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Comparison of elastic buckling loads for liquid storage tanks

P. Mirfakhraei† and D. Redekop‡

Department of Mechanical Engineering, University of Ottawa, Ottawa, Canada K1N 6N5

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Abstract. The problem of the elastic buckling of a cylindrical liquid-storage tank subject to horizontal earthquake loading is considered. An equivalent static loading is used to represent the dynamic effect. A theoretical solution based on the nonlinear Flügge shell equations is developed, and numerical results are found using the new differential quadrature method. A second solution is obtained using the finite element package ADINA. A major motivation of the study was to show that the new method can serve to verify finite element solutions for cylindrical shell buckling problems. For this purpose the paper concludes with a comparison of buckling results for a number of cases covering a wide range in tank geometry.

Key words: buckling; tanks; earthquake loading; differential quadrature method; finite element method.

1. Introduction

The problem of the buckling of cylindrical liquid-storage tanks subject to seismic action continues to be an active concern in structural engineering research (Mirfakhraei *et al*. 1996, Cho *et al*. 1999, Ishida *et al*. 1999, Mirfakhraei and Redekop 1999). Horizontal seismic excitation is the major cause of damage. It serves to accelerate the liquid in the tank, which then exerts a large nonsymmetric pressure on the tank wall that can lead to buckling.

While the response of the tank is clearly dynamic, experimental and theoretical studies (Mirfakhraei *et al.* 1996) have shown that the characteristic tank behavior can be determined through a static analysis. Such an analysis greatly reduces the size of the computational problem. Crucial to the success of the analysis, however, is the accounting for the boundary conditions that apply, and the development of an appropriate but practical system of equations.

In this study the nonlinear Flügge theory (Yamaki 1984, Flügge 1973) is used to develop a set of stability equations for cylindrical shells. The solution is in two steps, in each of which the new differential quadrature method (DQM) is used. A second solution is found using the ADINA finite element method (FEM) package. The paper concludes with a comparison of buckling results for a number of shell cases, covering a wide range of tank geometry.

2. Geometry and loading

A vertical cylindrical tank (Fig. 1) has a length L, a radius R, a thickness h, and is assumed made of

†Research Associate
‡Professor

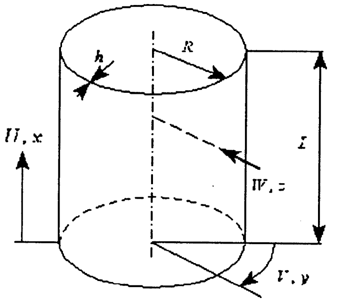

Fig. 1 Geometry

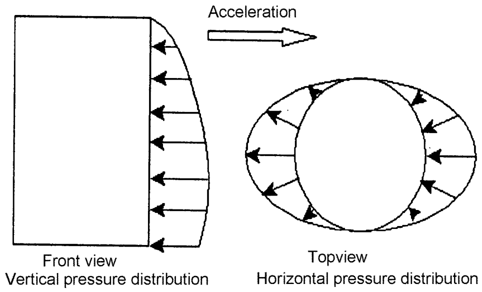

Fig. 2 Equivalent static loading

steel. A loading $P_s=\rho_l\, g\, (L-x)$ arises from hydrostatic pressure, and a loading $P_d = \alpha(x)\, \rho_l\, R\, G \cos\theta\, A(t)$ is assumed to arise from horizontal seismic action. Here ρ_l is the liquid mass density, g the acceleration of gravity, and G the acceleration factor. The $\cos\theta$ variation of P_d provides outward compression over one half of the circumference and inward tension over the other. $A(t)$ is a function dependent on the magnitude of acceleration and $\alpha(x)$ gives the longitudinal variation of the dynamic pressure, where x is the longitudinal position variable. This function is given in graphical form by Malhotra and Veletsos (1994) for some special values of length/radius (L/R) and radius/thickness (R/h) ratios. By curve fitting relations for $\alpha(x)$ have been derived for three sets of ratios as

$$
\begin{aligned}
L/R &= 0.5;\ \alpha(x) = 0.0532\, x^3 - 0.7129\, x^2 + 0.2812\, x + 0.3799 \\
L/R &= 1.0;\ \alpha(x) = -0.886\, x^3 + 0.0421\, x^2 + 0.2009\, x + 0.6516 \\
L/R &= 3.0;\ \alpha(x) = -13.4455\, x^5 + 26.2304\, x^4 - 21.0359\, x^3 + 8.108\, x^2 - 0.175\, x + 0.3239 \qquad (1)
\end{aligned}
$$

A plot of the loading distribution on the shell surface is given in Fig. 2.

3. Nonlinear Flügge shell theory

The nonlinear Flügge equations are used as the basis of the theoretical work. The effects of prebuckling deformations are accounted in the derivation of equation both in the domain and in the boundary conditions. The resulting set of stability equations is not restricted to axisymmetric loading as are those in the standard reference by Yamaki (1984). Based on an engineering consideration of the magnitude of terms, a final linearized set of stability equations is obtained, permitting the use of standard eigenvalue procedures. Final stability equations are linear in terms of buckling variables but have weighting coefficients from prebuckling terms.

A two-step procedure is used in the stability analysis. In the first step, the pre-buckling analysis, the membrane and bending resultants are found for the hydrostatic and equivalent static lateral loadings using the linear Flügge equations. In the second step, the buckling load analysis, the newly derived stability equations are used to find the lowest eigenvalue $\lambda_{\min}$.

The governing equations are found by minimizing the total energy through the variational principle (Yamaki 1984) and are given by

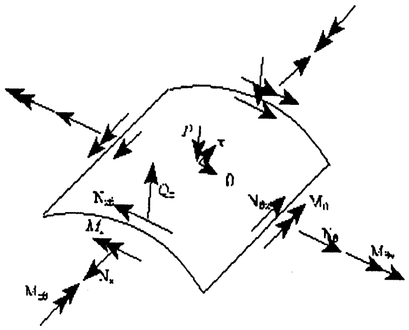

Fig. 3 Stress and moment resultants

$$
\begin{aligned}
&[N_x (1+U_{,x})]_{,x} + [N_{yx} (1 + U_{,x})]_{,y} + (N_y\, U_{,y})_{,y} + (N_{xy}\, U_{,y})_{,x} + p_x - p\, W_{,x} = 0 \\
&[N_{xy} (1+V_{,y} - W/R)]_{,x} + N_y (1+V_{,y} - W/R)_{,y} - (M_{y,y} + M_{xy,x})/R + (N_x\, V_{,x})_{,x} + (N_{yx}\, V_{,x})_{,y} - N_{yx}\, W_{,x}/R + p_y \\
&\qquad - (p + N_y/R)\,(W_{,y} + V/R) = 0 \\
&M_{x,xx} + (M_{xy} + M_{yx})_{,xy} + M_{y,yy} + N_y\, (1+V_{,y} - W/R)/R + [N_x\, W_{,x} + N_{xy}\, (W_{,y} + V/R)]_{,x} \\
&\qquad + [N_{yx}\, W_{,x} + N_y\, (W_{,y} + V/R)]_{,y} + N_{yx}\, V_{,x}/R + p\,(1 + U_{,x} + V_{,y} - W/R) = 0 \qquad (2)
\end{aligned}
$$

where p is the normal pressure, and p_x, p_y are the load components per unit area in the axial and circumferential directions. The displacement components and resultants appearing in these equations are shown in their positive sense in Figs. 1, 3. The stress and moment resultants are found from

$$
\begin{aligned}
N_x &= J[U_{,x} + \upsilon\, (V_{,y} - W/R) + \varepsilon_{xo} + \upsilon\, \varepsilon_{yo}] + D\, W_{,xx}/R \\
N_y &= J[V_{,y} - W/R + \upsilon\, U_{,x} + \varepsilon_{yo} + \upsilon\, \varepsilon_{xo}] - D\, (W_{,yy} + W/R^2)/R \\
N_{xy} &= \upsilon_1[J(U_{,y} + V_{,x} + \gamma_{xyo} + D(V_{,x}/R + W_{,xy})/R] \\
N_{yx} &= \upsilon_1[J(U_{,y} + V_{,x} + \gamma_{xyo} + D(U_{,y}/R - W_{,xy})/R] \\
M_x &= -D[W_{,xx} + \upsilon\, W_{,yy} + (U_{,x} + \upsilon\, V_{,y})/R] \\
M_y &= -D[W_{,yy} + \upsilon\, W_{,xx} + W/R^2] \\
M_{xy} &= -(1-\upsilon)\, D\, [W_{,xy} + V_{,x}/R] \\
M_{yx} &= -(1-\upsilon)\, D\, [W_{,xy} + (V_{,x} - U_{,y})/2R] \qquad (3)
\end{aligned}
$$

where $J = Eh/(1-\upsilon^2)$, $D = Eh^3/[12(1-\upsilon^2)]$, E, υ are the Young's modulus and Poisson ratio, $\upsilon_1 = (1-\upsilon)/2$, and the mid-surface nonlinear strain components are given by

$$
\begin{aligned}
\varepsilon_{xo} &= [U^2_{,x} + V^2_{,x} + W^2_{,x}]/2 \\
\varepsilon_{yo} &= [U^2_{,y} + (V_{,y} - W/R)^2 + (W_{,y} + V/R)^2]/2 \\
\gamma_{xyo} &= U_{,x}\, U_{,y} + V_{,x}(V_{,y} - W/R) + W_{,x}\, (W_{,y} + V/R) \qquad (4)
\end{aligned}
$$

Eqs. (2-3) contain 11 relations involving 11 unknowns, requiring reductions to obtain a practical system. The solution is subject to boundary conditions at the top and bottom edges of the shell.

In the determination of the buckling load a deformed state adjacent to the equilibrium state is sought. The displacements and resultants are thus assumed to have the form

$$(U, V, W) = (U_o, V_o, W_o) + (U_1, V_1, W_1)$$

$$(N_x, N_{xy}, N_{yx}, N_y) = (N_{xo}, N_{xyo}, N_{yxo}, N_{yo}) + (N_{x1}, N_{xy1}, N_{yx1}, N_{y1})$$
$$(M_x, M_{xy}, M_{yx}, M_y) = (M_{xo}, M_{xyo}, M_{yxo}, M_{yo}) + (M_{x1}, M_{xy1}, M_{yx1}, M_{y1}) \quad (5)$$

where the subscripts o and 1 correspond to prebuckling and infinitesimal incremental states respectively.

The expressions (5) are substituted into the governing Eqs. (2) and the expressions for the resultants (3-4). Simplification leads to the governing system of equations for the buckling problem. Since the incremental stress resultants include some prebuckling terms there are some nonlinear prebuckling terms in the final equations. The prebuckling terms are considered small so that the nonlinear prebuckling terms are dropped from the final equations. The final equations are lengthy and are given in full by Mirfakhraei (1999).

4. Boundary conditions

The solution defined in the preceding section is subject to boundary conditions on the shell edges. For the present problem clamped conditions apply for the bottom edge and free conditions for the top edge. The clamped conditions enforced at the bottom edge are given simply as $u = v = w = w_{,x} = 0$. The free boundary conditions enforced at the top edge are

$$N_x = 0;\ T_x \equiv N_{xy} - M_{xy}/R = 0$$
$$M_x = 0;\ S_x \equiv M_{x,x} + (M_{xy}+M_{yx})_{,y} = 0 \quad (6)$$

where S_x and T_x are effective shear forces. The first and third conditions are immediately converted into displacement form using the linear parts of the expressions for the resultants (3). The T_x and S_x conditions are converted respectively using;

$$J[U_{1,y}(1 + U_{o,x}) + V_{1,x}(1 + V_{o,y} - W_o/R) + U_{1,x}U_{o,y} + V_{1,y}V_{o,x} - W_1 V_{o,x}/R + W_{1,y}W_{o,x}$$
$$+ W_{1,x}(W_{o,y} + V_o/R) + V_1 W_{o,x}/R] + 3D(W_{1,xy} + V_{1,x}/R)/R = 0$$
$$W_{1,xxx} + (2 - \upsilon)W_{1,xyy} + U_{1,xx} - \upsilon_1 U_{1,yy} + \upsilon_2 V_{1,xy} = 0 \quad (7)$$

where $\upsilon_2 = (3 - \upsilon)/2$.

5. DQM solution

The basis of the DQM is the representation of the derivatives of a function $f(x)$ by a weighted sum of trial function values in the domain, i.e.,

$$\left.\frac{d^r f}{dx^r}\right|_{x = x_1} = \sum_{j=1}^{M} A_{ij}^{(r)} f(x_j) \quad (8)$$

Here the $A_{ij}^{(r)}$ are the unknown weighting coefficients of the r-th order derivative at the i-th sampling point in the domain, and M is the number of sampling points in the x direction. For the current study sets of trial functions are required for both of the coordinate directions x and θ.

Polynomial test functions are used here in the longitudinal direction. The functions are taken

$$f(x) = 1, x, x^2, \ldots, x^{M-1} \tag{9}$$

For these functions explicit formulas for the weighting coefficients in (8) are given as

$$A_{ij}^{(1)} = \frac{\pi(x_i)}{(x_i - x_j)\pi(x_j)}; i, j = 1, 2, \ldots, M;\ i \neq j$$

$$\pi(x_i) = \prod_{j=1}^{M}(x_i - x_j);\ i \neq j$$

The weighting coefficients for higher order derivatives may be obtained through recurrence relationships (Bert and Malek 1996).

A spacing of sampling points that proved successful in the solution of earlier shell problems (Bert and Malek 1996, Mirfakhraei and Redekop 1998) is used in the axial direction. At each sampling point either the DQM analogue of a governing equation for the domain is represented, or a boundary equation. For shells there are four conditions at each boundary, while there are only three governing equations. It is necessary to enforce one of the boundary equations at an interior point. This point, a 'δ point', is taken a short distance ($\delta \cong 10^{-4}$ on a unit domain) from the boundary point. For the present problem the first three of the conditions (6) replace respectively the first, second and third domain equations at the sampling point m, while the final condition replaces the third domain equation at the sampling point m-1. As the DQM approach is displacement-based these conditions must be converted into displacement form.

Harmonic test functions (Bert and Malek 1996, Mirfakhraei and Redekop 1998) are used in the meridional direction in this problem of cyclic periodicity. Continuity conditions across $\theta = 360^{\circ}$ are then satisfied identically. The test functions are taken as

$$f(\theta) = \cos[2(k-1)\pi\theta];\ k = 1, 2, 3, \ldots, N/2+1$$
$$f(\theta) = \sin[2k - N/2-1)\pi\theta];\ k = N/2+2, N/2+3, \ldots, N \tag{11}$$

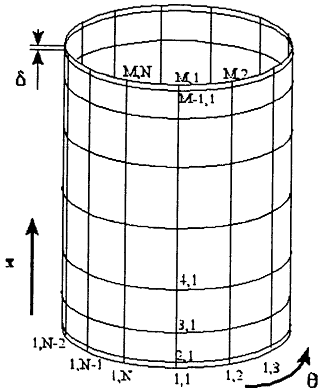

Fig. 4 DQM mesh

where N is an even number. For equally spaced sampling points the weighting coefficients may readily be found from the inverse of a Vandermonde matrix. A sample DQM mesh is given in Fig. 4.

5. Pre-buckled state

Prebuckling displacements and stress resultants are present in both the buckling equations and the boundary conditions. They are found as a preliminary step of the buckling analysis. The weighting coefficients of the DQM are found at this stage, and these same weighting coefficients are used later for the buckling load determination. As it can be assumed that prebuckling displacements are small the linear equilibrium equations of Flügge are adopted. These are given by

$$\begin{gathered} U,_{xx} + \upsilon_1\, U,_{yy} + \upsilon_3 V,_{xy} - \upsilon W,_x/R + \kappa(\upsilon_1\, U,_{yy} + RW,_{xxx} - \upsilon_1\, RW,_{xyy}) = 0 \\ \upsilon_3\, U,_{xy} + \upsilon_1\, V,_{xx} + V,_{yy} - W,_y/R + \kappa(3\upsilon_1\, U,_{xx} + \upsilon_2\, RW,_{xxy}) = 0 \\ \upsilon U,_x + V,_y - W/R + \kappa R^2(\upsilon_1\, U,_{xyy} - U,_{xxx} - \upsilon_2\, V,_{xxy} \\ - RW,_{xxx} - 2RW,_{xxyy} - RW,_{yyyy} - 2W/,_{yy}/R - W/R^3) = -P \end{gathered} \tag{12}$$

where $\upsilon_3 = (1 + \upsilon)/2$, $\kappa = h^2/(12R^2)$, and P is the normal pressure. Eqs. (12) are used to determine the prebuckling displacements first for the hydrostatic pressure loading, and then for the equivalent static loading.

6. Buckling load analysis

The governing equations are enforced at each of the domain mesh points with relations (8) used to replace the derivativers. A set of algebraic equations is set up in terms of the displacement commponents at the mesh points. Considering 'M' mesh points in the logitudinal direction, and 'N' points in the circumferential direction, there will be 3 MN algebraic equations, including the boundary equations. There will also be 3 MN unknown displacement components, and the buckling load parameter λ. The assembly of the domain and boundary equations yields a matrix equation of the form

$$[K_b + K_{p1}]\begin{Bmatrix} (\Delta_b) \\ (\Delta_d) \end{Bmatrix} = \lambda[K_{bg} + K_{p2}]\begin{Bmatrix} (\Delta_b) \\ (\Delta_d) \end{Bmatrix} \tag{13}$$

The matrix K_b consists of the terms for the DQM analogue of the buckling equations and the boundary conditions. Matrix K_{bg} includes the displacement terms which have a coefficient of the buckling load. Matrix K_{p1} includes the prebuckling terms due to hydrostatic pressure, and K_{p2} the prebuckling terms due to the equivalent static load arising from unit acceleration. The resultant matrices on the two sides are full, and thus static condensation, often used in the DQM, is not possible here. Each matrix is of size 3 MN. The vector Δ_b contains the displacements corresponding to the boundary points, while Δ_d the displacements corresponding to the domain points. The smallest eigenvalue $\lambda_{\min}$ may be found directly using standard eigenvalue extraction routines.

Based on the procedure outlined in the preceding a Matlab™ computer program named *tankeq.m* was developed. The DQM results given in the following are based on this program.

7. ADINA FEM analysis

The ADINA (Bathe 1996, ADINA 1998) program version 7.3 was used to find FEM results. Isoparametric eight-noded shell elements were used. A linearized buckling analysis was carried out, using starting vectors generated using the Lanczos method. The non-uniform load was input using a tabular form of the spatial functions for the distribution given in Fig. 2.

8. Validation

The validation analysis is for a tank which has $L = 7.5$ m, $R = 15$ m, $h = 15$ mm. These data are representative of short tanks. For this tank and others considered in this study a Poisson ratio of $\upsilon = 0.3$ and a Young's modulus of $E = 200$ GPa was used. The loading considered for the validation was that of the equivalent static loading. A convergence study was conducted for each of the two methods considered.

The validation results are presented in Tables 1-2. Table 1 gives the results in the DQM analysis, while Table 2 the results for the FEM analysis. There is a steady convergence for both methods,

Table 1 Convergence of DQM results

Solution	$N:M$	12	14	16
1	30	5.4724	5.4835	5.4919
2	34	3.7383	3.7905	3.7997
3	38	3.1311	3.1765	3.1849
4	40	2.9942	3.0360	3.0439
5	42	2.9130	2.9514	2.9587
6	46	2.8389	2.8721	2.8784

Table 2 Convergence of ADINA results

Solution	Mesh	$\lambda_{\min}$
1	60×20	2.920
2	80×30	2.841
3	100×40	2.817

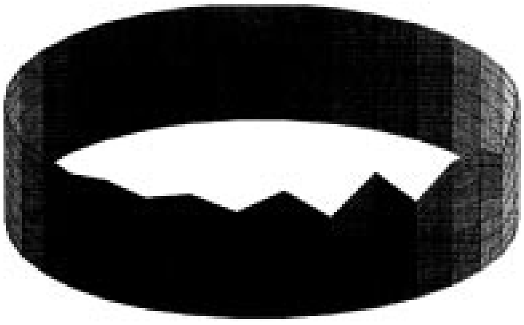

Fig. 5 DQM buckling mode

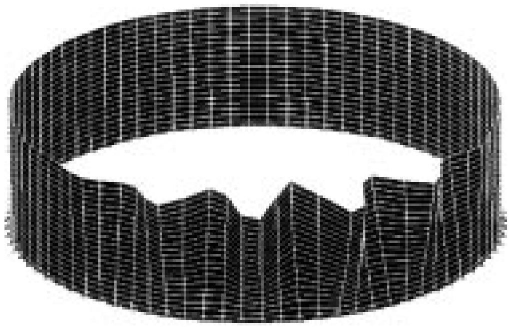

Fig. 6 FEM buckling mode

although for the DQM analysis the convergence is not as fast as was reported for the one-dimensional DQM (Mirfakhraei 1999). The converged results of the two methods show close agreement.

Figs. 5 and 6 show the buckling modes for the tank as determined by the DQM and FEM methods. As expected buckling occurs on the side where the load produces an inward pressure. The deformation for the tank in the axial direction resembles a quarter sine wave in the buckled part. Buckling is seen to be due to excessive circumferential stress.

9. Parametric study

Results were computed for a number of tanks covering a wide range of the geometric parameters. Two loading cases were considered for each tank. The first loading case (Set 1) consists of the equivalent static load. The second loading case (Set 2) consists of the hydrostatic pressure of the liquid (water) together with the equivalent static load. The hydrostatic pressure causes a prestress in the tank, and the equivalent static load causes buckling.

Table 3 gives results for the Set 1 loading case, while Table 4 gives results for the Set 2 loading case. For each load case results are given for three *L/R* ratios, three *R/h* ratios, and three values of the shell radius. Both DQM and FEM results are given for each tank, but only DQM solutions are given for the Set 2 loading case. A blank entry in Table 4 indicates that acceptable convergence was not obtained for that geometric case.

In Table 3 there are three parameters that determine the buckling load; *L/R*, *R/h* and *R*. In each instance when the *L/R* and *R/h* parameters are kept constant the buckling load decreses with an increase of *R*. There is a nearly linear trend in the reduction of the buckling load. When the *L/R* and *R* parameters are kept constant the buckling load decreases nonlinearly with an increase of *R/h*. Finally when the *R/h* and *R* parameters are kept constant the buckling load decreases nonlinearly with an increase of *L/R*. For the most part the DQM and FEM results of Table 3 agree within 5% although a maximum difference of 16% is observed. Except for one instance the FEM results are more conservative than the DQM results.

The stabilizing effect of internal hydrostatic pressure on buckling is evident on comparing the results of Tables 3 and 4. The effect is small for short tanks of small redius, but significant for long tanks of

Table 3 Comparison of results for set 1 loading case

L/R	*R/h*	*R* = 1 m		*R* = 5 m		*R* = 10 m	
		DQM	FEM	DQM	FEM	DQM	FEM
0.5	750	85.58	83.08	17.65	16.92	8.77	8.26
	1000	42.78	40.88	8.52	8.15	4.23	4.08
	1500	15.67	14.94	3.13	2.97	1.76	1.48
1.0	750	23.46	21.84	4.59	4.40	2.30	2.18
	1000	11.35	10.68	2.21	2.13	1.10	1.07
	1500	3.89	3.86	0.78	0.69	0.39	0.39
3.0	750	4.24	4.06	0.85	0.83	0.43	0.41
	1000	1.98	1.98	0.40	0.40	0.20	0.20
	1500	0.66	0.71	0.14	0.14	0.07	0.07

Table 4 DQM results for set 2 loading case

L/R	*R/h*	*R* = 1 m	*R* = 5 m	*R* = 10 m
0.5	750	95.08	24.42	16.13
	1000	50.31	15.42	10.91
	1500	23.15	9.93	8.18
1.0	750	30.12	10.84	8.56
	1000	18.16	8.27	7.12
	1500	9.80	6.78	-
3.0	750	11.22	7.00	5.50
	1000	9.04	5.94	-
	1500	6.66	-	4.75

large radius. The increase in buckling load due to internal pressure is seen to vary from 1.11 to 67 times.

The issue of the relative efficiency of the DQM and FEM approaches was not addressed comprehensively in this study. Analyses using the two methods were carried out on the same computer, with the FEM generally requring somewhat shorter calculation times. Howere there was no dedicated effort to optimize the DQM calculation process, and the Matlab^{TM} built-in computational routine intended for relatively small matrices was used to extract the eigenvalues. It is believed that the current study demonstrates sufficiently the usefulness of the DQM as a supportive computational resource. The definitive determination of its relative efficiency with respect to the FEM is reserved for a future study.

10. Conclusions

A solution for the elastic buckling problem of a seismically excited liquid-storage tank has been presented. Stability equations stemming from the nonlinear Flügge shell theory were derived. These equations include prebucking terms in the boundary conditions, and are more general than those of Yamaki in that they can account for nonsymmetric cases. The equations were solved using the differential quadrature approach. Numerical results obtained using the solution compare well with results found using the finite element method. The study demonstrates the usefulness of the differential quadrature method as a supportive compuational resource in cylindrical shell buckling analysis

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