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Size-dependent mechanical behavior of functionally graded trigonometric shear deformable nanobeams including neutral surface position concept

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Abstract. A nonlocal trigonometric shear deformation beam theory based on neutral surface position is developed for bending, buckling, and vibration of functionally graded (FG) nanobeams using the nonlocal differential constitutive relations of Eringen. The present model is capable of capturing both small scale effect and transverse shear deformation effects of FG nanobeams, and does not require shear correction factors. The material properties of the FG nanobeam are assumed to vary in the thickness direction. The equations of motion are derived by employing Hamilton's principle, and the physical neutral surface concept. Analytical solutions are presented for a simply supported FG nanobeam, and the obtained results compare well with those predicted by the nonlocal Timoshenko beam theory.

Keywords: nanobeam; nonlocal elasticity theory; bending; buckling; vibration; functionally graded materials; neutral surface position

1. Introduction

Recently, nanoscale structures have attracted considerable attention among the researchers community for the future application of nano electro-mechanical systems (NEMS) and atomic force microscopy (AFM) (Dai *et al*. 1996, Lourie *et al*. 1998). To accomplish the design of nanostructures and systems, an essential study of their mechanical behavior seems necessary. Size effects are significant in the mechanical behavior of these structures in which dimensions are small and comparable to molecular distances. Therefore, size dependent theories of continuum mechanics have received increasing attention in recent years due to the need to model and analyze very small sized mechanical structures and devices in the rapid developments of micro- or

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nanotechnologies (Amara *et al*. 2010).

One of the well-known models is the non-local elasticity theory (Eringen 1972, 1983). Unlike the local theories which assume that the stress at a point is a function of strain at that point, the nonlocal elasticity theory assumes that the stress at a point is a function of strains at all points in the continuum. Non-local elasticity has been extensively applied to analyze the bending, buckling, vibration and wave propagation of beam-like elements in micro- or nanoelectromechanical devices (Peddieson *et al*. 2003, Lu *et al*. 2006, Wang and Varadan 2006, Reddy and Pang 2008, Benzair *et al*. 2008, Murmu and Pradhan 2009a, b, c, Adda Bedia *et al*. 2015, Aissani *et al*. 2015, Besseghier *et al*. 2015). Sudak (2003) studied infinitesimal column buckling of carbon nanotubes (CNTs), incorporating the van der Waals (vdW) forces and small scale effect, and showed that the critical axial strain decreases compared with the results of classical beams. Wang (2005) discussed the molecular dispersion relationships for CNTs by taking into account the small scale effect. Wang and Hu (2005) studied flexural wave propagation in a SWCNT by using the continuum mechanics and dynamic simulation. Lu *et al*. (2007) investigated the wave propagation and vibration properties of single- or multi-walled CNTs based on nonlocal beam model. More recently, Tounsi and his co-workers (Heireche *et al*. 2008) investigated the sound wave propagation in single- and double-walled CNTs taking into account the nonlocal effect, temperature and initial axial stress. Roque *et al*. (2011) used the nonlocal elasticity theory of Eringen to study bending, buckling and free vibration of Timoshenko nanobeams by using a meshless method. Nami and Janghorban (2013) studied the static response of rectangular nanoplates using trigonometric shear deformation theory based on nanolocal elasticity theory. Pour *et al*. (2015) presented a nonlocal sinusoidal shear deformation beam theory for the nonlinear vibration of single walled carbon nanotubes. By utilizing Euler–Bernoulli beam theory, Hajnayeb and Khadem (2015) studied free vibrations of a clamped-clamped double-walled carbon nanotube (DWNT) under axial force. Bagdatli (2015) investigated nonlinear transverse vibrations of tensioned Euler-Bernoulli nanobeams using nonlocal beam theory.

Developments in the field of materials engineering lead to a new type of materials with smooth and continuous variation of the material properties that called functionally graded materials (FGMs) (Attia *et al*. 2015, Bouchafa *et al*. 2015, Bennai *et al*. 2015, Ebrahimi and Dashti 2015, Kar and Panda 2015, Darılmaz 2015). Nanotechnology is also concerned with fabrication of functionally graded (FG) materials and engineering structures at a nanoscale, which enables a new generation of materials with revolutionary properties and devices with enhanced functionality. Recently, the application of FG materials has broadly been spread in micro- and nano-structures such as micro- and nano-electromechanical systems (MEMS and NEMS) (Witvrouw and Mehta 2005, Lee *et al*. 2006, Hasanyan *et al*. 2008, Mohammadi-Alasti *et al*. 2011, Zhang and Fu 2012), thin films in the form of shape memory alloys (Fu *et al*. 2003, Lu *et al*. 2011), and atomic force microscopes (AFMs) to achieve high sensitivity and desired performance (Rahaeifard *et al*. 2009). In such applications, size effects have been experimentally observed (Fleck *et al*. 1994, Stolken and Evans 1998, Chong *et al*. 2001, Lam *et al*. 2003). Since the dimension of these structural devices typically falls below micron- or nano-scale in at least one direction, an essential feature triggered in these devices is that their mechanical properties such as Young’s modulus, flexural rigidity, and so on are size-dependent. So far, only a few works have been reported for FG nanobeams based on the nonlocal elasticity theory. Janghorban and Zare (2011) investigated nonlocal free vibration axially FG nanobeams by using differential quadrature method. Eltaher *et al*. (2012) studied free vibration of FG nanobeam based on the nonlocal Euler-Bernoulli beam theory. Recently, Larbi Chaht *et al*. (2015) studied the static bending and buckling of a FG nanobeam using the nonlocal sinusoidal beam theory. Sobhy (2015) investigated the bending response, free vibration, mechanical buckling and thermal buckling of FG nanoplates embedded in an elastic medium. The four-unknown shear deformation theory incorporated in Eringen's nonlocal elasticity theory is employed for this end. Kolahchi *et al*. (2015) studied the bending behavior of FG nanoplates based on a new sinusoidal shear deformation theory. Zenkour and Abouelregal (2015) investigated the vibration phenomenon of a FG nanobeam subjected to a time-dependent heat flux. Based on a refined nonlocal shear deformation theory beam theory, Zemri *et al*. (2015) discussed the mechanical response of FG nanoscale beam. Belkorissat *et al*. (2015) investigated the vibration properties of FG nano-plate using a new nonlocal refined four variable theory. Al-Basyouni *et al*. (2015) studied the size dependent bending and vibration response of FG micro beams based on modified couple stress theory and neutral surface position. Tagrara *et al*. (2015) investigated the bending, buckling and free vibration analysis of carbon nanotube-reinforced composite beams resting on elastic foundation using a trigonometric refined beam theory. Bounouara *et al*. (2016) used a nonlocal zeroth-order shear deformation theory for free vibration analysis of FG nanoscale plates resting on elastic foundation.

As one may note, the most cited references deal the modeling of micro/nano-beams are based on the assumptions that the material is homogeneous. A very limited literature is available for micro/nano-scale structures use FGM. That gives us a potential to investigate the bending, buckling and dynamic behavior of functionally graded nanobeams.

In this paper, a nonlocal trigonometric beam theory is proposed for bending, buckling, and vibration of FG nanobeams. This theory is based on assumption that the in-plane and transverse displacements consist of bending and shear parts. In addition, it is also based on the assumption that the transverse shear stress vanishes on the top and bottom surfaces of the beam and is nonzero elsewhere. Thus there is no need to use shear correction factors as in the case of Timoshenko beam theory (TBT). Material properties of FG nanobeam are assumed to vary according to power law distribution of the volume fraction of the constituents. In addition, the small scale effect is taken into account by using the nonlocal constitutive relations of Eringen. To simplify the governing equations for the FG nanobeam, the coordinate system is located at the physical neutral surface of the beam. This is due to the fact that the stretching – bending coupling in the constitutive equations of an FG nanobeam does not exist when the physical neutral surface is considered as a coordinate system (Ould Larbi *et al*. 2013, Yahoobi and Feraidoon 2010). Thus, the present nonlocal trigonometric theory based on the exact position of neutral surface together with Hamilton principle are employed to extract the motion equations of the FG nanobeam. Analytical solutions for the deflection, buckling load, and natural frequency are presented for simply supported FG nanobeams, and the obtained results are compared with those available in literature to verify the accuracy of the present solution. The effects of nonlocal parameter, aspect ratio and power law index on the static, stability and dynamic responses of the FG nanobeam are discussed.

2. Theoretical formulations

Consider a uniform FG nanobeam of thickness h, length L, and width b made by mixing two distinct materials (metal and ceramic) is studied here. The coordinate x is along the longitudinal direction and z is along the thickness direction. For such beams, the neutral surface may not coincide with its geometric mid-surface (Ould Larbi *et al*. 2013, Yahoobi and Feraidoon 2010). The applied compressive force may be assumed to act at the mid-surface of the beam for all the

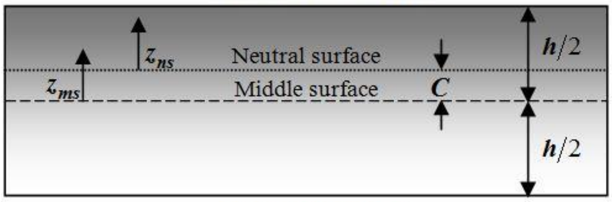

Fig. 1 The position of middle surface and neutral surface for a functionally graded beam

practical purposes, but the in-plane stress resultants act along the neutral surface. The noncoincidence of line of action of stress resultant and applied compressive force results in a couple as schematically shown in Fig. 1. The present study attempts to investigate the position of neutral surface and the deflection characteristics under in-plane loads.

Here, two different datum planes are considered for the measurement of z, namely, z_{ms} and z_{ns} measured from the middle surface, and the neutral surface of the beam, respectively (Fig. 1). The volume-fraction of ceramic V_C is expressed based on z_{ms} and z_{ns} coordinates (Fig. 1) as

$$V = \left(\frac{z_{ms}}{h} + \frac{1}{2} \right)^k = \left(\frac{z_{ns} + C}{h} + \frac{1}{2} \right)^k \tag{1}$$

where k is the material distribution parameter which takes the value greater or equal to zero and C is the distance of neutral surface from the mid-surface. Material non-homogeneous properties of a functionally graded material beam may be obtained by means of the Voigt rule of mixture (Eltaher *et al*. 2012, Bourada *et al*. 2012, Larbi Chaht *et al*. 2015, Tounsi *et al*. 2013a, Bouderba *et al*. 2013, Hebali *et al*. 2014, Zidi *et al*. 2014, Bakora and Tounsi 2015, Hamidi *et al*. 2015, Mahi *et al*. 2015, Akbaş, 2015, Bennoun *et al*. 2016, Salima *et al*. 2016). Thus, using Eq. (1), the material non-homogeneous properties of FG nanobeam P, such as Young's modulus (E), Poisson's ratio (v), the shear modulus (G), and the mass density (ρ), can be described by

$$P(z_{ns}) = (P_t - P_b)\left(\frac{z_{ns} + C}{h} + \frac{1}{2} \right)^k + P_b \tag{2}$$

where P_t and P_b are the corresponding material property at the top and bottom surfaces of the nanobeam.

The position of the neutral surface of the FG nanobeam is determined to satisfy the first moment with respect to Young's modulus being zero as follows (Ould Larbi *et al*. 2013, Bousahla *et al*. 2014, Fekrar *et al*. 2014, Bourada *et al*. 2015)

$$\int_{-h/2}^{h/2} E(z_{ms})(z_{ms} - C)dz_{ms} = 0 \tag{3}$$

Consequently, the position of neutral surface can be obtained as

$$C = \frac{\int_{-h/2}^{h/2} E(z_{ms}) z_{ms} dz_{ms}}{\int_{-h/2}^{h/2} E(z_{ms}) dz_{ms}} \tag{4}$$

2.1 Basic assumptions

The displacement field of the proposed theory is chosen based on the following assumptions:

(i) The origin of the Cartesian coordinate system is taken at the neutral surface of the FG nanobeam.
(ii) The displacements are small in comparison with the nanobeam thickness and, therefore, strains involved are infinitesimal.
(iii) The transverse displacement w includes two components of bending w_b, and shear w_s. These components are functions of coordinate x only.

$$w(x, z_{ns}) = w_b(x) + w_s(x) \tag{5}$$

(iv) The transverse normal stress σ_z is negligible in comparison with in-plane stresses σ_x.
(v) The displacement u in x-direction consists of extension, bending, and shears components.

$$u = u_0 + u_b + u_s, \tag{6}$$

The bending component u_b is assumed to be similar to the displacement given by the classical beam theory. Therefore, the expression for u_b can be given as

$$u_b = -z_{ns} \frac{\partial w_b}{\partial x}, \tag{7}$$

The shear component u_s gives rise, in conjunction with w_s, to a sinusoidal variations of shear strain γ_{xz} and hence to shear stress τ_{xz} through the thickness of the nanobeam in such a way that shear stress τ_{xz} is zero at the top and bottom faces of the nanobeam. Consequently, the expression for u_s can be given as

$$u_s = -f(z_{ns}) \frac{\partial w_s}{\partial x} \tag{8}$$

where

$$f(z_{ns}) = (z_{ns} + C) - \frac{h}{\pi} \sin\left(\frac{\pi (z_{ns} + C)}{h} \right) \tag{9}$$

2.2 Kinematics

Based on the assumptions made in the preceding section, the displacement field can be obtained using Eqs. (5)-(9) as

$$u(x, z_{ns}, t) = u_0(x, y) - z_{ns} \frac{\partial w_b}{\partial x} - f(z_{ns}) \frac{\partial w_s}{\partial x} \tag{10a}$$

$$w(x, z_{ns}, t) = w_b(x,t) + w_s(x,t) \tag{10b}$$

The strains associated with the displacements in Eq. (10) are

$$\varepsilon_x = \varepsilon_x^0 + z_{ns}\, k_x^b + f(z_{ns})\, k_x^s \quad \text{and} \quad \gamma_{xz} = g(z_{ns})\, \gamma_{xz}^s \tag{11}$$

where

$$\begin{gathered} \varepsilon_x^0 = \frac{\partial u_0}{\partial x}, \quad k_x^b = -\frac{\partial^2 w_b}{\partial x^2}, \quad k_x^s = -\frac{\partial^2 w_s}{\partial x^2} \\ \gamma_{xz}^s = \frac{\partial w_s}{\partial x}, \quad g(z_{ns}) = 1 - f'(z_{ns}) \quad \text{and} \quad f'(z_{ns}) = \frac{df(z_{ns})}{dz_{ns}} \end{gathered} \tag{12}$$

2.3 Constitutive relations

Response of materials at the nanoscale is different from those of their bulk counterparts. In the theory of nonlocal elasticity Eringen (1972, 1983), the stress at a reference point x is considered to be a functional of the strain field at every point in the body. For example, in the non – local elasticity, the uniaxial constitutive law is expressed as elasticity Eringen (1983).

$$\sigma_x - \mu \frac{d^2 \sigma_x}{dx^2} = E(z_{ns}) \varepsilon_x \tag{13a}$$

$$\tau_{xz} - \mu \frac{d^2 \tau_{xz}}{dx^2} = G(z_{ns}) \gamma_{xz} \tag{13b}$$

and $\mu = (e_0 a)^2$ is a nonlocal parameter revealing the nanoscale effect on the response of nanobeams, e_0 is a constant appropriate to each material and a is an internal characteristic length. In general, a conservative estimate of the nonlocal parameter is $e_0 a < 2.0$ nm for a single wall carbon nanotube (Wang 2005, Heireche *et al*. 2008, Tounsi *et al*. 2013b, c, Berrabah *et al*. 2013, Benguediab *et al*. 2014, Zidour *et al*. 2014, Semmah *et al*. 2014, Bessaim *et al*. 2015).

2.4 Equations of motion

Using the dynamic version of principle of virtual work (Belabed *et al*. 2014, Draiche *et al*. 2014, Ait Amar Meziane *et al*. 2014, Ait Yahia *et al*. 2015), variationally consistent governing differential equations for the FG nanobeam under consideration are obtained. The principle of virtual work when applied to the FG nanobeam leads to

$$\begin{gathered} \int_0^L \int_A \left(\sigma_x \delta\, \varepsilon_x + \tau_{zx} \delta\, \gamma_{zx} \right) dA\, dx - \int_0^L \int_A \rho \left[\ddot{u} \delta\, u + (\ddot{w}_b + \ddot{w}_s) \delta\, (w_b + w_s) \right] dA\, dx \\ - \int_0^L q \delta (w_b + w_s) dx - \int_0^L N_0 \frac{d(w_b + w_s)}{dx} \frac{d\delta(w_b + w_s)}{dx} dx = 0 \end{gathered} \tag{14}$$

Collecting the coefficients of δu_0, δw_b and δw_s in Eq. (14), equations of motion are obtained as

$$\delta u_0: \ \frac{dN}{dx} = I_0 \ddot{u}_0 - I_1 \frac{d\ddot{w}_b}{dx} - J_1 \frac{d\ddot{w}_s}{dx} \tag{15a}$$

$$\delta w_b: \ \frac{d^2 M_b}{dx^2} + q - N_0 \frac{d^2(w_b + w_s)}{dx^2} = I_0(\ddot{w}_b + \ddot{w}_s) + I_1 \frac{d\ddot{u}_0}{dx} - I_2 \frac{d^2\ddot{w}_b}{dx^2} - J_2 \frac{d^2\ddot{w}_s}{dx^2} \tag{15b}$$

$$\delta w_s: \ \frac{d^2 M_s}{dx^2} + \frac{dQ}{dx} + q - N_0 \frac{d^2(w_b + w_s)}{dx^2} = I_0(\ddot{w}_b + \ddot{w}_s) + J_1 \frac{d\ddot{u}_0}{dx} - J_2 \frac{d^2\ddot{w}_b}{dx^2} - K_2 \frac{d^2\ddot{w}_s}{dx^2} \tag{15c}$$

where N, M_b, M_s and Q are the stress resultants defined as

$$(N, M_b, M_s) = \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} (1, z_{ns}, f)\, \sigma_x dz_{ns} \qquad \text{and} \qquad Q = \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} g\, \tau_{xz} dz_{ns} \tag{16}$$

and (I_0, I_1, J_1, I_2, J_2, K_2) are the mass inertias defined as

$$\left(I_0, I_1, J_1, I_2, J_2, K_2\right) = \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} \left(1, z_{ns}, f, z_{ns}^2, z_{ns}\ f, f^2\right) \rho(z_{ns})\, dz_{ns} \tag{17}$$

when the shear deformation effect is neglected ($w_s = 0$), the equilibrium equations in Eq. (15) recover those derived from the Euler–Bernoulli beam theory into Eq. (16), the stress resultants are obtained as

$$N - \mu \frac{d^2 N}{dx^2} = A \frac{du_0}{dx} - B_s \frac{d^2 w_s}{dx^2} \tag{18a}$$

$$M_b - \mu \frac{d^2 M_b}{dx^2} = -D \frac{d^2 w_b}{dx^2} - D_s \frac{d^2 w_s}{dx^2} \tag{18b}$$

$$M_s - \mu \frac{d^2 M_s}{dx^2} = B_s \frac{du_0}{dx} - D_s \frac{d^2 w_b}{dx^2} - H_s \frac{d^2 w_s}{dx^2} \tag{18c}$$

$$Q - \mu \frac{d^2 Q}{dx^2} = A_s \frac{dw_s}{dx} \tag{18d}$$

where

$$\left(A, D, B_s, D_s, H_s\right) = \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} E(z_{ns}) \left(1, z_{ns}^2, f, z_{ns}\ f, f^2\right) dz_{ns}, \qquad A_s = \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} G(z_{ns}) g^2 dz_{ns} \tag{19}$$

By substituting Eq. (18) into Eq. (15), the nonlocal equations of motion can be expressed in terms of displacements (u_0, w_b, w_s) as

$$A\frac{d^2u_0}{dx^2}-B_s\frac{d^3w_s}{dx^3}=I_0\left(\ddot{u}_0-\mu\frac{d^2\ddot{u}_0}{dx^2}\right)-I_1\left(\frac{d\ddot{w}_b}{dx}-\mu\frac{d^3\ddot{w}_b}{dx^3}\right)-J_1\left(\frac{d\ddot{w}_s}{dx}-\mu\frac{d^3\ddot{w}_s}{dx^3}\right) \tag{20a}$$

$$\begin{aligned}&-D\frac{d^4w_b}{dx^4}-D_s\frac{d^4w_s}{dx^4}+q-\mu\frac{d^2q}{dx^2}-N_0\left(\frac{d^2(w_b+w_s)}{dx^2}-\mu\frac{d^4(w_b+w_s)}{dx^4}\right)\\&=I_0\left((\ddot{w}_b+\ddot{w}_s)-\mu\frac{d^2(\ddot{w}_b+\ddot{w}_s)}{dx^2}\right)+I_1\left(\frac{d\ddot{u}_0}{dx}-\mu\frac{d^3\ddot{u}_0}{dx^3}\right)\\&-I_2\left(\frac{d^2\ddot{w}_b}{dx^2}-\mu\frac{d^4\ddot{w}_b}{dx^4}\right)-J_2\left(\frac{d^2\ddot{w}_s}{dx^2}-\mu\frac{d^4\ddot{w}_s}{dx^4}\right)\end{aligned} \tag{20b}$$

$$\begin{aligned}&B_s\frac{d^3u_0}{dx^3}-D_s\frac{d^4w_b}{dx^4}-H_s\frac{d^4w_s}{dx^4}+A_s\frac{d^2w_s}{dx^2}+q-\mu\frac{d^2q}{dx^2}-N_0\left(\frac{d^2(w_b+w_s)}{dx^2}-\mu\frac{d^4(w_b+w_s)}{dx^4}\right)\\&=I_0\left((\ddot{w}_b+\ddot{w}_s)-\mu\frac{d^2(\ddot{w}_b+\ddot{w}_s)}{dx^2}\right)+J_1\left(\frac{d\ddot{u}_0}{dx}-\mu\frac{d^3\ddot{u}_0}{dx^3}\right)\\&-J_2\left(\frac{d^2\ddot{w}_b}{dx^2}-\mu\frac{d^4\ddot{w}_b}{dx^4}\right)-K_2\left(\frac{d^2\ddot{w}_s}{dx^2}-\mu\frac{d^4\ddot{w}_s}{dx^4}\right)\end{aligned} \tag{20c}$$

The equations of motion of local beam theory can be obtained from Eq. (20) by setting the nonlocal parameter μ equal to zero.

3. Analytical solution of simply supported FG nanobeam

The above equations of motion are analytically solved for bending, buckling and free vibration problems. The Navier solution procedure is used to determine the analytical solutions for a simply supported FG nanobeam. The solution is assumed to be of the form

$$\begin{Bmatrix}u_0\\w_b\\w_s\end{Bmatrix}=\sum_{m=1}^{\infty}\begin{Bmatrix}U_n\cos(\alpha\,x)\,e^{i\,\omega\,t}\\W_{bn}\sin(\alpha\,x)\,e^{i\,\omega\,t}\\W_{sn}\sin(\alpha\,x)\,e^{i\,\omega\,t}\end{Bmatrix} \tag{21}$$

where U_n, W_{bn}, and W_{sn} are arbitrary parameters to be determined, ω is the eigenfrequency associated with nth eigenmode, and $\alpha = n\pi/L$. The transverse load q is also expanded in the Fourier sine series as

$$q(x)=\sum_{n=1}^{\infty}Q_n\sin\alpha\,x,\qquad Q_n=\frac{2}{L}\int_0^L q(x)\sin(\alpha\,x)\,dx \tag{22}$$

The Fourier coefficients Q_n associated with some typical loads are given

$$Q_n = q_0, \quad n = 1 \quad \text{for sinusoidal load,} \tag{23a}$$

$$Q_n = \frac{4q_0}{n\pi}, \quad n = 1,3,5..... \text{ for uniform load,} \tag{23b}$$

$$Q_n = \frac{2q_0}{L}\sin\frac{n\pi}{2}, \quad n = 1,2,3.... \text{ for point load } Q_0 \text{ at the midspan,} \tag{23c}$$

Substituting the expansions of u_0, w_b, w_s, and q from Eqs. (21) and (22) into Eq. (20), the analytical solutions can be obtained from the following equations

$$\left(\begin{bmatrix} S_{11} & 0 & S_{13} \\ 0 & S_{22}-\xi & S_{23}-\xi \\ S_{13} & S_{23}-\xi & S_{33}-\xi \end{bmatrix} - \lambda\omega^2 \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{bmatrix}\right) \begin{Bmatrix} U_n \\ W_{bn} \\ W_{sn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \lambda Q_n \\ \lambda Q_n \end{Bmatrix} \tag{24}$$

where

$$\begin{gathered} S_{11} = A\alpha^2, \quad S_{13} = -B_s\alpha^3, \quad S_{22} = D\alpha^4, \quad S_{23} = D_s\alpha^4, \quad S_{33} = H_s\alpha^4 + A_s\alpha^2, \\ m_{11} = I_0, \quad m_{12} = -I_1\alpha, \quad m_{13} = -J_1\alpha, \\ m_{22} = I_0 + I_2\alpha^2, \quad m_{23} = I_0 + J_2\alpha^2, \quad m_{33} = I_0 + K_2\alpha^2, \\ \xi = \lambda N_0\alpha^2, \quad \lambda = 1 + \mu\alpha^2 \end{gathered} \tag{25}$$

4. Numerical results

In this section, analytical solutions obtained in the previous sections are presented. The obtained results are compared with those computed independently based on the Euler–Bernoulli beam theory (EBT) and Timoshenko beam theory (TBT) for a wide range of nonlocal parameter (e_0a), the material distribution parameter (k) and thickness ratio (L/H). In the following analysis, two FG nanobeams are investigated. The first FG nanobeam has the following material properties: E_t = 0.25 TPa, E_b = 1 TPa, $v_t = v_b$ = 0.3 (Larbi Chaht *et al*. 2014). The second FG nanobeam is composed of steel and alumina (Al_2O_3). The bottom surface of the beam is pure steel, whereas the top surface of the beam is pure alumina. The material properties are as follows: E_t = 390 GPa, E_b = 210 GPa, ρ_t = 3960 kg/m^3, ρ_b = 7800 kg/m^3, $v_t = v_b$ = 0.3 (Eltaher *et al*. 2012). The shear correction factor is taken as 5/6 for Timoshenko beam theory. For convenience, the following nondimensionalizations are used:

- $\overline{w} = 100w\dfrac{E_tI}{q_0L^4}$ for uniform load;
- $\overline{\omega} = \omega L^2\sqrt{\dfrac{\rho_tA}{E_tI}}$ frequency parameter;
- $\overline{N} = N_{cr}\dfrac{L^2}{E_tI}$ critical buckling load parameter:

Table 1 Dimensionless transverse deflections $(\bar{w})$ of the FG nanobeam for uniform load

L/h	k	Nonlocal parameter, e_0a (nm) 0			0.5			1			1.5			2		
		EBT[(a)]	TBT[(a)]	Present	EBT[(a)]	TBT[(a)]	Present	EBT[(a)]	TBT[(a)]	Present	EBT[(a)]	TBT[(a)]	Present	EBT[(a)]	TBT[(a)]	Present
	0	5.2083	5.3383	5.3381	5.3333	5.4659	5.4657	5.7083	5.8487	5.8485	6.3333	6.4867	6.4865	7.2083	7.3798	7.3796
	0.3	3.1401	3.2169	3.2181	3.2154	3.2938	3.2945	3.4415	3.5245	3.5257	3.8183	3.9090	3.9103	4.3459	4.4472	4.4487
10	1	2.3674	2.4194	2.4193	2.4242	2.4772	2.4772	2.5946	2.6508	2.6508	2.8787	2.9401	2.9400	3.2765	3.3451	3.3450
	3	1.8849	1.9249	1.9233	1.9302	1.9710	1.9693	2.0659	2.1091	2.1073	2.2921	2.3393	2.3374	2.6088	2.6615	2.6594
	10	1.5450	1.5799	1.5790	1.5821	1.6176	1.6168	1.6933	1.7310	1.7300	1.8787	1.9190	1.9188	2.1383	2.1843	2.1832
	0	5.2083	5.2227	5.2228	5.2222	5.2366	5.2367	5.2638	5.2784	5.2784	5.3333	5.3480	5.3480	5.4305	5.4455	5.4455
	0.3	3.1401	3.1486	3.1475	3.1484	3.1570	3.1559	3.1736	3.1822	3.1811	3.2154	3.2241	3.2230	3.2740	3.2829	3.2818
30	1	2.3674	2.3732	2.3732	2.3737	2.3795	2.3795	2.3926	2.3985	2.3985	2.4242	2.4301	2.4301	2.4684	2.4744	2.4744
	3	1.8849	1.8894	1.8892	1.8900	1.8944	1.8943	1.9050	1.9095	1.9094	1.9302	1.9347	1.9346	1.9654	1.9700	1.9698
	10	1.5450	1.5489	1.5488	1.5491	1.5530	1.5529	1.5615	1.5654	1.5653	1.5821	1.5860	1.5860	1.6109	1.6149	1.6149
	0	5.2083	5.2096	5.2096	5.2095	5.2108	5.2109	5.2133	5.2146	5.2146	5.2195	5.2208	5.2209	5.2283	5.2296	5.2296
	0.3	3.1401	3.1408	3.1395	3.1408	3.1416	3.1402	3.1431	3.1438	3.1425	3.1468	3.1476	3.1463	3.1521	3.1529	3.1516
100	1	2.3674	2.3679	2.3679	2.3679	2.3685	2.3685	2.3696	2.3702	2.3702	2.3725	2.3730	2.3731	2.3765	2.3770	2.3770
	3	1.8849	1.8853	1.8854	1.8854	1.8858	1.8858	1.8867	1.8871	1.8872	1.8890	1.8894	1.8894	1.8922	1.8926	1.8926
	10	1.5450	1.5453	1.5454	1.5454	1.5457	1.5457	1.5465	1.5468	1.5469	1.5483	1.5487	1.5487	1.5509	1.5513	1.5513

[(a)] Şimşek and Yurtçu (2013)

Table 2 Dimensionless critical buckling load $(\bar{N})$ of the FG nanobeam

L/h	k	Nonlocal parameter, e_0a (nm) 0			0.5			1			1.5			2		
		EBT[(a)]	TBT[(a)]	Present	EBT[(a)]	TBT[(a)]	Present	EBT[(a)]	TBT[(a)]	Present	EBT[(a)]	TBT[(a)]	Present	EBT[(a)]	TBT[(a)]	Present
	0	2.4674	2.4056	2.4058	2.4079	2.3477	2.3478	2.2457	2.1895	2.1897	2.0190	1.9685	1.9686	1.7690	1.7247	1.7248
	0.3	4.0925	3.9921	3.9906	3.9940	3.8959	3.8946	3.7249	3.6335	3.6322	3.3488	3.2667	3.2655	2.9341	2.8621	2.8611
10	1	5.4282	5.3084	5.3086	5.2975	5.1805	5.1808	4.9406	4.8315	4.8317	4.4418	4.3437	4.3440	3.8918	3.8059	3.8061
	3	6.8176	6.6720	6.6780	6.6534	6.5113	6.5172	6.2051	6.0727	6.0781	5.5787	5.4596	5.4645	4.8879	4.7835	4.7879
	10	8.3176	8.1289	8.1338	8.1173	7.9332	7.9379	7.5704	7.3987	7.4031	6.8062	6.6518	6.6558	5.9633	5.8281	5.8316
	0	2.4674	2.4603	2.4604	2.4606	2.4536	2.4537	2.4406	2.4336	2.4337	2.4079	2.4011	2.4012	2.3637	2.3570	2.3570
	0.3	4.0925	4.0811	4.0826	4.0813	4.0699	4.0714	4.0481	4.0368	4.0383	3.9940	3.9828	3.9843	3.9205	3.9096	3.9110
30	1	5.4282	5.4146	5.4147	5.4134	5.3998	5.3999	5.3694	5.3559	5.3560	5.2975	5.2843	5.2843	5.2001	5.1871	5.1872
	3	6.8176	6.8011	6.8018	6.7989	6.7825	6.7832	6.7436	6.7273	6.7280	6.6534	6.6373	6.6380	6.5311	6.5153	6.5160
	10	8.3176	8.2962	8.2968	8.2949	8.2735	8.2741	8.2274	8.2062	8.2068	8.1173	8.0964	8.0970	7.9681	7.9476	7.9482
	0	2.4674	2.4667	2.4668	2.4667	2.4661	2.4662	2.4649	2.4643	2.4643	2.4619	2.4613	2.4613	2.4576	2.4570	2.4571
	0.3	4.0925	4.0915	4.0933	4.0915	4.0905	4.0923	4.0885	4.0874	4.0893	4.0834	4.0824	4.0843	4.0764	4.0754	4.0772
100	1	5.4282	5.4270	5.4271	5.4269	5.4257	5.4257	5.4229	5.4217	5.4217	5.4162	5.4150	5.4150	5.4069	5.4057	5.4057
	3	6.8176	6.8161	6.8162	6.8159	6.8144	6.8145	6.8108	6.8094	6.8095	6.8025	6.8010	6.8011	6.7908	6.7893	6.7894
	10	8.3176	8.3157	8.3158	8.3155	8.3136	8.3137	8.3094	8.3075	8.3076	8.2992	8.2972	8.2973	8.2849	8.2830	8.2831

[(a)] Şimşek and Yurtçu (2013)

Table 1 shows the nondimensional maximum deflections $\overline{w}$ of a simply supported FG nanobeam subjected to uniform load. The calculated values are obtained using 100 terms in series in Eqs. (21) and (22). It should be noted that $e_0a = 0$ corresponds to local beam theory. It can be seen that the results of the present beam theory based on neutral surface position are in excellent agreement with those predicted by TBT (Şimşek and Yurtçu 2013) for all values of thickness ratio L/h, material distribution parameter k and nonlocal parameter e_0a. A significant change in the maximum deflection is observed when varying the material distribution parameter k. One also can note that as the nonlocal parameter increases, the maximum deflection increases, which highlight the significance of the nonlocal effect. It is noted that there is no effect of thickness ratio on $\overline{w}$ when the local EBT is used. This is due to that the local EBT neglects the transverse shear deformation effect.

Table 2 presents the nondimensional critical buckling loads for different values of thickness ratio L/h, material distribution parameter k and nonlocal parameter e_0a. As can be noted, the obtained results are in good agreement with those of Şimşek and Yurtçu (2013). The critical buckling load decreases as the nonlocal parameter increases. This emphasizes the significance of the nonlocal effect on the buckling response of beams. The variation of the material distribution parameter k leads to a significant change in the buckling load. It is noted that there is no effect of thickness ratio on critical buckling loads when the local EBT is used. This is due to that the local EBT neglects the transverse shear deformation effect.

Table 3 presents the fundamental nondimensional frequencies while varying the nonlocal parameter and the material distribution for a thickness ratio of 10, 30, and 100, respectively. The material properties of the FG nanobeam are according to those used by Eltaher *et al*. (2012). The present results are compared with those computed using both EBT and TBT and an excellent

Table 3 Dimensionless fundamental frequency $(\overline{\varpi})$ of the FG nanobeam

L/h	k	Nonlocal parameter, e_0a (nm)														
		0			0.5			1			1.5			2		
		EBT	TBT	Present	EBT	TBT	Present	EBT	TBT	Present	EBT	TBT	Present	EBT	TBT	Present
10	0	9.8293	9.7075	9.7077	9.7102	9.5899	9.5901	9.3774	9.2612	9.2614	8.8915	8.7813	8.7815	8.3228	8.2196	8.2198
	0.3	8.2694	8.1700	8.1710	8.1692	8.0711	8.0720	7.8892	7.7944	7.7954	7.4804	7.3905	7.3914	7.0019	6.9178	6.9187
	1	6.9650	6.8814	6.8816	6.8807	6.7981	6.7982	6.6448	6.5651	6.5652	6.3005	6.2249	6.2250	5.8975	5.8267	5.8268
	3	6.1575	6.0784	6.0755	6.0829	6.0048	6.0019	5.8744	5.7990	5.7962	5.5700	5.4985	5.4958	5.2137	5.1468	5.1443
	10	5.6544	5.5794	5.5769	5.5859	5.5118	5.5093	5.3945	5.3229	5.3205	5.1150	5.0470	5.0448	4.7878	4.7242	4.7221
30	0	9.8651	9.8511	9.8511	9.8516	9.8376	9.8376	9.8114	9.7975	9.7975	9.7456	9.7318	9.7318	9.6556	9.6419	9.6419
	0.3	8.3015	8.2901	8.2902	8.2902	8.2787	8.2789	8.2564	8.2450	8.2451	8.2010	8.1897	8.1898	8.1252	8.1140	8.1141
	1	6.9929	6.9832	6.9833	6.9833	6.9737	6.9737	6.9548	6.9453	6.9453	6.9082	6.8987	6.8987	6.8444	6.8349	6.8350
	3	6.1806	6.1715	6.1712	6.1722	6.1631	6.1627	6.1470	6.1380	6.1376	6.1058	6.0968	6.0964	6.0494	6.0405	6.0401
	10	5.6744	5.6658	5.6655	5.6667	5.6581	5.6578	5.6436	5.6350	5.6347	5.6057	5.5972	5.5969	5.5540	5.5455	5.5452
100	0	9.8692	9.8679	9.8679	9.8680	9.8667	9.8667	9.8643	9.8631	9.8631	9.8583	9.8570	9.8570	9.8498	9.8485	9.8485
	0.3	8.3052	8.3042	8.3042	8.3042	8.3031	8.3032	8.3011	8.3001	8.3001	8.2960	8.2950	8.2950	8.2889	8.2878	8.2878
	1	6.9961	6.9952	6.9952	6.9952	6.9943	6.9943	6.9926	6.9917	6.9917	6.9883	6.9874	6.9874	6.9823	6.9814	6.9814
	3	6.1833	6.1825	6.1824	6.1825	6.1817	6.1817	6.1802	6.1794	6.1794	6.1764	6.1756	6.1756	6.1711	6.1703	6.1703
	10	5.6767	5.6760	5.6759	5.6761	5.6753	5.6752	5.6740	5.6732	5.6731	5.6705	5.6697	5.6697	5.6656	5.6648	5.6648

agreement is observed with TBT. From obtained results, it can be seen that the fundamental nondimensional frequency is reduced with the increase of the nonlocal parameter and the material distribution parameter.

In general, the effect of transverse shear deformations and the nonlocal parameter is to increase the deflections and reduce the buckling loads as well as natural frequencies, as can be seen from the results presented in Tables 1-3. The increase of the material distribution parameter leads to a decrease of both the dimensionless deflections and fundamental frequencies contrary to the dimensionless buckling load. This is due to the fact that an increase in the material distribution parameter yields an increase in the stiffness of the FG nanobeam.

Numerical results are plotted in Figs. 2-7 using the present theory and the material properties of the FG nanobeam are according to those used by Eltaher *et al*. (2012). Fig. 2 depicts the nonlocal scale parameter effects on the nondimensional deflection of FG nanobeam for different thickness ratios. w_{NL} and w_L represent the nonlocal and local deflection, respectively. It can also be seen that the deflection increases with the nonlocal scale parameter. Also, it can be found from the results

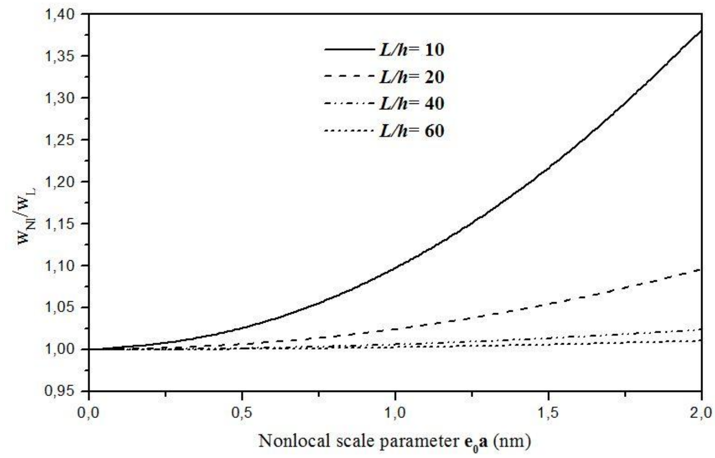

Fig. 2 The effect of nonlocal parameter on deflection for uniform load with different thickness ratios (k = 1)

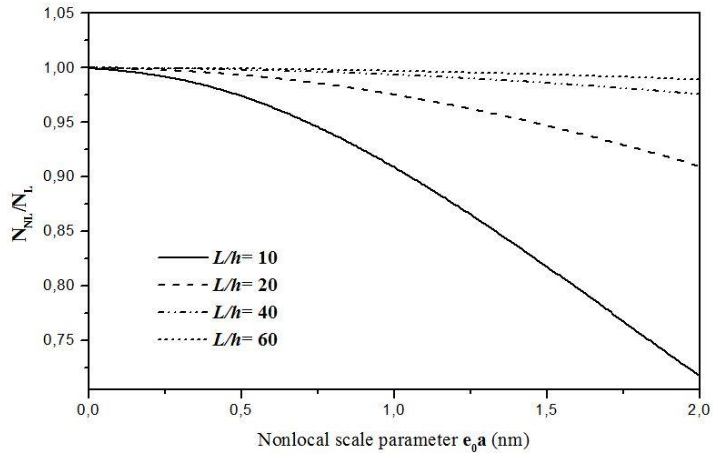

Fig. 3 The effect of nonlocal parameter on deflection for uniform load with different thickness ratios (k = 1)

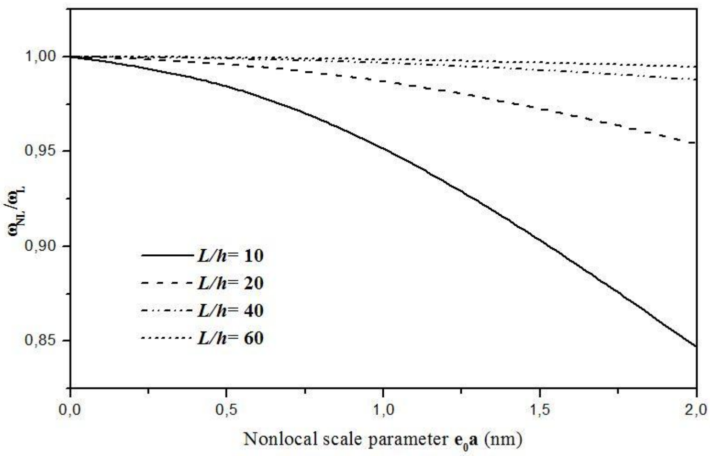

Fig. 4 The effect of nonlocal parameter on fundamental frequency for FG nanobeam with different thickness ratios (k = 1)

that the effect of nonlocality is more significant for lower values of thickness ratio (L/h), and this effect is very negligible for long FG nanobeams.

The effect of the nonlocal scale parameter on the buckling and dynamic responses of FG nanobeam is demonstrated in Figs. 3 and 4, respectively. These figures show that the responses vary nonlinearly with the nonlocal scale parameter. It can be observed that the nonlocal scale parameter strongly affects the nondimensional buckling loads and natural frequencies. Furthermore, it can be observed that when the thickness ratio is small, the scale effects are significant. However, the scale effects on the both critical buckling load and fundamental frequency will diminish with the ratio (i.e., L/h) increasing. It implies that the scale effects on the buckling and dynamic properties are not obvious for slender FG nanobeam but should be taken into account for short FG nanobeam.

The influence of the material distribution parameter on the dimensionless deflection, buckling

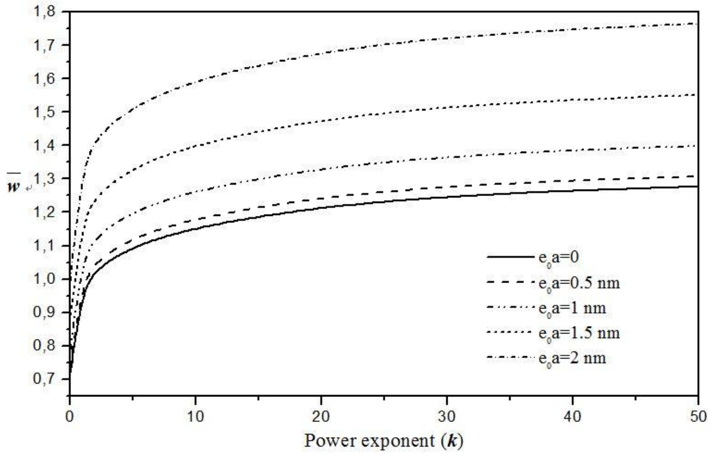

Fig. 5 Effect of the material distribution parameter on dimensionless deflection ($\bar{w}$) for uniform load with L/h = 10

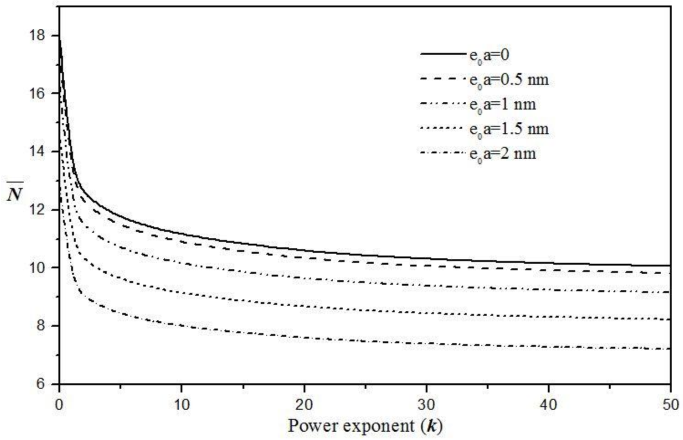

Fig. 6 Effect of the material distribution parameter on dimensionless buckling load $(\overline{N})$ with L/h = 10

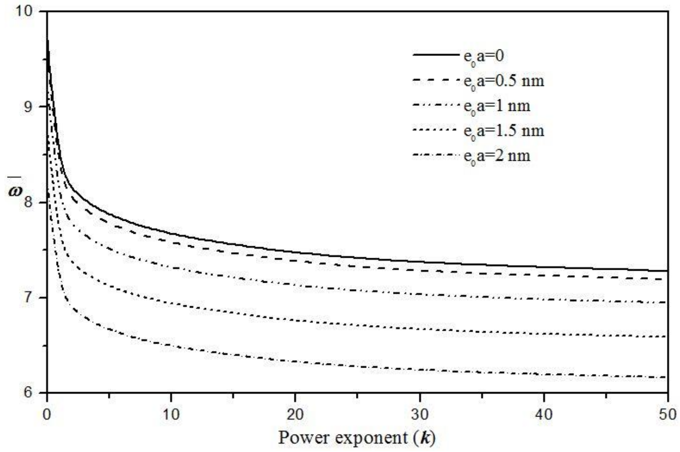

Fig. 7 Effect of the material distribution parameter on dimensionless fundamental frequency $(\overline{\omega})$ with L/h = 10

load and fundamental frequency of FG nanobeam is presented in Figs. 5 to 7 for various values of the nonlocal parameter with L/h = 10. It can be observed that both the dimensionless buckling load and fundamental frequencies decrease whereas the dimensionless deflection increases as the material distribution parameter increases. This is due to the fact that an increase in the material distribution parameter yields a decrease in the stiffness of the FG nanobeam.

Finally, the mechanical response of FG nanobeam is carried out using the present and other nonlocal beam theories because in these theories, the interaction of atoms with each other is incorporated into the equations of motion via the so-called, small-scale parameter. Indeed, the effect of inter-atomic bonds on the vibration behavior of beam-like nanostructures is taken into account by a small-scale parameter without serious difficulty in solving the governing equations (Peddieson *et al*. 2003). In addition, the applicability and the reliability of these nonlocal beam theories are justified by several authors such as Wang and Hu (2005), Harik (2001, 2002) and Tounsi *et al*. (2013b). Harik (2001, 2002) reported ranges of applicability for the continuum beam model in the mechanics of carbon nanotubes and nanorods. Wang and Hu (2005) present a rigorous study, in which they check the validity of the beam model in studying the flexural waves, simulated by the molecular dynamics (MD), in a single – walled carbon nanotube. Tounsi *et al*. (2013b) investigated the critical buckling strain and the obtained results are compared with those obtained from MD simulations

5. Conclusions

A nonlocal trigonometric shear deformation beam theory based on neutral surface position is proposed for bending, buckling, and free vibration of FG nanobeams. The present model is capable of capturing both small scale and shear deformation effects of FG nanobeams, and does not require shear correction factors. In addition, the displacement field proposed in the present theory is based on the assumption that the transverse displacements consist of bending and shear components in which the bending components do not contribute toward shear forces and, likewise, the shear components do not contribute toward bending moments. Based on the nonlocal differential constitutive relation of Eringen and the neutral surface concept, the nonlocal equations of motion of the proposed theory are derived from Hamilton’s principle. There is no stretching–bending coupling effect in the neutral surface-based formulation, and consequently, the governing equations and boundary conditions of FG nanobeams based on neutral surface have the simple forms as those of isotropic nanobeams. Numerical examples show that the present theory gives solutions which are almost identical with those generated by TBT. The obtained results show that, the material-distribution parameter may be manipulated to change the maximum deflection, to select a specific design frequency and maximize the critical buckling load. It is also shown that, the nonlocal parameter has a notable effect on the deflection, the fundamental frequencies and buckling of FG nanobeams. This model can be used in the analysis and design of nanobeams, such as nanosensors and nanoactuators.

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