j = 1,2,6 \tag{6}$$

and

$$A^c_{ij} = K\sum_k \overline{Q}^{(k)}_{ij}(z_k - z_{k-1}) \qquad \text{for} \qquad i,j = 4,5 \tag{7}$$

where K is the shear correction factor depends on lamina properties and lamination scheme (Pai and Schulz 1999) and calculated by various static and dynamic methods, z_{k-1} and z_k are the boundaries of the k-th layer. The quantities $\overline{Q}^{(k)}_{ij}$ are defined in the reference Viswanathan and Kim (2008). In the case of anti-symmetric angle-ply lamination the coefficients A_{16}, A_{26}, A_{45}, B_{11}, B_{12}, B_{22}, B_{66}, D_{16}, D_{26} are equal to zero (George 1999).

2.1 Thickness variation

Case (i): If $C_e = C_s = 0$, then the thickness variation becomes linear. In this case it can be easily shown that $C_\ell = \frac{1}{\eta} - 1$, where η is the taper ratio $h_k(0) / h_k(1)$.

Case (ii): If $C_\ell = C_s = 0$, then the thickness varies exponentially.

Case (iii): If $C_\ell = C_s = 0$, then the thickness varies in sinusoidal.

It may be noted that the thickness of any layer at the end $X = 0$ is h_{0k} for the cases (i) and (iii), but is h_{0k} $(1 + C_e)$ for the case (ii).

The displacement components u_0, v_0, w and shear rotations ψ_x, ψ_θ are assumed in the form of

$$\begin{aligned}
u_0(x,\theta,t) &= U(x)e^{n\theta}\,e^{i\omega t}\\
v_0(x,\theta,t) &= V(x)e^{n\theta}\,e^{i\omega t}\\
w(x,\theta,t) &= W(x)e^{n\theta}\,e^{i\omega t}\\
\psi_x(x,\theta,t) &= \Psi_x(x)e^{n\theta}\,e^{i\omega t}\\
\psi_\theta(x,\theta,t) &= \Psi_\theta(x)e^{n\theta}\,e^{i\omega t}
\end{aligned} \tag{8}$$

where ω is the angular frequency of vibration, t is the time and n is the circumferential node number.

Using the above Eq. (8) the resulting equation becomes in the matrix form as

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} & L_{15}\\ L_{21} & L_{22} & L_{23} & L_{24} & L_{25}\\ L_{31} & L_{32} & L_{33} & L_{34} & L_{35}\\ L_{41} & L_{42} & L_{43} & L_{44} & L_{45}\\ L_{51} & L_{52} & L_{53} & L_{54} & L_{55} \end{bmatrix} \begin{Bmatrix} U\\ V\\ W\\ \Psi_x\\ \Psi_\theta \end{Bmatrix} = \begin{Bmatrix} 0\\0\\0\\0\\0 \end{Bmatrix} \tag{9}$$

where the differential operators L_{ij}'s are given in Appendix A.

The non-dimensional parameters are introduced as follows

$$
\begin{aligned}
&X = \frac{x}{\ell}, && \text{a distance coordinate, and } X \in [0,1]; \\
&\lambda = \omega \ell \sqrt{\frac{I_1}{A_{11}}}, && \text{a frequency parameter;} \\
&L = \frac{\ell}{r}, && \text{a length parameter;} \\
&H = \frac{h}{r}, && \text{ratio of total thickness to radius;} \\
&\delta_k = \frac{h_k}{h}, && \text{relative layer thickness of the } k\text{-th layer}
\end{aligned}
\tag{10}
$$

I_1 is the normal inertia coefficients defined by

$$I_1 = \int \rho^{(k)} dz$$

We obtain the resulting equation in the matrix form as follows.

$$
\begin{bmatrix}
L_{11}^* & L_{12}^* & L_{13}^* & L_{14}^* & L_{15}^* \\
L_{21}^* & L_{22}^* & L_{23}^* & L_{24}^* & L_{25}^* \\
L_{31}^* & L_{32}^* & L_{33}^* & L_{34}^* & L_{35}^* \\
L_{41}^* & L_{42}^* & L_{43}^* & L_{44}^* & L_{45}^* \\
L_{51}^* & L_{52}^* & L_{53}^* & L_{54}^* & L_{55}^*
\end{bmatrix}
\begin{Bmatrix} \bar{U} \\ \bar{V} \\ \bar{W} \\ \bar{\Psi}_X \\ \bar{\Psi}_\Theta \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}
\tag{11}
$$

The differential operators L_{ij}^* of the matrix are given in Appendix B.

2.2 Spline collocation procedure

The displacement functions $\bar{U}, \bar{V}, \bar{W}$ and rotational functions $\bar{\Psi}_X$, $\bar{\Psi}_\Theta$ are approximated by cubic spline functions in the range of $X \in [0,1]$ as

$$
\begin{aligned}
\bar{U}(X) &= \sum_{i=0}^{2} a_i X^i + \sum_{j=0}^{N-1} b_j (X - X_j)^3 \, H(X - X_j) \\
\bar{V}(X) &= \sum_{i=0}^{2} c_i X^i + \sum_{j=0}^{N-1} d_j (X - X_j)^3 \, H(X - X_j)
\end{aligned}
\tag{12}
$$

$$\bar{W}(X) = \sum_{i=0}^{2} e_i X^i + \sum_{j=0}^{N-1} f_j (X - X_j)^3 \, H(X - X_j)$$

$$\bar{\Psi}_X(X) = \sum_{i=0}^{2} o_i X^i + \sum_{j=0}^{N-1} p_j (X - X_j)^3 \, H(X - X_j) \tag{12}$$

$$\bar{\Psi}_\Theta(X) = \sum_{i=0}^{2} l_i X^i + \sum_{j=0}^{N-1} q_j (X - X_j)^3 \, H(X - X_j)$$

Here, $H(X - X_j)$ is the Heaviside step functions. The range of X is divided in to N subintervals, at the points $X = X_s$, $s = 1, 2, 3, ..., N - 1$. The width of each subinterval is $1/N$ and $X_s = s/N$ ($s = 0, 1, 2, ..., N$), since the knots X_s are chosen equally spaced.

The assumed spline functions given in Eq. (12) are approximated at the nodes (coincide with the knots) and these splines satisfy the differential equations given in Eq. (11), at all X_s and resulting into the homogeneous system of ($5N + 5$) equations in the ($5N + 15$) unknown spline coefficients.

The following boundary conditions are considered to analyze the problem.

(i) Clamped-Clamped (C-C) (both the ends are clamped)
(ii) Simply-Supported (S-S) (both the ends are simply supported)

By applying each of this boundary condition separately, we can obtain 10 more equations on spline coefficients. Combining these 10 equations with the earlier ($5N + 5$) equations, we get ($5N + 15$) homogeneous equations in the same number unknowns. Thus, we have a generalized eigenvalue problem in the form

$$[M]\{q\} = \lambda^2 [P]\{q\} \tag{13}$$

where $[M]$ and $[P]$ are the square matrices, $\{q\}$ is the column matrix of the spline coefficients and λ is the eigenfrequency parameter.

3. Results and discussion

In this problem, the frequencies are analysed for two and four layered anti-symmetric angle-ply cylindrical shells of non-uniform thickness under Clamped-Clamped (C-C) and simply supported (S-S) boundary conditions using Kevler-49 Epoxy (KE) and E-glass/epoxy (EGE) materials arranging them in different orders. Convergence study has been made for the frequency parameters of two and four layered shells under various boundary conditions with fixing the length ratio, thickness ratio and circumferential node number. The program is performed for N (number of knots) = 2 onwards and finally it is seen that $N = 14$ would be enough to achieve the change in percentage.

The reliability of the present formulation in providing vibration results for angle-ply laminated cylindrical shells are shown in Table 1. The material properties are assumed to be: $L/R = 4$; $h/R = 0.01$; $E_1 = 20E_2$; $G_{12}/E_2 = 0.65$; $\upsilon_{12} = 0.25$). In this table the comparison of fundamental frequency parameter λ for different symmetric and antisymmetric lamination angles 15°, 30° and 45° under S-S boundary condition are shown. Present results obtained using first order shear deformation

Table 1 Comparative study of fundamental frequency parameter λ with Narita *et al*. (1992) and Soldatos and Messina (2001) for symmetric and antisymmetric angle-ply lamination under S-S boundary condition. The circumferential node number n as superscript (L/R = 4; h/R = 0.01; $E_1 = 20E_2$; G_{12}/E_2 = 0.65; v_{12} = 0.25; CST)

Boundary conditions	Lamination angle θ (°)	Fundamental frequency λ
SS[$\theta/-\theta/-\theta/\theta$]	15 [Present]	0.110805^5
	30 [Present]	0.116103^4
	45 [Present]	0.107200^3
	15 [Ref *1]	0.120280^5
	30 [Ref *1]	0.123330^4
	45 [Ref *1]	0.119430^3
	30 [Ref *2]	0.123200
	45 [Ref *2]	0.119300
SS[$\theta/-\theta/\theta/-\theta$]	15 [Present]	0.117200^5
	30 [Present]	0.113753^4
	45 [Present]	0.105346^3
	15 [Ref *1]	0.119430^5
	30 [Ref *1]	0.120980^4
	45 [Ref *1]	0.117200^3
	30 [Ref *2]	0.120900
	45 [Ref *2]	0.117100

*1 Soldatos and Messina (2001)
*2 Narita *et al*. (1992)

theory are compared with Soldatos and Messina (2001) and Narita (1992) using classical shell theory (CST). The shells are considered to have constant thickness. It is evident from the results that the present values of the frequency parameter are lower as compare to Soldatos and Messina (2001) and Narita (Narita *et al*. 1992), these results were expected because the inclusion of shear deformation theory is more significant to analyze the vibration of layered shell structure since which yields lower values on the frequency parameters when we compared to the values predicted by classical shell theory. It can also be seen, from the Table 1, that the maximum percentage changes between present value and available result is 10%. The agreement of the current result is quite good.

Table 2 shows the effect of coefficient of linear variation in thickness η ($0.5 \le \eta \le 2.1$) on the fundamental frequency parameter λ of two and four layered cylindrical shells under C-C boundary conditions. The parameters, circumferential node number n = 2, length parameter L = 1 and thickness ratio H = 0.02 are fixed. The two layered shells consists of KE-KE and EGE-EGE materials and four layered shells consists of KE-EGE-EGE-KE materials combination. For both types of layers 45° angle is used which is arranged in anti-symmetric manner. The frequency changes as the coefficient of linear variation in thickness increases. The results shows that two layered shells consisting of EGE-EGE material showed the highest frequency followed by four layered shells and then two layered shells comprising of KE-KE material.

Table 2 Variation of fundamental frequency parameter λ with respect to coefficient of linear thickness variation η of two and four layered cylindrical shells under C-C boundary conditions

η	45° / –45° (KE-KE)	45° / –45° (EGE-EGE)	45° / –45° / 45° / –45° (KE-EGE-EGE-KE)
	λ	λ	λ
0.5	0.576111	1.028459	0.776881
0.7	0.569463	1.023132	0.771009
0.9	0.568895	1.022861	0.770563
1.1	0.568949	1.022940	0.770689
1.3	0.569622	1.023800	0.771386
1.5	0.570905	1.025233	0.772645
1.7	0.572777	1.027238	0.774454
1.9	0.575209	1.029815	0.776789
2.1	0.578090	1.032964	0.779626

Table 3 Variation of fundamental frequency parameter λ with respect to coefficient of exponential thickness variation C_e of two and four layered cylindrical shells under C-C boundary conditions

C_e	45° / –45° (KE-KE)	45° / –45° (EGE-EGE)	45° / –45° / 45° / –45° (KE-EGE-EGE-KE)
	λ	λ	λ
-0.2	0.569396	1.023531	0.771160
-0.1	0.570022	1.024259	0.771783
0	0.568844	1.022724	0.770555
0.2	0.569142	1.022946	0.770747
0.1	0.569712	1.023296	0.771218

The variation of fundamental frequency parameter λ with respect to coefficient of exponential variation in thickness C_e ($-0.2 \leq C_e \leq 0.2$) for C-C boundary condition is depicted in Table 3. The fundamental frequency parameter λ is depicted for two and four layered shells having ply-angle equal to 45° arrange in anti-symmetric manner. Further, for two layered shells two different materials KE-KE and EGE-EGE are used and combination of these two materials KE-EGE-EGE-KE is used for four layered shells. The frequency changes as the coefficient of exponential variation in thickness increases. It can be seen that shells comprising of KE-KE material showed least frequency values followed by four layered shells and two layered shells consisting of EGE-EGE material.

The variation between fundamental frequency parameter λ and coefficient of sinusoidal variation in thickness C_s ($-0.5 \leq C_s \leq 0.5$) is presented in Table 4 for C-C boundary condition. The two layered shells having (EGE-EGE) material shows highest fundamental frequency followed by four layered shells and shell consisting of (KE-KE) material shows least fundamental frequency. In general, there is a slight change in the frequency with the increase of coefficient of exponential variation in thickness.

Table 4 Variation of fundamental frequency parameter λ with respect to coefficient of exponential thickness variation C_s of two and four layered cylindrical shells under C-C boundary conditions

C_s	45° / –45° (KE-KE)	45° / –45° (EGE-EGE)	45° / –45° / 45° / –45° (KE-EGE-EGE-KE)
	λ	λ	λ
-0.5	0.586743	1.042683	0.788115
-0.3	0.575308	1.031686	0.77616
-0.1	0.569526	1.023173	0.771062
0.1	0.569693	1.023883	0.771456
0.3	0.575795	1.030436	0.777351
0.5	0.586743	1.042683	0.788115

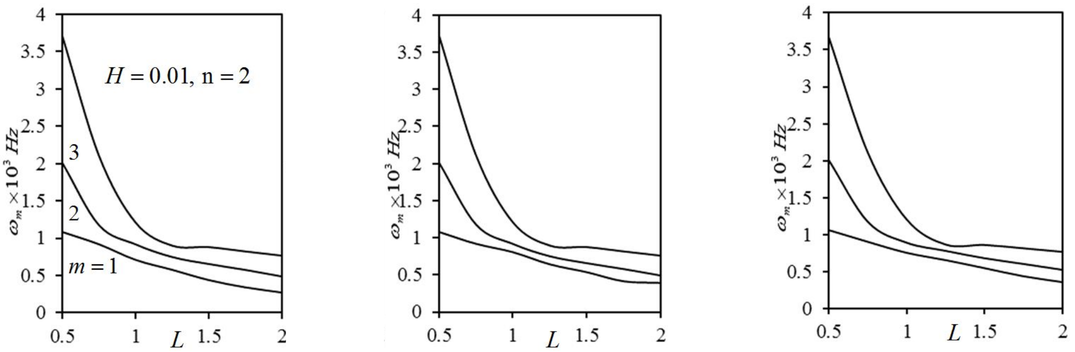
Table 5 Variation of fundamental frequency parameter λ with respect to circumferential node number n of two and four layered cylindrical shells under C-C boundary conditions

n	30° / –30° (KE-KE)	30° / –30° / 30° / –30° (KE-EGE-EGE-KE)	30° / –45° / 45° / –30° (KE-EGE-EGE-KE)
	λ	λ	λ
1	0.445583	0.562747	0.663766
2	0.455291	0.607571	0.701867
3	0.447001	0.603695	0.658966
4	0.403274	0.555496	0.576541
5	0.329800	0.464789	0.421435

Table 5 depicts the effect of circumferential node number n on the fundamental frequency parameter λ of two and four layered shells. The parameters, coefficient of linear thickness in variation η = 0.7, length parameter L = 1 and thickness ratio H = 0.02 are fixed. The fundamental frequency differs with the increase of circumferential node number increases. Further, as the number of layers increases the fundamental frequency also increases. In addition to that four layered shells 30° / –60° / 60° / –30° (KE-EGE-EGE-KE) showed highest frequency, followed by 30° / –45° / 45° / –30° (KE-EGE-EGE-KE), 30° / –30° / 30° / –30° (KE-EGE-EGE-KE) and then two layered shells showed least fundamental frequency 30° / –30° (KE-KE).

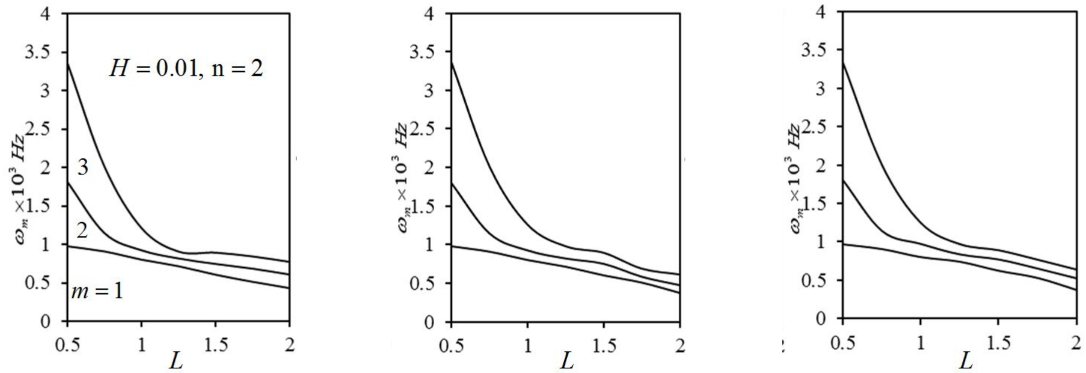
Fig. 2 shows variation of angular frequency ω_m (m = 1, 2, 3) with respect to length parameter L of two layered shells having ply orientation 30° / –30° (KE-KE) under C-C boundary conditions. The parameters circumferential node number n = 2 and thickness ratio H = 0.02 are fixed. Further, for Fig. 2(a)-(c) coefficient of linear thickness variation η = 0.7, coefficient of exponential thickness variation C_e = 0.2, and coefficient of sinusoidal thickness variation C_s = 0.5 is fixed respectively. Figs. 2(a)-(c) shows that the value of angular frequency decreases as the length parameter increases. The decrease is significant between 0.5 < L < 1 and it became steady between 1 < L < 2. Moreover, the value of angular frequency is higher for higher modes.

The effect of length parameter L on the angular frequency ω_m (m = 1, 2, 3) is depicted in Fig. 3 for C-C boundary conditions. The two layered shells having ply orientation 30° / –30° (EGE-EGE) are considered. The circumferential node number n = 2 and thickness ratio H = 0.02 are fixed.



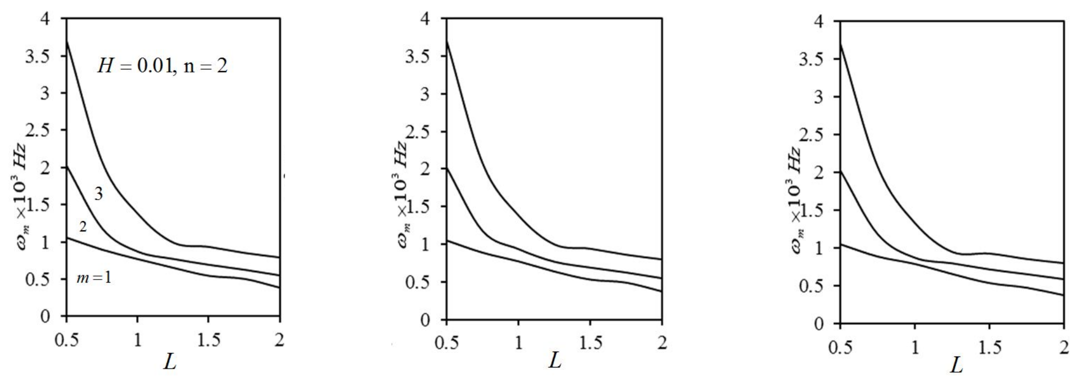
(a) $\eta = 0.7$ (b) $C_e = 0.2$ (c) $C_s = 0.5$

Fig. 2 The variation of angular frequency ω_m (m = 1, 2, 3) with respect to length parameter L for two layered cylindrical shells having ply angles 30° / –30° (KE-KE) under C-C boundary conditions



(a) $\eta = 0.7$ (b) $C_e = 0.2$ (c) $C_s = 0.5$

Fig. 3 The variation of angular frequency ω_m (m = 1, 2, 3) with respect to length parameter L for two layered cylindrical shells having ply angles 30° / –30° (EGE-EGE) under C-C boundary conditions



(a) $\eta = 0.7$ (b) $C_e = 0.2$ (c) $C_s = 0.5$

Fig. 4 The variation of angular frequency ω_m (m = 1, 2, 3) with respect to length parameter L for four layered cylindrical shells having ply angles 30° / –30° / 30° / –30° (KE-EGE-EGE-KE) under C-C boundary conditions

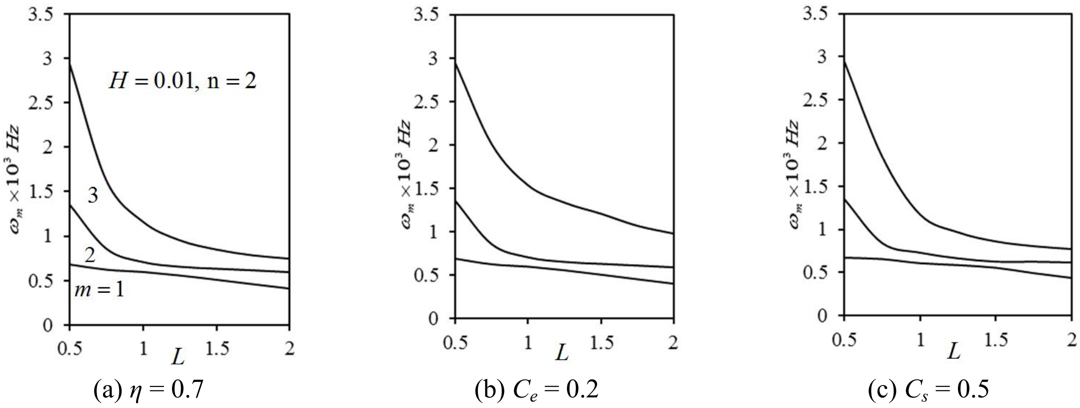


Fig. 5 The variation of angular frequency ω_m (m = 1, 2, 3) with respect to length parameter L for two layered cylindrical shells having ply angles 30° / –30° (KE-KE) under S-S boundary conditions

Further for Figs. 3(a)-(c) coefficient of linear thickness variation η = 0.7, coefficient of exponential thickness variation C_e = 0.2 and coefficient of sinusoidal thickness variation C_s = 0.5 respectively are fixed. Figs. 3(a)-(c) shows that the angular frequency decreases with the increase of length parameter. The decrease is significant between 0.5 < L < 1 and it became almost linear between 1 < L < 2.

In Fig. 4 four layered shells are studied for the effect of length parameter L on the angular frequency ω_m (m = 1, 2, 3). The layers of shell having ply orientation 30° / –30° / 30° / –30° and materials are arranged in the order of (KE-EGE-EGE-KE). The circumferential node number n = 2 and thickness ratio H = 0.02 are fixed. Figs. 4(a)-(c) depicts that as the angular frequency decreases considerably between 0.5 < L < 1 and it became steady between 1 < L < 2 with the increase of length parameter.

The S-S boundary condition is considered in Fig. 5. The variation of angular frequency ω_m (m = 1, 2, 3) with respect to length parameter L is studied for two layered shells having ply orientation 30° / –30° (KE-KE). The circumferential node number n = 2 and thickness ratio H = 0.02 are fixed.

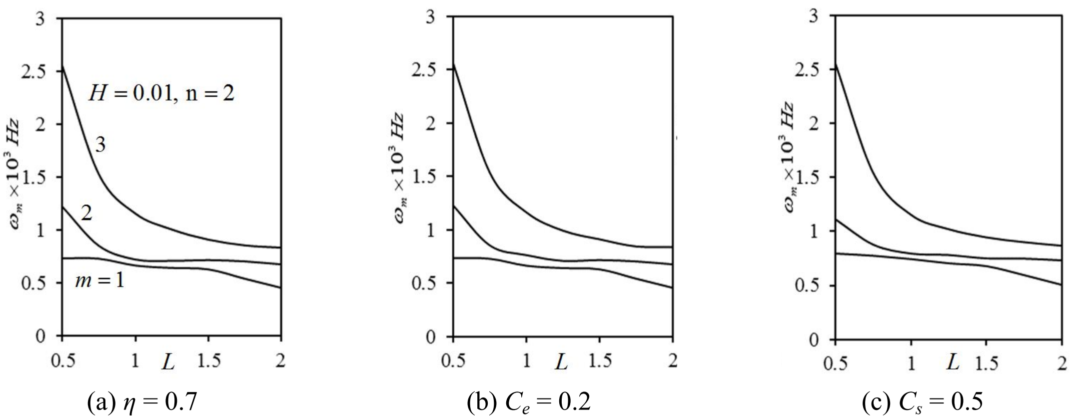
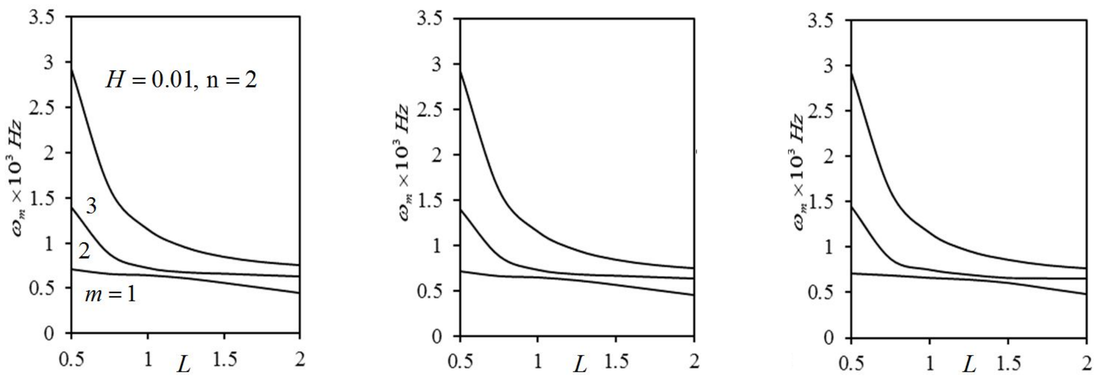


Fig. 6 The variation of angular frequency ω_m (m = 1, 2, 3) with respect to length parameter L for two layered cylindrical shells having ply angles 30° / –30° (EGE-EGE) under S-S boundary conditions

Further for Figs. 5(a-c) coefficient of linear thickness variation η = 0.7, coefficient of exponential thickness variation C_e = 0.2 and coefficient of sinusoidal thickness variation C_s = 0.5 respectively are fixed. Figs. 5(a)-(c) depicts that as the length parameter increases the value of angular frequency decreases.

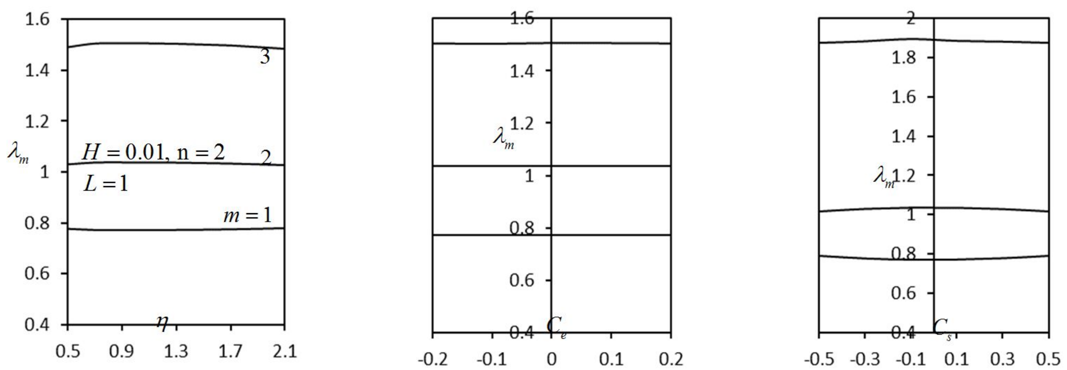
The variation of angular frequency ω_m (m = 1, 2, 3) with respect to length parameter L is studied for two layered shells under S-S boundary condition in Fig 6. The shell having ply orientation 30° / –30° (EGE-EGE) are considered. The circumferential node number n = 2 and thickness ratio H = 0.02 are fixed. The same trend of curvature can be seen as in Figs. 3(a)-(c) the only difference is that the value of angular frequency is significantly lower for S-S boundary condition as compare to C-C boundary condition.

In Fig. 7 four layered shells are studied for the effect of length parameter L on the angular frequency ω_m (m = 1, 2, 3) under S-S boundary condition. The layers of shell having ply



(a) η = 0.7 (b) C_e = 0.2 (c) C_s = 0.5

Fig. 7 The variation of angular frequency ω_m (m = 1, 2, 3) with respect to length parameter L for four layered cylindrical shells having ply angles 30° / –30° / 30° / –30° (KE-EGE-EGE-KE) under S-S boundary conditions



(a) (b) (c)

Fig. 8 The variation of frequency parameter λ_m (m = 1, 2, 3) with respect to (a) coefficient of linear thickness variation (b) coefficient of exponential thickness variation and (c) coefficient of sinusoidal thickness variation for four layered cylindrical shells having ply angles 45° / –45° / 45° / –45° (KE-EGE-EGE-KE) under C-C boundary condition orientation 30° / –30° / 30° / –30° and materials are arranged in the order of (KE-EGE-EGE-KE). The circumferential node number n = 2 and thickness ratio H = 0.02 are fixed. The same trend of curvature can be seen as for Figs. 4(a)-(c). Fig. 8 depicts the effect of three thickness coefficients i.e., linear, exponential and sinusoidal variation in thickness on the value of the frequency parameter λ_m (m = 1, 2, 3) under C-C boundary condition. The layers of the shells having ply orientation 45° / –45° / 45° / –45° and materials are arranged in the order of (KE-EGE-EGE-KE). The circumferential node number n = 2, length parameter L = 1 and thickness ratio H = 0.02 are fixed. The value of the frequency parameter remains almost same with the change of linear and exponential variation in thickness. Further, the value of the frequency parameter forms concave or convex curve with the change of sinusoidal variation in thickness.

4. Conclusions

The free vibration of anti-symmetric angle-ply cylindrical shells of non-uniform thickness is analysed including shear deformation. The variation of frequencies with respect to the length parameter, thickness coefficients and circumferential node number are studied under C-C and S-S boundary conditions. The frequencies of the shells are significantly affected by layered materials, ply orientation, length parameter, different coefficient of thickness variations and boundary conditions. It is concluded from the results that:

- The simply supported boundary condition significantly lowers the frequency as compare to clamped-clamped boundary condition.
- The frequency decreases as the length of the cylinder increases for different number of layers, materials, coefficients of thickness variations and boundary conditions considered.
- The frequency differs with the increase of circumferential node number.
- The frequency differs with the increase of different coefficients of thickness variations.

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