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A new higher order shear and normal deformation theory for functionally graded beams

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Abstract. In this scientific work, constructing of a novel shear deformation beam model including the stretching effect is of concern for flexural and free vibration responses of functionally graded beams. The particularity of this model is that, in addition to considering the transverse shear deformation and the stretching effect, the zero transverse shear stress condition on the beam surface is assured without introducing the shear correction parameter. By employing the Hamilton's principle together with the concept of the neutral axe's position for such beams, the equations of motion are obtained. Some examples are performed to demonstrate the effects of changing gradients, thickness stretching, and thickness to length ratios on the bending and vibration of functionally graded beams.

Keywords: functionally graded beam; shear deformation theory; stretching effect; neutral surface position

1. Introduction

Functionally graded materials (FGMs) are generally metal–matrix composites (MMCs) in which material properties change in thickness direction from one surface to the other. The ceramic constituent provides high-temperature resistance due to its low thermal conductivity. The ductile metal constituent on the other hand, prevents fracture caused by stresses due to high temperature gradient in a very short span of time. FGMs are considered for the first time in Japanese (Koizumi 1993, 1997), and are applied as thermal barrier materials in space planes, space structures and nuclear reactors. Consequently, the mechanical response of structural components with FGMs is of

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highly importance in both research and industrial fields (Talha and Singh 2010, Benachour *et al*. 2011, El Meiche *et al*. 2011, Chakraverty and Pradhan 2014, Ait Amar Meziane *et al*. 2014).

It can be outlined that functionally graded materials, a great advance has been carried out for developing elasticity theory as well as the plates and shells models. However, studies on functionally graded (FG) beams are very limited in the scientific literature. Benatta *et al*. (2009) and Sallai *et al*. (2009) studied the bending response of simply supported FG hybrid beams subjected to uniformly distributed transverse loads by employing a higher-order shear deformation theory. The finite element method and the third-order shear deformation theory (TSDT) are used by Kadoli *et al*. (2008) to investigate the bending of FG beams by considering different boundary conditions (BCs) at the edges. Sankar (2001) developed a beam model to analyse the bending problem of a simply supported beam. Li (2008) considered the bending and transverse vibrations problem of FG Timoshenko beams. Ke *et al*. (2009a, b) as well as Yang and Chen (2008) investigated free vibrations, buckling and post-buckling of an exponential FG Timoshenko beams with the presence of open cracks. In most shear deformation theories, FG beams have been analysed ignoring the thickness stretching (ε_z).This effect, has been considered by Carrera *et al*. (2011) in FG plates by employing the finite elements method. Neves *et al*. (2011, 2012a, b) proposed an interesting hyperbolic sine shear deformation theory to study the bending and free vibration behaviours of FG plates. Houari *et al*. (2013) developed a new higher-order shear and normal deformation theory for the thermo-elastic bending investigation of FG sandwich plates. The same approach was employed by Bessaim *et al*. (2013) for the static and vibration analysis of FG sandwich plates. Saidi *et al*. (2013) used the new hyperbolic shear deformation theory in which the stretching effect is included to investigate the thermo-mechanical bending response of FG sandwich plates. Hebali *et al*. (2014) proposed a new quasi-three-dimensional (3D) hyperbolic shear deformation theory for the bending and free vibration analysis of FG plate. Belabed *et al*. (2014) developed an efficient and simple higher order shear and normal deformation theory for FG plates. Larbi Chaht *et al*. (2014) studied bending and buckling behaviors of size-dependent nanobeams made of functionally graded materials including the thickness stretching effect. Bourada *et al*. (2014) presented a new simple and refined trigonometric higher-order beam theory for bending and vibration of FG beams with including the thickness stretching effect.

Since, the mechanical properties of functionally graded beam can be change continuously and gradually along the thickness direction, the neutral surface of such beam may not confused with its geometric median axis. Therefore, stretching and bending deformations of FG beam are coupled. Some researchers (Morimoto *et al*. 2006, Ould Larbi *et al*. 2013, Bouremana *et al*. 2013, Bousahla *et al*. 2014, Fekrar *et al*. 2014) have shown that when the reference axis is properly chosen, the stretching-bending coupling will be avoided in constitutive equations. Based on neutral surface position, Ould Larbi *et al*. (2013) studied the static and dynamic behavior of FG beams.

This article tries to present a novel shear deformation beam theory for FG beams by including the so-called “stretching effect”. By superposing the deflection into bending, shear and stretching parts, the motion equations of the functionally graded beams are obtained based on the exact position of neutral axis together with Hamilton’s principle. Numerical examples are proposed to demonstrate the effects of varying gradients, thickness stretching, and thickness to length ratios on the bending and free vibration of functionally graded beams.

2. Theoretical formulations

2.1 Physical neutral surface

Consider a straight FG beam of area A, height h, and length L. The reference system, (x, z), with the origin at the left end of the beam is employed in this investigation. The x axis coincides with the median axis of the beam, and the z axis is considered to be perpendicular to this axis. Due to asymmetry of material properties of FG beams with respect to middle plane, the stretching and bending relations are coupled. But, if the origin of the coordinate system is properly chosen along the thickness direction of the FG beam so as to be the neutral axis, the properties of the FG beam being symmetric with respect to it. To specify the position of neutral axis of FG beams, two different axis are selected for the measurement of z, namely, z_{ms} and z_{ns} measured from the median axis and the neutral axis of the beam, respectively, as depicted in Fig. 1.

The volume-concentration of ceramic V_C is written in terms of coordinates z_{ms} and z_{ns} as

$$V_C = \left(\frac{z_{ms}}{h} + \frac{1}{2}\right)^k = \left(\frac{z_{ns} + C}{h} + \frac{1}{2}\right)^k \tag{1}$$

where k ($k \geq 0$) is the volume fraction exponent and C is the distance of neutral axis from the centroidal axis. In this work, material properties are considered to vary in accordance with the rule of mixture (Suresh and Mortensen 1998, Tounsi *et al*. 2013a, Bouderba *et al*. 2013, Bachir Bouiadjra *et al*. 2013, Kettaf *et al*. 2013, Klouche Djedid *et al*. 2014, Zidi *et al*. 2014). Hence, by considering Eq. (1), the mechanical properties of FG beam (P), in term of thickness coordinate are expressed as

$$P(z) = P_M + P_{CM}\left(\frac{z_{ns} + C}{h} + \frac{1}{2}\right)^k, \quad P_{CM} = P_C - P_M \tag{2}$$

Here, P_M and P_C denote the values of the material properties of the metal and ceramic constituents of the FG beam respectively. In this investigation, the the modulus of elasticity E and the mass density ρ are expressed according to Eq. (2), while Poisson's ratio v, is assumed to be constant (Sallai *et al*. 2009). Based on the physical neutral surface concept put forward by Ould Larbi *et al*. (2013), the physical neutral axis of an FG beam is expressed as

$$C = \frac{\int_{-h/2}^{h/2} E(z_{ms}) z_{ms} dz_{ms}}{\int_{-h/2}^{h/2} E(z_{ms}) dz_{ms}} \tag{3}$$

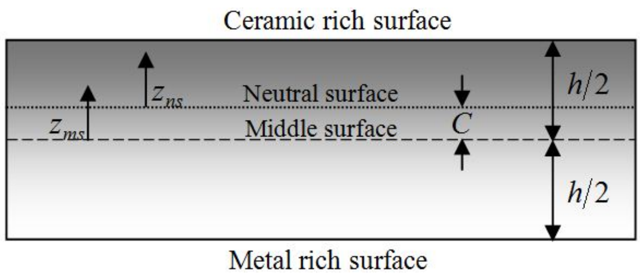

Fig. 1 The position of middle surface and neutral surface for a functionally graded beam

It can be noted that distance (C) becomes zero for homogeneous beams.

2.2 Basic hypotheses

The main hypotheses of this investigation are as follows:

(i) The origin of the *Cartesian Coordinate System* is considered at the neutral axis of the FG beam.
(ii) The displacements are small in comparison with the plate thickness and, therefore, strains involved are infinitesimal.
(iii) The transverse displacement w includes three components of bending w_b, shear w_s, and stretching effect w_{st}. The two first components are functions of coordinate x only and the third one is function of x and z_{ns}.

$$w(x, z_{ns}, t) = w_b(x,t) + w_s(x,t) + w_{st}(x, z_{ns}, t) \tag{4}$$

(iv) The displacement u along x-direction is composed of three parts namely: extension, bending, and shear components.

$$u = u_0 + u_b + u_s, \tag{5}$$

The bending component u_b is considered to be analogue to the displacement used in the classical beam theory. Thus, u_b becomes

$$u_b = -z_{ns}\frac{\partial w_b}{\partial x}, \tag{6}$$

The shear component u_s gives rise, together with w_s, to the sinusoidal distributions of shear strain γ_{xz} and thus to shear stress τ_{xz} within the thickness of the beam in such a way that shear stress τ_{xz} becomes zero at the top and bottom faces of the beam. Therefore, the relation for u_s can be expressed by

$$u_s = -f(z_{ns})\frac{\partial w_s}{\partial x}, \tag{7}$$

where

$$f(z_{ns}) = \left[(z_{ns} + C) - \frac{h}{\pi}\sin\left(\frac{\pi}{h}(z_{ns} + C)\right)\right] \tag{8}$$

The component due to the stretching effect w_{st} can be given as

$$w_{st}(x, z_{ns}, t) = g(z_{ns})\,\varphi(x,t) \tag{9}$$

The additional displacement φ, accounts for the stretching effect and $g(z_{ns})$ is written as follows

$$g(z_{ns}) = \cos\left[\frac{\pi}{h}(z_{ns} + C)\right] \tag{10}$$

2.3 Kinematics and constitutive equations

By considering the above hypotheses, the displacement field can be expressed by employing Eqs. (4)-(10) as

$$u(x,z_{ns},t)=u_0(x,t)-z_{ns}\frac{\partial w_b}{\partial x}-f(z_{ns})\frac{\partial w_s}{\partial x} \tag{11a}$$

$$w(x,z_{ns},t)=w_b(x,t)+w_s(x,t)+g(z_{ns})\,\varphi(x,t) \tag{11b}$$

The non-zero linear strains derived from Eq. (11) are

$$\varepsilon_x=\varepsilon_x^0+z_{ns}\,k_x^b+f(z_{ns})\,k_x^s \tag{12a}$$

$$\gamma_{xz}=g(z_{ns})\,\gamma_{xz}^0 \tag{12b}$$

$$\varepsilon_z=g'(z_{ns})\,\varepsilon_z^0 \tag{12c}$$

where

$$\varepsilon_x^0=\frac{\partial u_0}{\partial x},\quad k_x^b=-\frac{\partial^2 w_b}{\partial x^2},\quad k_x^s=-\frac{\partial^2 w_s}{\partial x^2},\quad \gamma_{xz}^0=\frac{\partial w_s}{\partial x}+\frac{\partial \varphi}{\partial x},\quad \varepsilon_z^0=\varphi \tag{12d}$$

$$g'(z_{ns})=\frac{dg(z_{ns})}{dz_{ns}} \tag{12e}$$

By using the Hooke's law, the stresses in the beam are expressed as follows

$$\sigma_x=Q_{11}(z_{ns})\varepsilon_x+Q_{13}(z_{ns})\varepsilon_z,\quad \tau_{xz}=Q_{55}(z_{ns})\,\gamma_{xz},\quad \text{and}\quad \sigma_z=Q_{13}(z_{ns})\varepsilon_x+Q_{33}(z_{ns})\varepsilon_z \tag{13a}$$

where

$$Q_{11}(z_{ns})=Q_{33}(z_{ns})=\frac{E(z_{ns})}{\left(1-\nu^2\right)},\quad Q_{13}(z_{ns})=\nu\,Q_{11}(z_{ns}),\quad \text{and}\quad Q_{55}(z_{ns})=\frac{E(z_{ns})}{2\left(1+\nu\right)} \tag{13b}$$

2.4 Equations of motion

Here, the governing equations are obtained by employing Hamilton's principle as (Reddy 2002, Draiche *et al*. 2014, Nedri *et al*. 2014)

$$\int_{t_1}^{t_2}\left(\delta U+\delta V-\delta K\right)dt=0 \tag{14}$$

where t is the time; t_1 and t_2 are the initial and end time, respectively; δU is the virtual variation of the strain energy; δV is the variation of work carried out by the applied forces; and δK is the virtual variation of the kinetic energy. The variation of the strain energy of the beam can be expressed as

$$\delta U = \int_0^L \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} \left(\sigma_x \delta\, \varepsilon_x + \sigma_z \delta\, \varepsilon_z + \tau_{xz} \delta\, \gamma_{xz}\right) dz_{ns} dx$$
$$= \int_0^L \left(N \frac{d\delta\, u_0}{dx} + N_z \delta\, \varphi - M_b \frac{d^2 \delta\, w_b}{dx^2} - M_s \frac{d^2 \delta\, w_s}{dx^2} + Q\left[\frac{d\delta\, w_s}{dx} + \frac{d\delta\, \varphi}{dx} \right] \right) dx \tag{15}$$

Where N, M_b, M_s, N_z and Q are the stress resultants defined as

$$(N, M_b, M_s) = \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} (1, z_{ns}, f(z_{ns}))\, \sigma_x dz_{ns}, \quad N_z = \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} \sigma_z g'(z_{ns}) dz_{ns}, \quad \text{and} \quad Q = \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} \tau_{xz} g(z_{ns}) dz_{ns} \tag{16}$$

The variation of work carried out by externally transverse loads q can be expressed as

$$\delta V = -\int_0^L q \delta\left(w_b + w_s\right) dx \tag{17}$$

The variation of the kinetic energy can be expressed as

$$\delta K = \int_0^L \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} \rho(z_{ns}) \left[\dot{u} \delta\, \dot{u} + \dot{w} \delta\, \dot{w} \right] dz_{ns} dx$$
$$= \int_0^L \Big\{ I_0 \left[\dot{u}_0 \delta \dot{u}_0 + \left(\dot{w}_b + \dot{w}_s\right)\left(\delta\, \dot{w}_b + \delta\, \dot{w}_s\right) \right] + J_0 \left[\left(\dot{w}_b + \dot{w}_s\right) \delta\, \varphi + \dot{\varphi}\, \delta\left(\dot{w}_b + \dot{w}_s\right) \right]$$
$$- I_1 \left(\dot{u}_0 \frac{d\delta \dot{w}_b}{dx} + \frac{d\, \dot{w}_b}{dx} \delta\, \dot{u}_0 \right) + I_2 \left(\frac{d\dot{w}_b}{dx} \frac{d\delta\, \dot{w}_b}{dx} \right) - J_1 \left(\dot{u}_0 \frac{d\delta \dot{w}_s}{dx} + \frac{d\, \dot{w}_s}{dx} \delta\, \dot{u}_0 \right)$$
$$+ K_2 \left(\frac{d\dot{w}_s}{dx} \frac{d\delta\, \dot{w}_s}{dx} \right) + J_2 \left(\frac{d\dot{w}_b}{dx} \frac{d\delta\, \dot{w}_s}{dx} + \frac{d\dot{w}_s}{dx} \frac{d\delta\, \dot{w}_b}{dx} \right) + K_0 \dot{\varphi}\, \delta\, \dot{\varphi} \Big\} dx \tag{18}$$

where dot-superscript convention denotes the differentiation with respect to the time variable t; and (I_i, J_i, K_i) are mass inertias expressed by

$$\left(I_0, I_1, I_2\right) = \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} \left(1, z_{ns}, z_{ns}^2\right) \rho(z_{ns}) dz_{ns} \tag{19a}$$

$$\left(J_0, J_1, J_2\right) = \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} \left(g, f, z_{ns} f\right) \rho(z_{ns}) dz_{ns} \tag{19b}$$

$$(K_0, K_2) = \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} \left(g^2, f^2\right) \rho(z_{ns}) dz_{ns} \tag{19c}$$

Substituting the relations for δU, δV, and δK from Eqs. (15), (17), and (18) into Eq. (14) and integrating by parts, and collecting the coefficients of δu_0, δw_b, δw_s and $\delta\varphi$, the following equations of motion of the FG beam are found

$$\delta u_0: \ \frac{dN}{dx} = I_0 \ddot{u}_0 - I_1 \frac{d\ddot{w}_b}{dx} - J_1 \frac{d\ddot{w}_s}{dx} \tag{20a}$$

$$\delta w_b: \ \frac{d^2 M_b}{dx^2} + q = I_0\left(\ddot{w}_b + \ddot{w}_s\right) + J_0 \ddot{\varphi} + I_1 \frac{d\ddot{u}_0}{dx} - I_2 \frac{d^2 \ddot{w}_b}{dx^2} - J_2 \frac{d^2 \ddot{w}_s}{dx^2} \tag{20b}$$

$$\delta w_s: \ \frac{d^2 M_s}{dx^2} + \frac{dQ}{\partial x} + q = I_0\left(\ddot{w}_b + \ddot{w}_s\right) + J_0 \ddot{\varphi} + J_1 \frac{d\ddot{u}_0}{dx} - J_2 \frac{d^2 \ddot{w}_b}{dx^2} - K_2 \frac{d^2 \ddot{w}_s}{dx^2} \tag{20c}$$

$$\delta \varphi: \ \frac{dQ}{dx} - N_z = J_0\left(\ddot{w}_b + \ddot{w}_s\right) + K_0 \ddot{\varphi} \tag{20d}$$

Eq. (20) can be written as functions of displacements (u_0, w_b, w_s and φ) by employing Eqs. (11), (12), (13) and (16) as follows

$$A_{11} \frac{\partial^2 u_0}{\partial x^2} - B_{11}^s \frac{\partial^3 w_s}{\partial x^3} + L \frac{d\varphi}{dx} = I_0 \ddot{u}_0 - I_1 \frac{d\ddot{w}_b}{dx} - J_1 \frac{d\ddot{w}_s}{dx} \tag{21a}$$

$$-D_{11} \frac{d^4 w_b}{dx^4} - D_{11}^s \frac{d^4 w_s}{dx^4} + L^a \frac{d^2 \varphi}{dx^2} + q = I_0\left(\ddot{w}_b + \ddot{w}_s\right) + J_0 \ddot{\varphi} + I_1 \frac{d\ddot{u}_0}{dx} - I_2 \frac{d^2 \ddot{w}_b}{dx^2} - J_2 \frac{d^2 \ddot{w}_s}{dx^2} \tag{21b}$$

$$\begin{aligned} & B_{11}^s \frac{d^3 u_0}{dx^3} - D_{11}^s \frac{d^4 w_b}{dx^4} - H_{11}^s \frac{d^4 w_s}{dx^4} + A_{55}^s \frac{d^2 w_s}{dx^2} + (R + A_{55}^s) \frac{d^2 \varphi}{dx^2} + q \\ & = I_0(\ddot{w}_b + \ddot{w}_s) + J_0 \ddot{\varphi} + J_1 \frac{d\ddot{u}_0}{dx} - J_2 \frac{d^2 \ddot{w}_b}{dx^2} - K_2 \frac{d^2 \ddot{w}_s}{dx^2} \end{aligned} \tag{21c}$$

$$L \frac{du_0}{dx} - L^a \frac{d^2 w_b}{dx^2} + \left(R + A_{55}^s\right) \frac{d^2 w_s}{dx^2} + R^a \varphi - A_{55}^s \frac{d^2 \varphi}{dx^2} = J_0\left(\ddot{w}_b + \ddot{w}_s\right) + K_0 \ddot{\varphi} \tag{21d}$$

where A_{11}, D_{11}, etc., are the beam stiffness, expressed by

$$\left(A_{11}, D_{11}, B_{11}^s, D_{11}^s, H_{11}^s\right) = \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} Q_{11}\left(1, z^2, f(z_{ns}), z_{ns}\, f(z_{ns}), f^2(z_{ns})\right) dz_{ns} \tag{22a}$$

and

$$A_{55}^{s}=\int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} Q_{55}\left[g(z_{ns})\right]^{2}dz_{ns},\quad \left[L,L^{a},R,R^{a}\right]=\int_{-\frac{h}{2}-C}^{\frac{h}{2}-C}\left[1,z_{ns},f(z_{ns}),g'(z_{ns})\right]g'(z_{ns})dz_{ns} \tag{22b}$$

3. Analytical solution

The equations of motion admit the Navier solutions for simply supported beams. The variables u_0, w_b, w_s and φ can be expressed by considering the following variations

$$\begin{Bmatrix} u_0 \\ w_b \\ w_s \\ \varphi \end{Bmatrix} == \sum_{m=1}^{\infty}\begin{Bmatrix} U_m \cos(\lambda\, x)\, e^{i\,\omega\, t} \\ W_{bm} \sin(\lambda\, x)\, e^{i\,\omega\, t} \\ W_{sm} \sin(\lambda\, x)\, e^{i\,\omega\, t} \\ \Phi_{stm} \sin(\lambda\, x)\, e^{i\,\omega\, t} \end{Bmatrix} \tag{23}$$

where U_m, W_{bm}, W_{sm} and Φ_{stm} are arbitrary coefficients to be found, ω is the frequency associated with mth eigenmode, and $\lambda = m\pi / L$. The transverse load q is also expanded in Fourier series as

$$q(x)=\sum_{m=1}^{\infty} Q_m \sin(\lambda\, x) \tag{24}$$

where Q_m is the load amplitude computed from

$$Q_m = \frac{2}{L}\int_{0}^{L} q(x)\sin(\lambda\, x)dx \tag{25}$$

The coefficients Q_m are given below for some typical loads. For the case of a sinusoidally varied load, we have

$$m=1 \quad \text{and} \quad Q_1 = q_0 \tag{26a}$$

and for the case of uniform varied load, we have

$$Q_m = \frac{4q_0}{m\pi},\quad (m=1,3,5....) \tag{26b}$$

Substituting Eqs. (23) and (24) into Eq. (21), the analytical solutions can be determined by

$$\left(\begin{bmatrix} a_{11} & 0 & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix} - \omega^2 \begin{bmatrix} m_{11} & m_{12} & m_{13} & 0 \\ m_{12} & m_{22} & m_{23} & m_{24} \\ m_{13} & m_{23} & m_{33} & m_{34} \\ 0 & m_{24} & m_{34} & m_{44} \end{bmatrix}\right)\begin{Bmatrix} U_m \\ W_{bm} \\ W_{sm} \\ \Phi_{stm} \end{Bmatrix} = \begin{Bmatrix} 0 \\ Q_m \\ Q_m \\ 0 \end{Bmatrix} \tag{27}$$

where

$$\begin{aligned}
&a_{11} = -A_{11}\lambda^2, && a_{13} = B_{11}^s\lambda^3, && a_{14} = L\lambda,\\
&a_{22} = -D_{11}\lambda^4, && a_{23} = -D_{11}^s\lambda^4, && a_{24} = -L^a\lambda^2,\\
&a_{33} = -\lambda^2(H_{11}^s\lambda^2 + A_{55}^s), && a_{34} = -\lambda^2(A_{55}^s + R), && a_{44} = -(A_{55}^s\lambda^2 + R^a)
\end{aligned} \tag{28a}$$

$$\begin{aligned}
&m_{11} = -I_0, && m_{12} = \lambda I_1, && m_{13} = \lambda J_1,\\
&m_{22} = -(I_0 + I_2\lambda^2), && m_{23} = -(I_0 + J_2\lambda^2), && m_{33} = -(I_0 + K_2\lambda^2),\\
&m_{24} = m_{34} = -J_0 && m_{44} = -K_0 &&
\end{aligned} \tag{28b}$$

4. Results and discussion

In this paper, bending and free vibration analysis of simply supported FG beams by the current shear and normal deformation beam model is considered for study.

For all numerical results reported here, the following values of mechanical properties were used:

Ceramic (P_C: Alumina, Al_2O_3): E_c = 380 GPa; ν = 0.3; ρ_c = 3960 kg/m^3.
Metal (P_M: Aluminium, Al): E_m = 70 GPa; ν = 0.3; ρ_m = 2707 kg/m^3.
Non-dimensional parameters of FG beam may be expressed as

$$\bar{w} = 100\frac{E_m h^3}{q_0 L^4} w\left(\frac{L}{2}\right),\quad \bar{u} = 100\frac{E_m h^3}{q_0 L^4} u\left(0, -\frac{h}{2} - C\right),\quad \bar{\sigma}_x = \frac{h}{q_0 L}\sigma_x\left(\frac{L}{2}, \frac{h}{2} - C\right),$$

$$\bar{\tau}_{xz} = \frac{h}{q_0 L}\tau_{xz}(0, -C),\quad \bar{\omega} = \frac{\omega L^2}{h}\sqrt{\frac{\rho_m}{E_m}}$$

4.1 Bending analysis

Table 1 contains nondimensional displacements and stresses of FG beams subjected to uniform load q_0. Different values of the volume fraction exponent k and span-to-depth ratio L / h are considered. The computed results are compared with those of Li *et al*. (2010) and Ould Larbi *et al*. (2013). It should be noted that the current solution are derived based on a trigonometric distribution of both axial and transverse displacements within the thickness, while the higher shear deformation theories (HSDT) of Ould Larbi *et al*. (2013) and Li *et al*. (2010) are obtained based on a hyperbolic and a cubic distribution of axial displacements and a constant deflection within the thickness (i.e., $\varepsilon_z = 0$). Since the effect of normal strain is neglected ($\varepsilon_z = 0$) in beam theories (Ould Larbi *et al*. 2013, Li *et al*. 2010), they lead to identical solutions, and their results are also in good agreement with the present theory which considers the thickness stretching effect (i.e., $\varepsilon_z \neq 0$). The difference between the results of the present beam theory and the other theories (Ould Larbi *et al*. 2013, Li *et al*. 2010) is due to the normal strain effect which is omitted in these latter ones.

In Figs. 2-4 we present the evolution of the axial displacement $\bar{u}$, axial stresses $\bar{\sigma}_x$ and

Table 1 Nondimensional deflections and stresses of FG beams under uniform load

k	Method	$L/h=5$				$L/h=20$			
		$\overline{w}$	$\overline{u}$	$\overline{\sigma}_x$	$\overline{\tau}_{xz}$	$\overline{w}$	$\overline{u}$	$\overline{\sigma}_x$	$\overline{\tau}_{xz}$
0	Li *et al*. (2010)	3.1657	0.9402	3.8020	0.7500	2.8962	0.2306	15.0130	0.7500
	Ould Larbi *et al*. (2013)	3.1651	0.9406	3.8043	0.7489	2.8962	0.2305	15.0136	0.7625
	Present $\varepsilon_z \neq 0$	3.1357	0.9261	3.8614	0.7438	2.8906	0.2300	15.2708	0.7656
0.5	Li *et al*. (2010)	4.8292	1.6603	4.9925	0.7676	4.4645	0.4087	19.7005	0.7676
	Ould Larbi *et al*. (2013)	4.8282	1.6608	4.9956	0.7660	4.4644	0.4087	19.7013	0.7795
	Present $\varepsilon_z \neq 0$	4.7584	1.6124	5.0789	0.7604	4.4292	0.4010	20.0787	0.7824
1	Li *et al*. (2010)	6.2599	2.3045	5.8837	0.7500	5.8049	0.5686	23.2054	0.7500
	Ould Larbi *et al*. (2013)	6.2590	2.3052	5.8875	0.7489	5.8049	0.5685	23.2063	0.7625
	Present $\varepsilon_z \neq 0$	6.1271	2.2162	5.9841	0.7438	5.7131	0.5515	23.6405	0.7656
2	Li *et al*. (2010)	8.0602	3.1134	6.8812	0.6787	7.4415	0.7691	27.0989	0.6787
	Ould Larbi *et al*. (2013)	8.0683	3.1146	6.8878	0.6870	7.4421	0.7691	27.1005	0.7005
	Present $\varepsilon_z \neq 0$	7.8501	2.9703	6.9957	0.6838	7.2688	0.7390	27.5763	0.7044
5	Li *et al*. (2010)	9.7802	3.7089	8.1030	0.5790	8.8151	0.9133	31.8112	0.5790
	Ould Larbi *et al*. (2013)	9.8345	3.7128	8.1187	0.6084	8.8186	0.9134	31.8151	0.6218
	Present $\varepsilon_z \neq 0$	9.6028	3.5488	8.2440	0.6079	8.6396	0.8798	32.3457	0.6271
10	Li *et al*. (2010)	10.8979	3.8860	9.7063	0.6436	9.6879	0.9536	38.1372	0.6436
	Ould Larbi *et al*. (2013)	10.9413	3.8898	9.7203	0.6640	9.6907	0.9537	38.1408	0.6788
	Present $\varepsilon_z \neq 0$	10.7561	3.7501	9.8597	0.6625	9.5715	0.9278	38.7327	0.6835

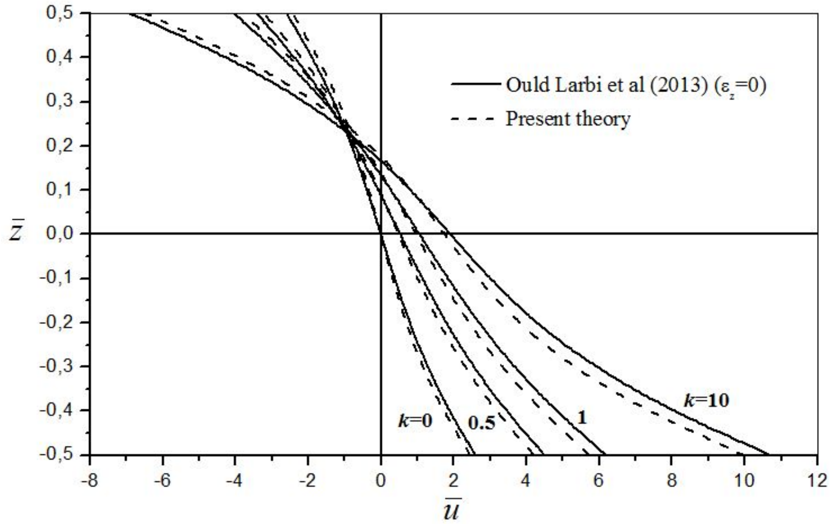

Fig. 2 The variation of the axial displacement $\overline{u}$ through-the-thickness of a FG beam ($L = 2h$)

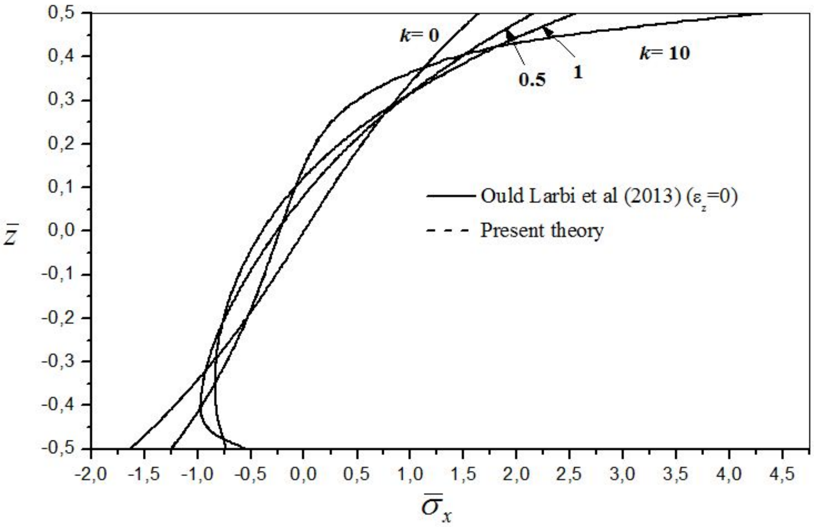

Fig. 3 The variation of the axial stress $\bar{\sigma}_x$ through-the-thickness of a FG beam (L = 2h)

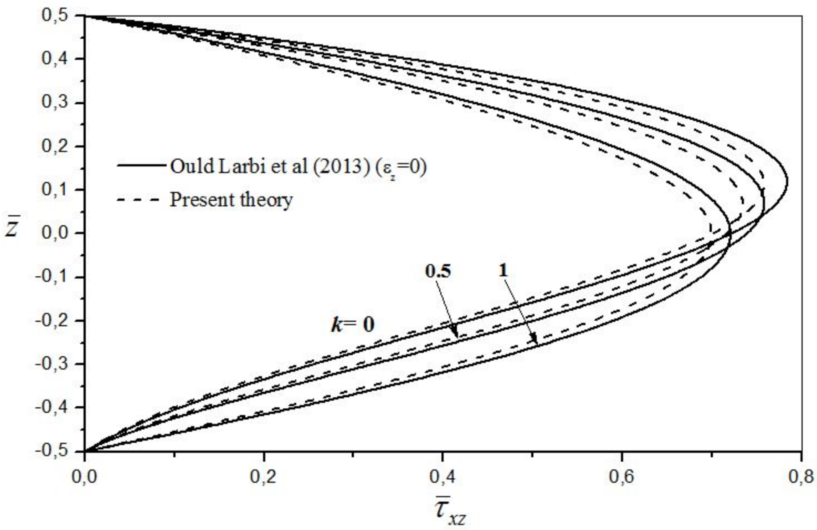

Fig. 4 The variation of the transverse shear stress $\bar{\tau}_{xz}$ through-the-thickness of a FG beam (L = 2h)

transverse shear stress $\bar{\tau}_{xz}$ within the depth of the FG beam. The case of uniform load is considered. A comparison with beam theory developed by Ould Larbi *et al*. (2013) is also shown in these figures using different values of the volume fraction exponent *k*. A good agreement between the present theory and the theory developed by Ould Larbi *et al*. (2013) is observed. Again, the difference between the results is due to the normal strain effect which is omitted in the beam theory developed by Ould Larbi *et al*. (2013). In general, a very good agreement between the solutions is observed, except the transverse shear stress $\bar{\tau}_{xz}$ where a small difference between the results is found (see Fig. 4). This is due to the effect of the normal strain which is important in

Table 2 Variation of fundamental frequency $\overline{\varpi}$ with the power-law index for FG beam

L/h	Theory	k					
		0	0.5	1	2	5	10
5	Ould Larbi *et al.* (2013)	5.1529	4.4108	3.9905	3.6263	3.4001	3.2812
	Present $\varepsilon_z \neq 0$	5.1665	4.4347	4.0271	3.6723	3.4374	3.3048
20	Ould Larbi *et al.* (2013)	5.4603	4.6511	4.2051	3.8361	3.6484	3.5389
	Present $\varepsilon_z \neq 0$	5.4650	4.6690	4.2383	3.8813	3.6858	3.5606

Table 3 First three nondimensional frequencies $\overline{\varpi}$ of FG beams

L/h	Mode	Theory	k					
			0	0.5	1	2	5	10
5	1	CBT	5.3953	4.5931	4.1484	3.7793	3.5949	3.4921
		Ould Larbi *et al.* (2013)	5.1529	4.4108	3.9905	3.6263	3.4001	3.2812
		Present $\varepsilon_z \neq 0$	5.1665	4.4347	4.0271	3.6723	3.4374	3.3048
	2	CBT	20.6187	17.5415	15.7982	14.3260	13.5876	13.2376
		Ould Larbi *et al.* (2013)	17.8844	15.4613	14.0121	12.6404	11.5349	11.0216
		Present $\varepsilon_z \neq 0$	17.9979	15.5965	14.1780	12.8232	11.6761	11.1231
	3	CBT	43.3483	36.8308	33.0278	29.7458	28.0850	27.4752
		Ould Larbi *et al.* (2013)	34.2248	29.8496	27.1085	24.3196	21.6987	20.5555
		Present $\varepsilon_z \neq 0$	34.5558	30.2017	27.4946	24.7075	21.9842	20.7758
20	1	CBT	5.4777	4.6641	4.2163	3.8472	3.6628	3.5547
		Ould Larbi *et al.* (2013)	5.4603	4.6511	4.2051	3.8361	3.6484	3.5389
		Present $\varepsilon_z \neq 0$	5.4650	4.6690	4.2383	3.8813	3.6858	3.5606
	2	CBT	21.8438	18.5987	16.8100	15.3334	14.5959	14.1676
		Ould Larbi *et al.* (2013)	21.5734	18.3964	16.6345	15.1617	14.3732	13.9257
		Present $\varepsilon_z \neq 0$	21.6003	18.4736	16.7706	15.3434	14.5227	14.0144
	3	CBT	48.8999	41.6328	37.6173	34.2954	32.6357	31.6883
		Ould Larbi *et al.* (2013)	47.5940	40.6534	36.7686	33.4681	31.5719	30.5342
		Present $\varepsilon_z \neq 0$	47.6822	40.8457	37.0853	33.8792	31.9080	30.7403

assessing the stress components in the transverse direction.

In Fig. 5 we present the evolution of the transverse normal stress $\overline{\sigma}_z$ across the depth of the FG beam for various values of the volume fraction exponent *k*. As can be shown in Fig. 5, the normal stress $\overline{\sigma}_z$ cannot be omitted for the present problem.

4.2 Free vibration

To check the accuracy of the method used in this investigation, the nondimensional fundamental frequencies $\overline{\varpi}$ computed by the present method are compared with those given by

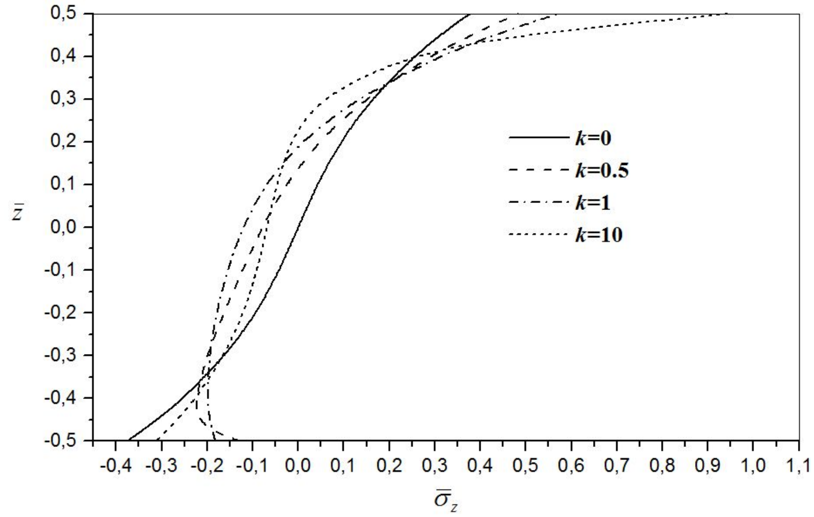

Fig. 5 The variation of the transverse normal stress $\bar{\sigma}_z$ through-the-thickness of a FG beam ($L = 2h$)

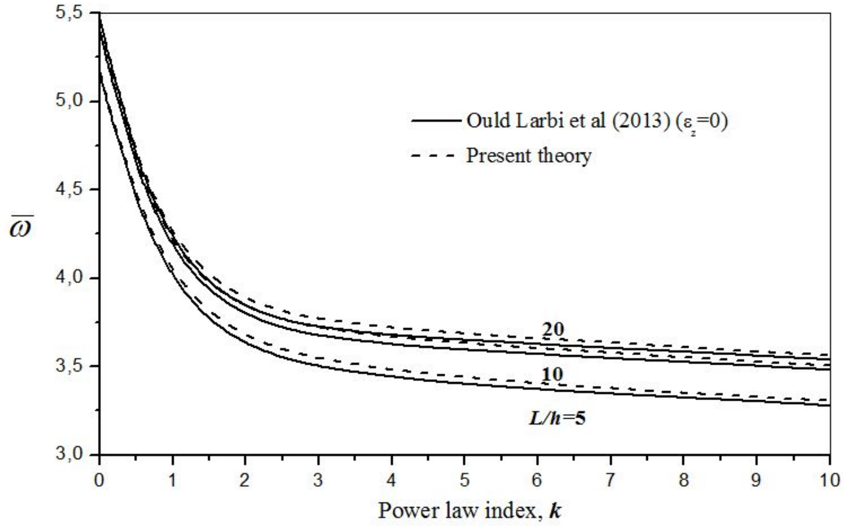

Fig. 6 Variation of the nondimensional fundamental frequency $\bar{\omega}$ of FG beam with power law index k and span-to-depth ratio L / h

Ould Larbi *et al*. (2013) of FG beams for different values of power law index k and span-to-depth ratio L / h and the results are presented in Table 2. In general, a good correlation between the results is observed. However, the HSDT solutions (Ould Larbi *et al*. 2013) slightly underestimate frequency due to omitting the thickness stretching effect.

The first three nondimensional frequencies $\bar{\omega}$ of FG beams predicted by the present method, the beam theory developed by Ould Larbi *et al*. (2013) and the classical beam theory (CBT) are presented in Table 3 for different values of power law index k and span-to-depth ratio L / h. It can be observed that the two shear deformation beam theories predict almost identical frequencies, whereas the CBT overestimates them. Again, it can be observed that the HSDT solutions (Ould Larbi *et al*. 2013) which ignores the thickness stretching effect, slightly underestimate frequency.

Fig. 6 shows the non-dimensional fundamental natural frequency $\bar{\omega}$ versus the volume fraction exponent k for different values of span-to-depth ratio L/h using both the present theory and Ould Larbi *et al*. (2013). A good correlation between the results is showed from Fig. 6. The small difference is due to the thickness stretching effect. The full ceramic beams (k = 0) lead to a highest frequency. However, the lowest frequency values are predicted for full metal beams ($k \rightarrow \infty$). This is due to the fact that an increase in the value of the volume fraction exponent results in a decrease in the value of the modulus of elasticity.

5. Conclusions

A novel higher-order shear and normal deformation beam theory based on the physical concept of neutral axis is proposed for bending and dynamic responses of FG beams. This method considers both the shear deformation and thickness stretching effects by a trigonometric distribution of all displacements through the thickness and without introducing a shear correction parameter. Based on the present method and the physical concept of neutral axis, the equations of motion are obtained from Hamilton's principle. Results show that the beam becomes stiffer when the thickness stretching effect is incorporated, and consequently, leads to a reduction of deflection and an increase of frequency. The formulation lends itself particularly well in analysing nanostructures (Heireche *et al*. 2008a, b, c, Tounsi *et al*. 2008, Benzair *et al*. 2008, Amara *et al*. 2010, Tounsi *et al*. 2010, Tounsi *et al*. 2013b, c, Berrabah *et al*. 2013, Gafour *et al*. 2013, Benguediab *et al*. 2014, Semmah *et al*. 2014) which will be considered in the near future.

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