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# Analytical solutions for bending of transversely or axially FG nonlocal beams

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**Abstract.** This paper presents the analytical solutions for the size-dependent static analysis of the functionally graded (FG) beams with various boundary conditions based on the nonlocal continuum model. The nonlocal behavior is described by the differential constitutive model of Eringen, which enables to this model to become effective in the analysis and design of nanostructures. The elastic modulus of beam is assumed to vary through the thickness or longitudinal directions according to the power law. The governing equations are derived by using the nonlocal continuum theory incorporated with Euler-Bernoulli beam theory. The explicit solutions are derived for the static behavior of the transversely or axially FG beams with various boundary conditions. The verification of the model is obtained by comparing the current results with previously published works and a good agreement is observed. Numerical results are presented to show the significance of the nonlocal effect, the material distribution profile, the boundary conditions, and the length of beams on the bending behavior of nonlocal FG beams.

**Keywords:** functionally graded beam; nonlocal theory; analytical solution; power series method

## 1. Introduction

In the local (classical) continuum theories based on the hyperelastic constitutive relations, the stress state at a reference point is only a function of strain state at that point. The nonlocal continuum theory initially developed by Eringen (1972a, b) indicates that the stress state at a reference point is a function of every strain state in the continuum body. Thus, the nonlocal theory can take the forces between atoms and small length scale into consideration.

Up to now, considerable research efforts have been made to deal with the static and dynamic behavior of beams considering the nonlocal continuum theory. Peddieson *et al.* (2003) utilized the nonlocal continuum theory to develop the nonlocal Euler-Bernoulli beam model. Reddy (2007) formulated various beam theories, including Euler-Bernoulli, Timoshenko, Reddy and Levinson beam theories based on the nonlocal theory. Reddy (2010) also reformulated the classical and first order shear deformation theories for beam and plate concerning with the nonlocal theory and the

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von Kármán nonlinear strains. Aydogdu (2009) presented a generalized nonlocal beam theory for different beam theories. The solutions for the static, buckling and vibration behaviors were obtained by Navier solutions for the simply supported beams. Phadikar and Pradhan (2010) used the finite element method with the Hermite cubic element to investigate the static, buckling and vibration of the nonlocal classical beam and plate. From their work, the solutions for four classical boundary conditions were accomplished and indicated that the nonlocal effect depends on boundary conditions. Alshorbagy *et al*. (2013) also carried out the finite element method for the nonlocal Euler-Bernoulli beam. Pradhan and Murmu (2010) investigated the flapwise bending vibration of the nonlocal Euler-Bernoulli cantilever beam. The governing equation was derived and solved by using differential quadrature method. Ghannadpour *et al*. (2013) performed a study on the bending, buckling and vibrational behaviors of the nonlocal Euler-Bernoulli beam by using the Ritz method and Wang *et al*. (2007) obtained the analytical solutions for the free vibration of the nonlocal Timoshenko beam. In addition, Zhang *et al*. (2010) employed the hybrid nonlocal theory introduced by Challamel and Wang (2008) to examine the bending, buckling and vibrational behaviors of beam. In their work, the obtained solutions from the hybrid nonlocal theory were remarkably different to those obtained by the nonlocal theory.

Functionally grade materials (FGMs) are a kind of composite materials in which the material properties are designed to vary continuously and gradually from one surface to the other. In contrast to the laminated composites, the FGMs eliminate the undesirable stress discontinuity existing between two surfaces in laminated composites. These FGMs are usually composed of the mixture of ceramics and metals due to the better thermal resistance of ceramic phase and the stronger mechanical performance of metal phase. Owing to this advantage, the FGMs are extensively used in many scientific and engineering fields, such as electronics, optics, chemistry, biomedical engineering, nuclear engineering.

During the last decade, many researchers have been attracted to understand the behavior of FG beams whose material properties vary through thickness direction. Sankar (2001) obtained the elasticity solution for the simply supported FG beam subjected to the sinusoidal transverse load. The Young's modulus varied exponentially through the thickness direction. Zhu and Sankar (2004) also dealt with the similar problem in which the Young's modulus was given by a polynomial function in the thickness. The differential governing equation was reduced by using the Fourier series method and solved by using the Galerkin method. Chakraborty *et al*. (2003) introduced the new finite beam element to study the thermoelastic behavior of the FG beam based on the Timoshenko beam theory, where material properties varied according to exponential and power-law functions through thickness direction. Zhong and Yu (2007) presented the analytical solutions of FG cantilever beams subjected to different loads with exponential variation of material through the thickness direction in term of the Airy stress function. Nie *et al*. (2013) continued to develop the solution for the arbitrary graded material and boundary conditions based on the Airy stress function. Li (2008) and Li *et al*. (2010) suggested the unified approach for analyzing the static and dynamic behavior of the FG beam by using the auxiliary function derived from governing equations. All parameters of the response of beam such as axial displacement, deflection, bending moment, shear force and internal stress could be described in term of the auxiliary function.

Until now, several researchers have performed the free vibration and stability analyses of the axially FG beams. Huang and Li (2010) studied the free vibration of the axially FG beam with non-uniform cross-section by transforming the differential governing equation into the Fredholm integral equation. Hein and Feklistova (2011) dealt with the free vibration of the non-prismatic and axially FG beam by using Euler-Bernoulli beam theory and Haar matrices. In their wok, the governing equation was transformed with the aid of a set of simplest wavelets. Shahba and Rajasekaran (2012) and Rajasekaran (2013) suggested the differential transform element method and the differential quadrature element method of lowest-order to analyze the free vibration and stability of the tapered axially FG beam based on the Euler-Bernoulli and Timoshenko beam theories, respectively. Shahba *et al*. (2011) exploited the super-convergent shape functions given in Reddy (2002) to examine the free vibration and stability behavior of the axially Timoshenko FG beam with classical and elastic supports. Based on the basic displacement functions obtained by using unit-dummy-load method, Shahba *et al*. (2011) introduced the shape functions for the finite element method to analyze the static, stability and free vibration responses of the axially Euler-Bernoulli FG beam.

With the speedy progress of technology, some researchers paid attention to study on the small-scale FG beams including the nonlocal theory. Eltaher *et al*. (2012, 2103a, b) exploited the general finite element method to figure out the static, stability and free vibration solutions of the nonlocal Euler-Bernoulli FG beam. The material property was assumed to vary through the thickness direction according to power-law form. Şimşek and Yurtcu (2013) examined the static and buckling responses of the nonlocal FG beam. In their study, the Navier solution was obtained for both Euler-Bernoulli and Timoshenko simply supported beams. Uymaz (2013) accomplished Navier solution for the forced vibration of the simply supported nonlocal FG beam. Various shear deformable beam theories were considered and the material properties varied through the thickness direction with the power-law form. Furthermore, Şimşek (2102) investigated the free longitudinal vibration of the axially nonlocal FG nanorods by using the Galerkin method. In his work, Young's modulus varied through the axial direction according to power-law form.

From the previously cited references, one can note that despite extensive researches for the analysis of FG beams as well as nonlocal FG beams, to the best of authors' knowledge, due to the appearance of nonlocal parameter in governing equations, there was no study reported on the exact explicit solutions for the static analysis of FG beams with various boundary conditions considering the nonlocal effect in the literature. To overcome, some researchers exploited Navier-type solution for only simply supported boundary condition. Therefore, that gives us a strong encouragement to obtain the exact solutions for the static behavior of the nonlocal FG beams with various boundary conditions.

The main objectives of this work are to present the explicit solutions for the static analysis of the transversely or axially FG beams with various boundary conditions considering the nonlocal effect and to show the significant influences of the nonlocal parameter, the material distribution profile, the boundary conditions, and the length of beams on the flexural behavior of nonlocal FG beams. The outline of this paper is as follows. The nonlocal elasticity theory is presented in Section 2. The mathematical model and the governing equations are explained in Section 3. After that, the explicit solutions for the bending of the transversely or axially FG beams are derived for various boundary conditions in Section 4. Section 5 is devoted to numerical results and parametric studies. The accuracy and reliability of this study is presented and verified by comparing the results with published works. Finally, concluding remarks are drawn.

## 2. Nonlocal elasticity theory

It is known that in contrast to the constitutive equation in classical elasticity, the nonlocal elasticity theory by Eringen (1983) states that the stress at a point **x** in an elastic continuum body depends not only on the strain at point **x** but also on those at all other points of the body. Therefore, the nonlocal stress tensor **σ** at point **x** is expressed as Reddy (2007, 2010), Şimşek and Yurtcu (2013).

$$\boldsymbol{\sigma}_{\mathbf{ij}} = \int_V \alpha\left(|\mathbf{x}' - \mathbf{x}|, \tau\right) \mathbf{t}_{\mathbf{ij}}(\mathbf{x}') d\mathbf{x}' \tag{1}$$

where $\mathbf{t}_{\mathbf{ij}}(\mathbf{x})$ are the components of the classical macroscopic stress tensor at point **x**; the kernel function $\alpha(|\mathbf{x}' - \mathbf{x}|, \tau)$ is the nonlocal modulus or the attenuation function specifying the nonlocal effect at a reference point **x** produced by the local strain at the source $\mathbf{x}'$, $|\mathbf{x}' - \mathbf{x}|$ being the distance (in Euclidean distance); $\tau$ is a material constant that depends on the internal and external characteristic lengths (such as the lattice spacing and wavelength, respectively). The integration is taken for total volume $V$ of elastic body. The macroscopic stress **t** at a point **x** in a Hookean solid is related to the strain **ε** at the point by the generalized Hooke's law as follows

$$\mathbf{t}(\mathbf{x}) = \mathbf{C}(\mathbf{x}) : \boldsymbol{\varepsilon}(\mathbf{x}) \quad \text{or} \quad \mathbf{t}_{\mathbf{ij}} = \mathbf{C}_{\mathbf{ijkl}} \boldsymbol{\varepsilon}_{\mathbf{kl}} \tag{2}$$

where $\mathbf{C}(\mathbf{x})$ is the fourth-order elasticity tensor and the colon denotes the 'double-dot product'. The integral constitutive relation in Eq. (1) makes the elasticity problems difficult to solve, in addition to possible lack of determinism. Therefore, Eringen (1983) discussed in detail properties of the nonlocal kernel $\alpha(|\mathbf{x}' - \mathbf{x}|)$ and proved that when a kernel takes a Green's function of the linear differential operator

$$L_a \alpha\left(|\mathbf{x}' - \mathbf{x}|\right) = \delta\left(|\mathbf{x}' - \mathbf{x}|\right) \tag{3}$$

The nonlocal constitutive relation in Eq. (1) is reduced to the following differential equation.

$$L_a \boldsymbol{\sigma}_{\mathbf{ij}} = \mathbf{t}_{\mathbf{ij}} \tag{4}$$

Thus, Eringen (1983) proposed a nonlocal model with the linear differential operator $L_a$ by matching the dispersion curves with lattice models as follows

$$L_a = 1 - (e_o a)^2 \nabla^2 \tag{5}$$

in where $\nabla^2$ is the Laplace operator; $e_o$ is a constant to adjust the model to match the reliable results by experiments or other models; $a$ is an internal characteristics length (e.g., granular distance, lattice parameter); $e_o a$ denotes the nonlocal parameter which reveals the small scale effect on the responses of structures of nanosize. Eringen (1983) proposed the parameter $e_o = \sqrt{(\pi^2 - 4)} / 2\pi \cong 0.39$ and $e_o = 1/\sqrt{12} \cong 0.288$ was given by Wang and Hu (2005). For a beam type structure, the nonlocal behavior can be neglected in the thickness direction. Thus, the constitutive relation for the nonlocal elasticity can be represented by following form

$$\sigma_{xx} - \mu \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E \varepsilon_{xx}, \quad \mu = (e_o a)^2 \tag{6}$$

where $\sigma_{xx}$ and $\varepsilon_{xx}$ are the axial normal stress and the axial strain, respectively, and $E$ is the elasticity modulus. When the nonlocal parameter is taken as $e_o a = 0$, the constitutive relation of the local theory is obtained. Integrating Eq. (6) over the area of cross-section of beam, we can obtain the axial force-strain and moment-curvature relations as follows

$$N_x - \mu \frac{\partial^2 N_x}{\partial x^2} = \int_A E\varepsilon_{xx} dA \tag{7}$$

$$M_x - \mu \frac{\partial^2 M_x}{\partial x^2} = \int_A zE\varepsilon_{xx} dA \tag{8}$$

where $N_x$ and $M_x$ are the axial force and the bending moment, respectively, and $A$ is the area of the cross-section.

## 3. Functionally graded beams

### *3.1 Material properties*

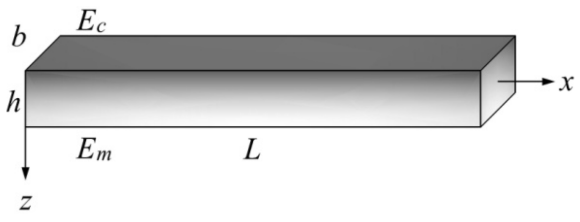
Figs. 1(a) and (b) show the transversely functionally graded (T-FG) beam and the axially functionally graded (A-FG) beam, respectively. It is assumed that the Poisson's ratio of beam through the thickness direction or the longitudinal direction is constant since the effect of Poisson's ratio on the deformation in static analysis is much less than that of Young's modulus from the study by Li (2008). According to the Voigt model introduced by Nakamura *et al*. (2000), the elasticity modulus of the FG material can be expressed as follows

$$E_f = E_m V_m + E_c V_c \tag{9}$$

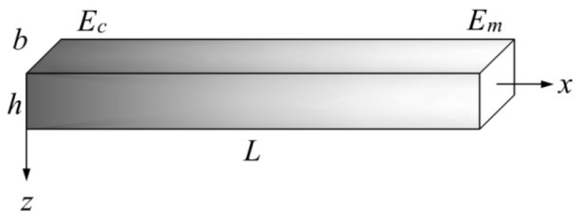
where $E_m$ and $E_c$ are the elasticity moduli of the metal and the ceramic, respectively; $V_m$ and $V_c$ are the volume fractions of the metal and the ceramic, respectively.

The volume fractions of the T-FG and A-FG materials are defined by the power-law form. The metal and ceramic volume fractions are assumed by

$$V_m = \left(\frac{z}{h} + \frac{1}{2}\right)^k, \; V_c = 1 - \left(\frac{z}{h} + \frac{1}{2}\right)^k \quad \frac{-h}{2} \le z \le \frac{h}{2} \quad \text{for T - FG material} \tag{10}$$

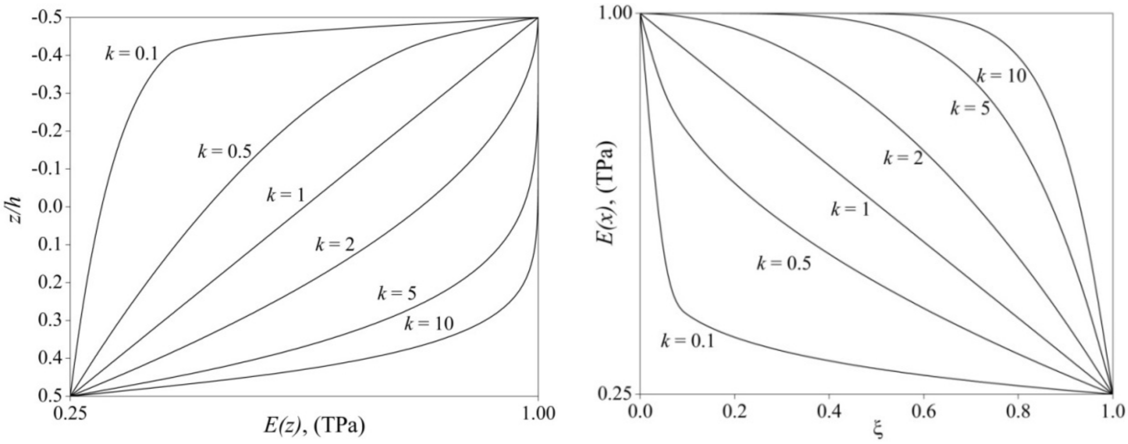



(a) T-FG beam




(b) A-FG beam

Fig. 1 Geometry and coordinate system of FG beams




(a) T-FG beam (b) A-FG beam

Fig. 2 Variation of Young's moduli of FG beams with respect to *k*

$$V_m = \left(\frac{x}{L}\right)^k \ , \ V_c = 1 - \left(\frac{x}{L}\right)^k \quad 0 \le x \le L \quad \text{for A - FG material} \tag{11}$$

where $k$ is a material non-homogeneity parameter which indicates the material variation profile along the depth or the length of the beam, and $k = 0$ corresponds to an isotropic homogeneous metal and $k = \infty$ corresponds to an isotropic homogeneous ceramic.

Substituting Eqs. (10) and (11) into Eq. (9), the elasticity moduli can be obtained for T-FG and A-FG beams, respectively, as follows

$$E(z) = (E_m - E_c)\left(\frac{z}{h} + \frac{1}{2}\right)^k + E_c \tag{12}$$

and

$$E(x) = (E_m - E_c)\left(\frac{x}{L}\right)^k + E_c \tag{13}$$

Figs. 2(a) and (b) show the variation of the elasticity moduli for T-FG and A-FG beams, respectively, with different values of $k$, where $\xi$ denotes $x/L$ in Fig. 2(b).

### 3.2 Governing equations

The Euler-Bernoulli beam theory is based on that plane section perpendicular to the axis of the beam before deformation remains plane, rigid, and perpendicular to the deformed axis after deformation. Owing to the distribution of material, the neutral axis would be disagreed with mid-plane for transversely FG beams as mentioned by Li *et al*. (2005), Kang and Li (2009) and Taeprasartsi (2012). However, let assume that the neutral axis is still kept at mid-plane which was employed by many researchers working on transversely FG beams. The kinematic relations according to the Euler-Bernoulli beam theory can be given by Reddy (1999).

$$u(x,z)=u_o(x)-z\frac{dw_o(x)}{dx} \tag{14}$$

$$w(x,z)=w_o(x) \tag{15}$$

where $u$ and $w$ are the axial and transverse displacements, respectively, at any general point in the deformed state of beam; $u_o$ and $w_o$ are their counterparts calculated at the mid-plane. The definition of the axial strain is written by

$$\varepsilon_{xx}=\frac{du}{dx}=\frac{du_o}{dx}-z\frac{d^2w_o}{dx^2}=\varepsilon_o+z\kappa \tag{16}$$

where $\varepsilon_o$ is the extensional strain and $\kappa$ is the bending strain.

Substituting Eq. (16) into Eqs. (7) and (8), the force-strain and moment-curvature relations of the nonlocal Euler-Bernoulli beam theory can be obtained as follows

$$N_x-\mu\frac{d^2N_x}{dx^2}=\int_A E\varepsilon_{xx}dA=A_{11}\frac{du_o}{dx}-B_{11}\frac{d^2w_o}{dx^2} \tag{17}$$

$$M_x-\mu\frac{d^2M_x}{dx^2}=\int_A zE\varepsilon_{xx}dA=B_{11}\frac{du_o}{dx}-D_{11}\frac{d^2w_o}{dx^2} \tag{18}$$

where $A_{11}$, $B_{11}$, and $D_{11}$ are the extensional, bending-extension coupling, and bending stiffnesses, respectively, and defied by

$$(A_{11},B_{11},D_{11})=\int_A E(1,z,z^2)dA \tag{19}$$

In this study, the FG beams are subjected to the uniformly distributed transverse load $q$. Thus, from the principle of virtual work as well as Reddy (2007), we obtain following equations

$$\frac{dN_x}{dx}=0 \tag{20}$$

$$\frac{d^2M_x}{dx^2}=-q \tag{21}$$

Substituting Eqs. (20) and (21) into Eqs. (17) and (18), the governing equation of the nonlocal FG beams can be presented as follows

$$A_{11}\frac{du_o}{dx}-B_{11}\frac{d^2w_o}{dx^2}=c_1 \tag{22}$$

$$B_{11}\frac{du_o}{dx}-D_{11}\frac{d^2w_o}{dx^2}=-\frac{1}{2}qx^2+c_2x+c_3+\mu q \tag{23}$$

where $c_1$, $c_2$ and $c_3$ are the integration constants.

It is noted that in governing equations Eqs. (22) and (23), the nonlocal effect is expressed in relation to distributed load. On the contrary, in the governing equations derived by Reddy (2007), Aydogdu (2009) and Şimşek and Yurtcu (2013) for bending problem, the nonlocal term is state in relation to second order derivative of distributed load. Thus, if beam is subjected to uniform loading, the nonlocal term would be eliminated from the governing equations. Due to this problem, almost researchers exploit the Fourier transformation to convert uniform load into sinusoidal load. In the present study, owing to the advantage of governing equations, the close-form solution can be directly obtained from governing equations which are presented in following section.

## 4. Analytical solution for bending of FG beams

In this section, the governing equations are analytically solved for the static bending of T-FG and A-FG beams with various boundary conditions.

### 4.1 Solutions for T-FG beams

The analytical solutions for the bending of T-FG beams are derived. In this case, the extensional, bending-extension coupling, and bending stiffnesses can be derived as follows

$$A_{11}^T = \int_A E(z)dA = A\left\{(E_m - E_c)\frac{1}{k+1} + E_c\right\} \tag{24}$$

$$B_{11}^T = \int_A zE(z)dA = Ah(E_c - E_m)\frac{k}{2(k+1)(k+2)} \tag{25}$$

$$D_{11}^T = \int_A z^2E(z)dA = I\left\{(E_c - E_m)\frac{3(k^2+k+2)}{4(k+1)(k+2)(k+3)} + E_c\right\} \tag{26}$$

where $A$, $I$ is the cross section area and the moment of inertia of beams.

By solving Eqs. (22) and (23), the following equations can be obtained.

$$\frac{du_o}{dx} = \frac{B_{11}^T}{A_{11}^T D_{11}^T - B_{11}^{T\,2}}\left\{\frac{1}{2}\left(qx^2 - 2c_2x - 2c_3 - 2\mu q\right) + c_1\frac{D_{11}^T}{B_{11}^T}\right\} \tag{27}$$

$$\frac{d^2w_o}{dx^2} = \frac{A_{11}^T}{A_{11}^T D_{11}^T - B_{11}^{T\,2}}\left\{\frac{1}{2}\left(qx^2 - 2c_2x - 2c_3 - 2\mu q\right) + c_1\frac{B_{11}^T}{A_{11}^T}\right\} \tag{28}$$

By introducing the dimensionless parameter $\xi = x/L$ and integrating Eqs. (27) and (28), the axial and transverse displacements can be obtained as follows

$$u_o = \frac{\Omega_1 L}{2}\left\{\frac{1}{3}qL^2\xi^3 - c_2L\xi^2 - 2\left(c_3 + \mu q - \frac{D_{11}^T}{B_{11}^T}c_1\right)\xi\right\} + c_4 \tag{29}$$

$$w_o = \frac{\Omega_2 L^2}{2}\left[\frac{1}{12}qL^2\xi^4 - \frac{1}{3}c_2 L\xi^3 - \left(c_3 + \mu q - \frac{B_{11}^T}{A_{11}^T}c_1\right)\xi^2\right] + c_5\xi + c_6 \tag{30}$$

where

$$\Omega_1 = \frac{B_{11}^T}{A_{11}^T D_{11}^T - {B_{11}^T}^2}, \quad \Omega_2 = \frac{A_{11}^T}{A_{11}^T D_{11}^T - {B_{11}^T}^2} \tag{31}$$

The six integration constants $c_1$~ $c_6$ in Eqs. (29) and (30) can be determined by using the boundary conditions, and the explicit expressions for the axial and transverse displacements of T-FG beams are presented in Appendix A for various boundary conditions. As can be seen in Appendix A, the axial displacement of HH beam is identical to that of CC beam.

### *4.2 Power series methodology for A-FG beams*

In this subsection, the power series methodology is presented in order to obtain the solutions of A-FG beams. In a similar way to the T-FG beam, the extensional, bending-extension coupling, and bending stiffnesses are obtain as follows

$$A_{11}^A = \int_A E(x)dA = A\left\{(E_m - E_c)\left(\frac{x}{L}\right)^k + E_c\right\} \tag{32}$$

$$B_{11}^A = \int_A zE(x)dA = 0 \tag{33}$$

$$D_{11}^A = \int_A z^2 E(x)dA = I\left\{(E_m - E_c)\left(\frac{x}{L}\right)^k + E_c\right\} \tag{34}$$

It is seen that the axial displacement and the transverse one can be evaluated separately since the bending-extension coupling stiffness is zero. For a beam under the uniformly distributed transverse load, the axial displacement at mid-plane $u_o$ can be omitted. For convenience, let us introduce the following denotation.

$$\beta = \frac{E_c - E_m}{E_c} \tag{35}$$

By substituting Eqs. (32)-(35) into Eq. (23), we can obtain the following equation

$$\frac{d^2 w_o}{d\xi^2} = \frac{L^2}{2E_c I}\left\{\frac{qL^2\xi^2 - 2c_2 L\xi - 2(c_3 + \mu q)}{(1-\beta\xi^k)}\right\} \tag{36}$$

In order to evaluate the transverse displacement $w_o$ in Eq. (36), the power series methodology is employed. The right terms of Eq. (36) can be transformed by using the Maclaurin expression as follows

$$\frac{1}{1-\beta\xi^k} = \sum_{n=0}^{\infty}\left(\beta\xi^k\right)^n \tag{37}$$

Substituting Eq. (37) into Eq. (36) and integrating Eq. (36), the transverse displacement of the A-FG beam can be expressed as follows

$$w_o = \Phi \sum_{n=0}^{\infty} \beta^n \left\{ \frac{\xi^{nk+4}}{(nk+3)(nk+4)} - \frac{2c_2}{qL} \frac{\xi^{nk+3}}{(nk+2)(nk+3)} - 2\left( \frac{c_3}{qL^2} + \frac{\mu}{L^2} \right) \frac{\xi^{nk+2}}{(nk+1)(nk+2)} \right\} + c_7 \xi + c_8 \quad (38)$$

where

$$\Phi = \frac{qL^4}{2E_c I} \quad (39)$$

In which $n$ is the series parameter and $c_7$ and $c_8$ are integral constants. Similar to the previous subsections, the four integration constants $c_2$, $c_3$, $c_7$, and $c_8$ in Eq. (38) can be determined by using the boundary conditions. The explicit expressions for the transverse displacements of A-FG beams with various boundary conditions are presented in Appendix B.

## 5. Results and discussion

In numerical examples, the flexural analysis of the T-FG and A-FG beams with various boundary conditions subjected to the uniformly distributed transverse unit load is performed. The results obtained from this study are compared with those from other available literatures. Especially, the influences of the nonlocal parameter $e_o a$ the non-homogeneity parameter $k$, and the boundary conditions (BCs), and the length of beams on the flexural behavior of the T-FG and A-FG beams are parametrically investigated.

### *5.1 Homogeneous beams*

For the purpose of verification, the transverse displacements of homogeneous beams considering the nonlocal effect are evaluated and compared with the existing data available in the literature. The geometry and material properties of beam as well as normalized displacement are used according to Aydogdu (2009). Moreover, from the derived solution in Appendix A, B, it is interesting to observe that the normalized displacement still base on the length $L$ of beam although the results are normalized by dimensionless quantities (for example, normalized deflection $w^* = 100 \times w_o \times \frac{EI}{qL^4}$ for simply supported homogeneous beams under uniform load). It can be explained from the assumption that the nonlocal effect is only valid in longitudinal direction for beam-type structures. As a result, the nonlocal effect is investigated for the variation of the length $L$ and the height is kept constant as $h = 1$ in this study. For the simply supported (HR or HH) beams, the normalized maximum transverse displacements are given in Table 1 for various values of the length of beams ($L$ = 10, 30, 100) and the nonlocal parameter. For comparison, the classical solutions which neglecting the nonlocal effect, the analytical solutions by Aydogdu (2009) using 100 terms in the series, and the Navier-type solutions by Şimşek and Yurtcu (2013) are presented. It can be found from Table 1 that results from this study are in excellent agreement with those from literature. It is noted that the present study obviously gives the exact values. Indeed, the well-known maximum displacement for the simply supported beam under the uniformly

Table 1 Normalized transverse displacements for homogeneous HR or HH beams ($h$ = 1)

| $L$ | $(e_oa)^2$ | Classical solutions | Aydogdu(2009) | Şimşek and Yurtcu (2013) | This study |
|---|---|---|---|---|---|
| 10 | 0 | 1.3021 | 1.3130 | 1.3020 | 1.3021 |
| | 1 | 1.3021 | 1.4487 | 1.4270 | 1.4271 |
| | 2 | 1.3021 | 1.5844 | 1.5520 | 1.5521 |
| | 3 | 1.3021 | 1.7201 | 1.6770 | 1.6771 |
| | 4 | 1.3021 | 1.8558 | 1.8020 | 1.8021 |
| 20 | 0 | 1.3021 | 1.3130 | 1.3020 | 1.3021 |
| | 1 | 1.3021 | 1.3469 | 1.3333 | 1.3333 |
| | 2 | 1.3021 | 1.3808 | 1.3645 | 1.3646 |
| | 3 | 1.3021 | 1.4148 | 1.3958 | 1.3958 |
| | 4 | 1.3021 | 1.4487 | 1.4270 | 1.4271 |
| 50 | 0 | 1.3021 | 1.3130 | 1.3020 | 1.3021 |
| | 1 | 1.3021 | 1.3184 | 1.3070 | 1.3071 |
| | 2 | 1.3021 | 1.3239 | 1.3120 | 1.3121 |
| | 3 | 1.3021 | 1.3313 | 1.3170 | 1.3171 |
| | 4 | 1.3021 | 1.3347 | 1.3220 | 1.3221 |

distributed load is $w_o = \dfrac{5qL^4}{384EI}$ from the classical beam theory. Thus, the normalized displacement for $L$ = 10 is $w_o^* = \dfrac{5\times 100}{384}$ = 1.3021 which is identical to the present result. It can also be found from Table 1 that the displacement increases as the nonlocal parameter increases, and its increase ratio of displacement is more significant in the small length.

### *5.2 T-FG beams*

To investigate the significance of using the T-FG material on the displacement of nonlocal beams, we consider a nanobeam that is composed of the metal and the ceramic where it changes across the beam's thickness according to a power-law given in Eq. (12). As a result, the bottom surface of the beam is pure metal and the top surface of the beam is pure ceramic. The elasticity moduli of the metal and the ceramic are $E_m$ = 0.25 TPa and $E_c$ = 1 TPa, respectively. Wang and Hu (2005) proposed a conservative estimate of the nonlocal parameter $0 \le e_oa \le 2$ nm for single-walled carbon nanotubes (SWCNTs). Therefore, in this study, the parameter is taken as $e_oa$ = 0, 0.5, 1.0, 1.5, and 2 nm to investigate the effects of $e_oa$, $k$, BCs, and the length of beams $L$ on the responses of T-FG nanobeams. For convenience, the displacements are normalized by following equation

$$w^* = 100\times w_o \times \frac{E_c I}{qL^4}, \quad u^* = 100\times u_o \times \frac{E_c I}{qL^3} \tag{40}$$

where the transverse and axial displacements are evaluated at mid-span of beams. As seen in Eqs. (A-5) and (A-22) in Appendix A, $u^*$ at mid-span are zero for HH and CC beams due to symmetric

Table 2 Normalized transverse displacements for HR T-FG beams ($h$ = 1)

| $L$ | $k$ | $e_o a$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 0 | | 0.5 | | 1.0 | | 1.5 | | 2.0 | |
| | | Şimşek (2013) | This study | Şimşek (2013) | This study | Şimşek (2013) | This study | Şimşek (2013) | This study | Şimşek (2013) | This study |
| 10 | 0 | 5.2083 | 5.2083 | 5.3333 | 5.3333 | 5.7083 | 5.7083 | 6.3333 | 6.3333 | 7.2083 | 7.2083 |
| | 0.3 | 3.1401 | 3.1387 | 3.2154 | 3.2140 | 3.4415 | 3.4400 | 3.8183 | 3.8167 | 4.3459 | 4.3440 |
| | 1 | 2.3674 | 2.3674 | 2.4242 | 2.4242 | 2.5946 | 2.5947 | 2.878 | 2.878 | 3.2765 | 3.2765 |
| | 3 | 1.8849 | 1.8850 | 1.9302 | 1.9302 | 2.0659 | 2.0659 | 2.2921 | 2.2921 | 2.6088 | 2.6088 |
| | 10 | 1.5450 | 1.5450 | 1.5821 | 1.5821 | 1.6933 | 1.6934 | 1.8787 | 1.8788 | 2.1383 | 2.1383 |
| 30 | 0 | 5.2083 | 5.2083 | 5.2222 | 5.2222 | 5.2638 | 5.2639 | 5.3333 | 5.3333 | 5.4305 | 5.4306 |
| | 0.3 | 3.1401 | 3.1387 | 3.1484 | 3.1471 | 3.1736 | 3.1722 | 3.2154 | 3.2140 | 3.2740 | 3.2726 |
| | 1 | 2.3674 | 2.3674 | 2.3737 | 2.3737 | 2.3926 | 2.3927 | 2.4242 | 2.4242 | 2.4684 | 2.2684 |
| | 3 | 1.8849 | 1.8850 | 1.8900 | 1.8900 | 1.9050 | 1.9051 | 1.9302 | 1.9302 | 1.9654 | 1.9654 |
| | 10 | 1.5450 | 1.5450 | 1.5491 | 1.5492 | 1.5615 | 1.5615 | 1.5821 | 1.5821 | 1.6109 | 1.6110 |
| 100 | 0 | 5.2083 | 5.2083 | 5.2095 | 5.2096 | 5.2133 | 5.2133 | 5.2195 | 5.2196 | 5.2283 | 5.2283 |
| | 0.3 | 3.1401 | 3.1387 | 3.1408 | 3.1395 | 3.1431 | 3.1417 | 3.1468 | 3.1455 | 3.1521 | 3.1508 |
| | 1 | 2.3674 | 2.3674 | 2.3679 | 2.3680 | 2.3696 | 2.3697 | 2.3725 | 2.3725 | 2.3765 | 2.3765 |
| | 3 | 1.8849 | 1.8850 | 1.8854 | 1.8854 | 1.8867 | 1.8868 | 1.8890 | 1.8890 | 1.8922 | 1.8922 |
| | 10 | 1.5450 | 1.5450 | 1.5454 | 1.5454 | 1.5465 | 1.5465 | 1.5483 | 1.5484 | 1.5509 | 1.5510 |

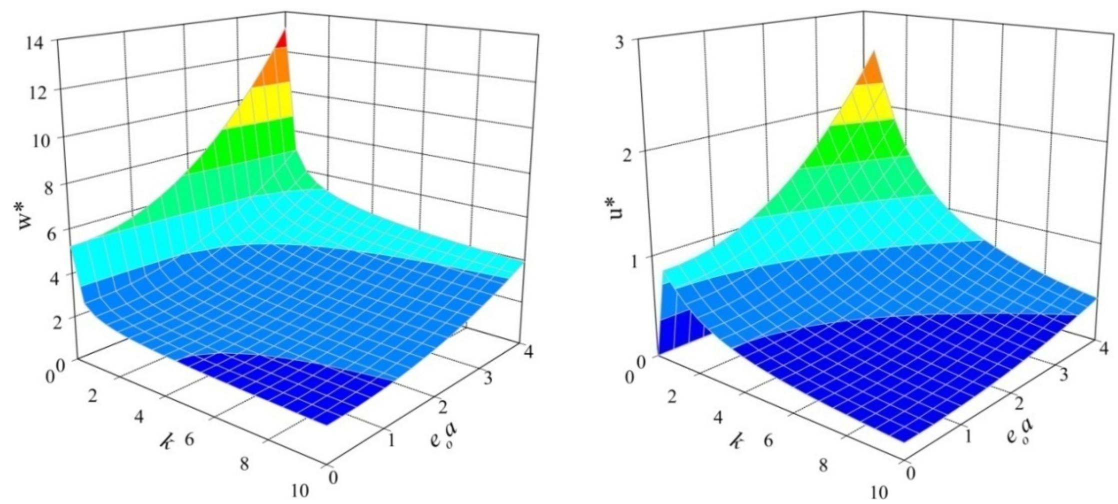
Table 3 Normalized axial displacements for HR T-FG beams ($h$ = 1)

| $L$ | $k$ | $e_o a$ | | | | |
|---|---|---|---|---|---|---|
| | | 0 | 0.5 | 1.0 | 1.5 | 2.0 |
| 10 | 0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| | 0.3 | 0.8932 | 0.9200 | 1.0004 | 1.1344 | 1.3220 |
| | 1 | 0.7576 | 0.7803 | 0.8485 | 0.9621 | 1.121. |
| | 3 | 0.4176 | 0.4301 | 0.4677 | 0.5303 | 0.6180 |
| | 10 | 0.1507 | 0.1553 | 0.1688 | 0.1914 | 0.2231 |
| 30 | 0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| | 0.3 | 0.8932 | 0.8962 | 0.9051 | 0.9200 | 0.9409 |
| | 1 | 0.7576 | 0.7601 | 0.7677 | 0.7803 | 0.7980 |
| | 3 | 0.4176 | 0.4190 | 0.4232 | 0.4301 | 0.4399 |
| | 10 | 0.1507 | 0.1512 | 0.1527 | 0.1553 | 0.1588 |
| 100 | 0 | 0.0000 | 0.000 | 0.0000 | 0.0000 | 0.0000 |
| | 0.3 | 0.8932 | 0.8935 | 0.8943 | 0.8956 | 0.8975 |
| | 1 | 0.7576 | 0.7578 | 0.7585 | 0.7596 | 0.7612 |
| | 3 | 0.4176 | 0.4177 | 0.4181 | 0.4187 | 0.4196 |
| | 10 | 0.1507 | 0.1508 | 0.1509 | 0.1511 | 0.1515 |

characteristic.

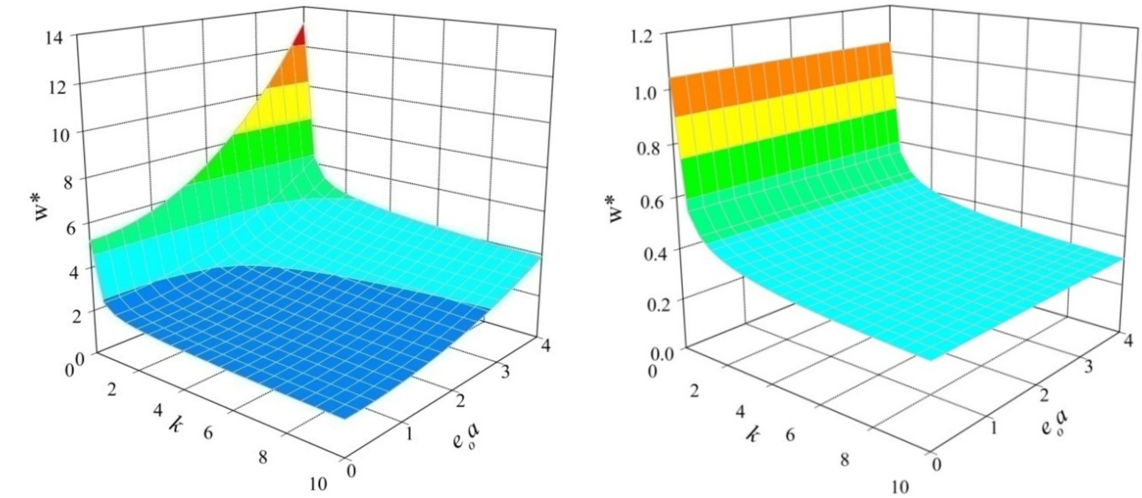
The variations of $w^*$ and $u^*$ are tabulated in Tables 2-3 with various values of $e_oa$, $k$, and $L$ for HR T-FG beams. As discussed in previous subsection, the height $h$ is kept as unit. Table 2 clearly shows that the present transverse displacements agree very well with the solutions of Şimşek and Yurtcu (2013). For $k = 0$, $u^*$ are zero for all BCs since there is no coupling effect between bending and extension. In addition, once again, if $e_oa$ is ignored, one can see that the presented results are completely agree with solutions of homogenous beams given by Budynas and Nisbett (2010). For example, for $k = 0$ and $e_oa = 0$, T-FG beams becomes the homogeneous metal beams as shown in Fig. 2; then the transverse displacement for HR beam is $w_o = \dfrac{5qL^4}{384E_mI}$. Thus, the normalized transverse displacement is $w^* = \dfrac{100\times 5\times E_c}{E_m}$ = 5.2083, which is exactly same with the result from this study in Table 2. It is seen in Tables 2-3 that the nonlocal effect is also significant in a small length on both of transverse and axial displacements for T-FG beams.

To show the effects of $e_oa$ and $k$, the variation of $w^*$ and $u^*$ are plotted in Figs. 3-5 for various BCs with $L = 10$ and $h = 1$. It is interesting to observe from Figs. 3-5 that as we expected, $w^*$ decreases as $k$ increases for all BCs since the elasticity modulus increases as shown in Fig. 2(a). On the other hand, $u^*$ increases with increase of $k$ and obtain the maximum value as $k$ of 0.378. This value of $k$ can be easily obtained by optimizing $\Omega_1$ in Eq. (31). It means that the coupling effect between bending and extension of T-FG beam is strongest for beam with $k$ of 0.378. After the maximum point, $u^*$ decreases to zero with further increase of $k$. The reason of this phenomenon is due to fact that as $k$ approaches zero or infinite, the T-FG beam approaches the homogenous metal or ceramic beams, respectively. It is remarkable observation that the nonlocal effect depends on BCs. The distinct features of the effect of the nonlocal parameter on BCs are summarized as follows: (1) for HR, HH, CR, and CH beams as shown in Figs. 3(a)-(c), 4, $w^*$ increases as $e_oa$




(a) HR beams-Transverse displacements (b) HR beams-Axial displacements

Fig. 3 Effects of a nonlocal parameter $e_oa$ and a material non-homogeneity parameter $k$ on the transverse and axial displacements of T-FG beams

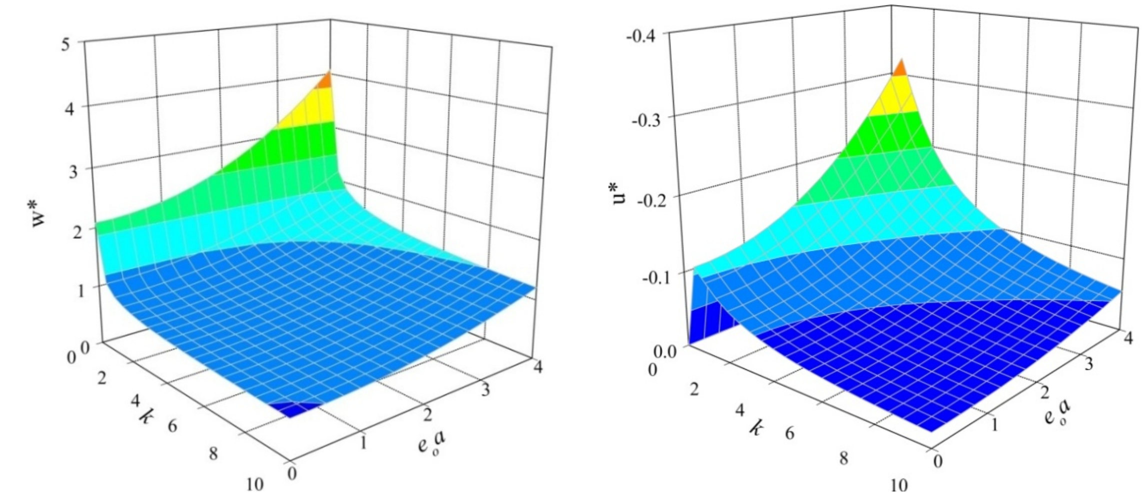



(c) HH beams-Transverse displacements (D) CC beams-Transverse displacements

Fig. 3 Continueds

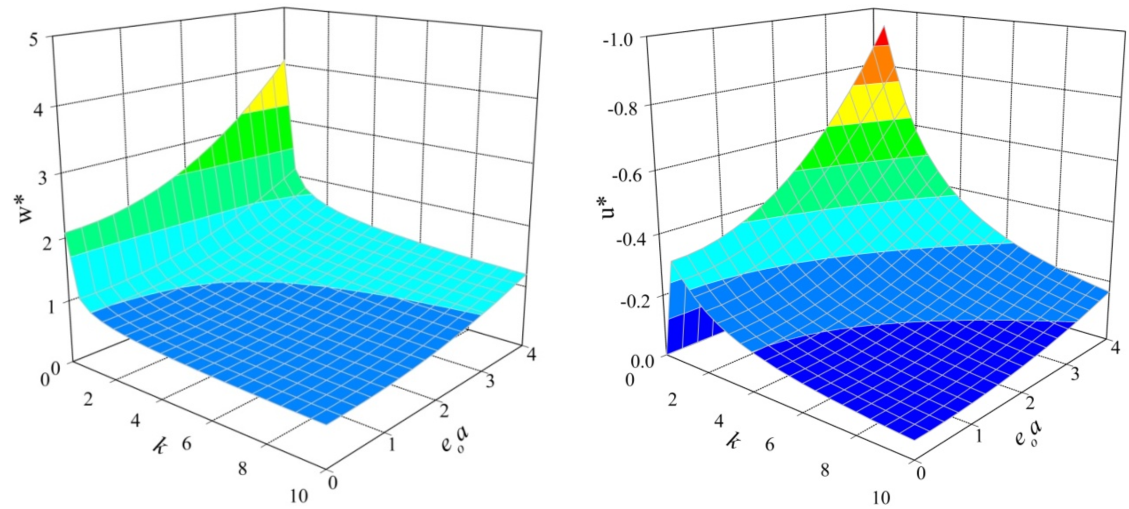
increases. That is to say, $e_oa$ softens nanobeams with above BCs. (2) On the contrary, $w^*$ and $u^*$ for CF beam decrease with increase of $e_oa$ as shown in Fig. 5. Thus, $e_oa$ affects the increase of stiffness of CF beam. (3) There is no effect of $e_oa$ on $w^*$ for CC beam as seen in Fig. 3(d) which is found to be consistent with the results of Alshorbagy *et al*. (2013) and Eltaher *et al*. (2013a, b). (4) Finally, the effect of $e_oa$ on $w^*$ is the largest for HR beam, followed by CR, CF and CC beams.

In order to investigate the effect of $k$ on the displacements of beams with the roller and hinged




(a) CR beams-Transverse displacements (b) CR beams-Axial displacements

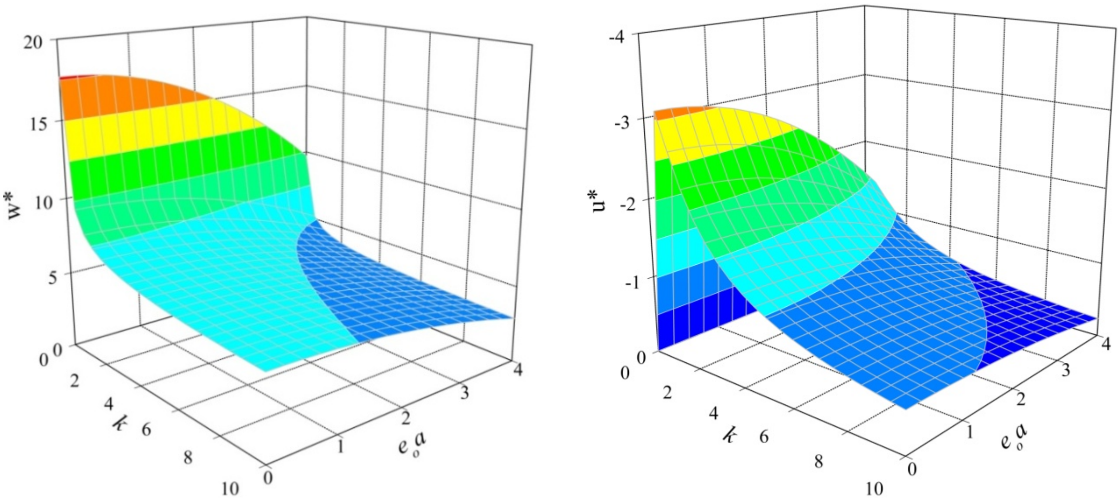
Fig. 4 Effects of a nonlocal parameter $e_oa$ and a material non-homogeneity parameter $k$ on the transverse and axial displacements of T-FG beams




(c) CH beams-Transverse displacements (d) CH beams-Axial displacements

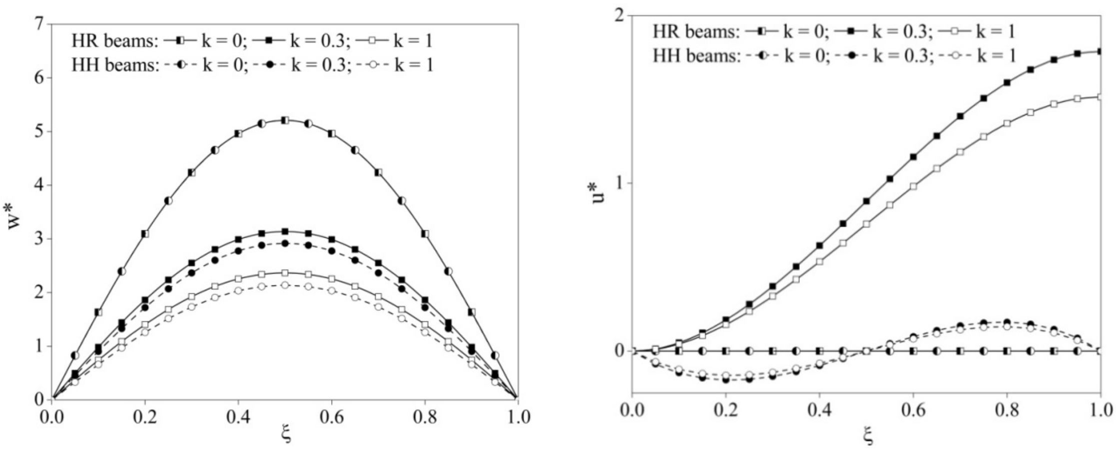
Fig. 4 Continued

BCs, $w^*$ and $u^*$ are presented in Fig. 6 for HR and HH beams along longitudinal direction of beams. In this case, $e_oa$, $L$ and $h$ are assumed to be 0, 10 and 1, respectively. It is seen that for the homogeneous metal beam ($k = 0$), $w^*$ of HR beam is the same as that of HH beam as shown in Fig. 6(a) and $u^*$ is zero for both HR and HH beams as shown in Fig. 6(b) since there is no coupling effect between bending and extension. However, for beams with $k > 0$, $w^*$ of HR beam is slightly larger than that of HH beam and $u^*$ of HR beam is significantly different from that of HH beam.




(a) CF beams-Transverse displacements (b) CF beams-Axial displacements

Fig. 5 Effects of a nonlocal parameter $e_oa$ and a material non-homogeneity parameter $k$ on the transverse and axial displacements of T-FG beams




(a) Transverse displacements (b) Axial displacements

Fig. 6 Transverse and axial displacements of HR T-FG beams and HH T-FG beams

### 5.3 A-FG beams

The A-FG beams as shown in Fig. 1(b) with various BCs are considered. The geometric and material properties of beam are the same as those in the previous example except that the elasticity modulus of A-FG beam is assumed to vary with the power-law function given by Eq. (13) through the axial direction of beam. In this case, as would be expected, the axial displacements at mid-plane are zero since there is no coupling effect between bending and extension.

In Table 4, the study of convergence quality of power series method for transverse displacement of HR A-FG beams are figured out for $L$ = 10, $h$ =1. One can see that the results from power series methodology converge monotonically for 25 term of series parameter $n$. On addition, the faster convergence results are obtained for the larger non-homogeneity parameter $k$. The results for transverse displacement of HR A-FG beams are presented in Table 5 by using 25 terms of series parameter $n$. It is clearly evident from Table 5 that the present results are great agreement with those given by Şimşek and Yurtcu (2013) in the case of $k$ = 0.

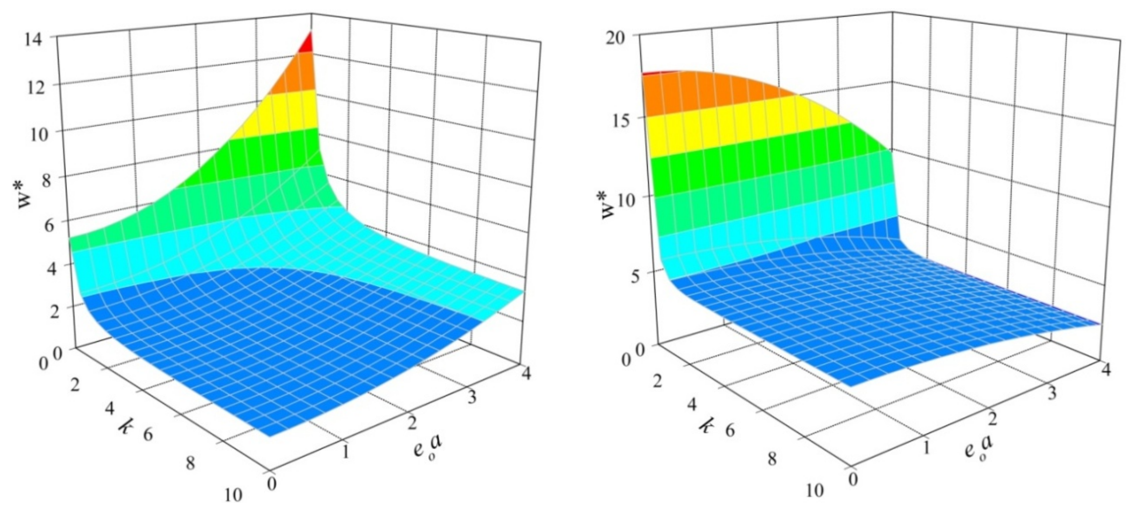
Table 4 Convergence study of power series methodology for transverse displacements of HR A-FG beams ($L$ = 10, $h$ =1)

| $e_oa$ | $k$ | Series parameter $n$ | | | | |
|---|---|---|---|---|---|---|
| | | 5 | 10 | 15 | 20 | 25 |
| 1.0 | 1 | 2.3673 | 2.3970 | 2.3992 | 2.3994 | 2.3995 |
| | 3 | 1.6765 | 1.6789 | 1.6791 | 1.6791 | 1.6791 |
| | 10 | 1.4512 | 1.4513 | 1.4513 | 1.4513 | 1.4513 |
| 2.0 | 1 | 2.9977 | 3.0383 | 3.0415 | 3.0419 | 3.0420 |
| | 3 | 2.1289 | 2.1328 | 2.1331 | 2.1332 | 2.1332 |
| | 10 | 1.8376 | 1.8379 | 1.8379 | 1.8379 | 1.8379 |

Table 5 Normalized transverse displacements for HR A-FG beams ($h$ = 1)

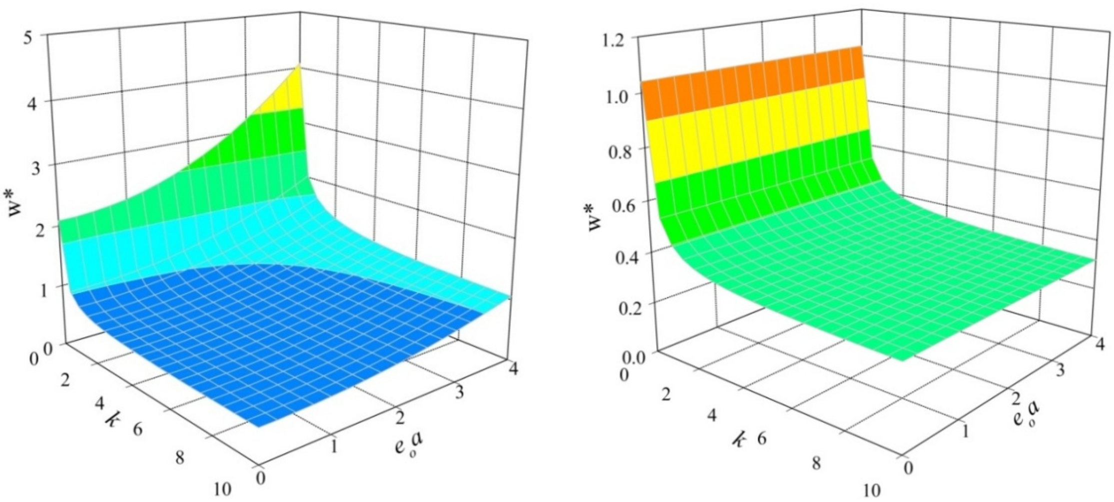
| $L$ | $k$ | $e_oa$ | | | | |
|---|---|---|---|---|---|---|
| | | 0 | 0.5 | 1.0 | 1.5 | 2.0 |
| 10 | 0 | 5.2083<br>(5.2083) | 5.3333<br>(5.3333) | 5.7083<br>(5.7083) | 6.3333<br>(6.3333) | 7.2083<br>(7.2083) |
| | 0.3 | 3.3520 | 3.4326 | 3.6744 | 4.0774 | 4.6416 |
| | 1 | 2.1853 | 2.2389 | 2.3995 | 2.6672 | 3.0420 |
| | 3 | 1.5278 | 1.5656 | 1.6791 | 1.8683 | 2.1332 |
| | 10 | 1.3225 | 1.3547 | 1.4513 | 1.6124 | 1.8379 |
| 30 | 0 | 5.2083<br>(5.2083) | 5.2222<br>(5.2222) | 5.2639<br>(5.2638) | 5.3333<br>(5.3333) | 5.4306<br>(5.4305) |
| | 0.3 | 3.3520 | 3.3609 | 3.3878 | 3.4326 | 3.4953 |
| | 1 | 2.1853 | 2.1913 | 2.2091 | 2.2389 | 2.2805 |
| | 3 | 1.5278 | 1.5320 | 1.5446 | 1.5656 | 1.5950 |
| | 10 | 1.3225 | 1.3261 | 1.3368 | 1.3547 | 1.3798 |
| 100 | 0 | 5.2083<br>(5.2083) | 5.2096<br>(5.2095) | 5.2133<br>(5.2133) | 5.2196<br>(5.2195) | 5.2283<br>(5.2283) |
| | 0.3 | 3.3520 | 3.3528 | 3.3552 | 3.3592 | 3.3649 |
| | 1 | 2.1853 | 2.1858 | 2.1875 | 2.1901 | 2.1939 |
| | 3 | 1.5278 | 1.5281 | 1.5293 | 1.5312 | 1.5338 |
| | 10 | 1.3225 | 1.3228 | 1.3238 | 1.3254 | 1.3276 |

Note: ( ) are the results from Şimşek and Yurtcu (2013)




(a) HR and HH beams (b) CF beams

Fig. 7 Effects of a nonlocal parameter $e_oa$ and a material non-homogeneity parameter $k$ on the transverse displacements of A-FG beams




(c) CR and CH beams (d) CC beams

Fig. 7 Continued

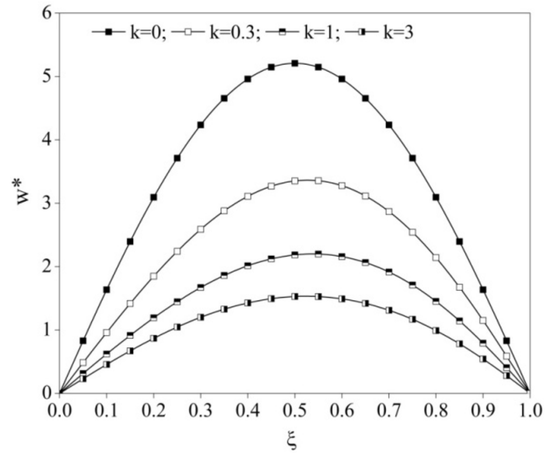



Fig. 8 Transverse displacements of HR and HH A-FG beams

Fig. 7 show the variation of $w^*$ at mid-span of beams with $L$ = 10 and $h$ = 1 for various BCs. It is seen that similar to the behavior of T-FG beams, $w^*$ decreases with increase of $k$ for all BCs. It is also seen that $e_oa$ reduces the bending stiffness of HR and CR beams but increases that of CF beam. Furthermore, the transverse displacements of CC beam are independent of $e_oa$. In contrary to the T-FG beams, $w^*$ of HR and CR beams are identical to those of HH and CH beams, respectively, since the coupling term between bending and extension of A-FG beam does not exist. In Fig. 8, the variation of $w^*$ of HR and HH beams are plotted with respect to $\xi$ for $e_oa$ = 0, $L$ = 10 and $h$ = 1. It is seen that the curves of $w^*$ of A-FG beams ($k$ > 0) are asymmetric and the value of $\xi$ corresponding to the maximum $w^*$ moves right side as $k$ increases. This is due to the fact that the material of the right end of beam is the full metal which is more flexible than that of the left end of beam.

## 6. Conclusions

In this paper, the static analysis of the transversely or axially functionally graded (T-FG or A-FG) beams is performed. The Euler-Bernoulli beam theory combined with the nonlocal continuum theory is used in the analysis. The explicit solutions for the transverse and axial displacements of the nonlocal FG beams subjected to the uniformly distributed transverse load with various boundary conditions are derived from the governing equations. The effects of the nonlocal parameter, the material non-homogeneity parameter, the boundary condition as well as the length of beam on the static behavior of FG beams are discussed. From the results analyzed above, the importance observations are summarized as follows:

- The nonlocal effect plays an important role on the static response of the FG nanobeams. Therefore, the small scale effect (or nonlocal effect) should be considered in the static analysis of the mechanical behavior of FG nanobeams.
- The nonlocal effect has inconsistent behavior for different boundary conditions. The transverse and axial displacements increase as the nonlocal parameter increases for HR, HH, CR, and CH beams. Whereas, two displacements decrease with increase of the nonlocal parameter for CF beam. This nonlocal effect on the transverse displacement is the largest for HR beam, followed by CR, CF and CC beams. In addition, this effect has no influence on the transverse displacement for CC beam and on the axial displacements for HH and CC beams.
- For A-FG beams, the transverse displacements of HR and CR beams are identical to those of HH and CH beams, respectively, which is contrary to the T-FG beams.
- The material distribution profile manipulates to change the maximum transverse and axial displacements. As the material non-homogeneity parameter increases, the transverse displacement decreases for all boundary conditions. On the other hand, the axial displacement has the maximum value at the material non-homogeneity parameter of 0.378.
- The length of beam influences on the nonlocal effect. In a small length, the nonlocal effect is significant on both of transverse and axial displacements. But its effect is negligible in a large length of beam.

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## References

Alshorbagy, A.E., Eltaher, M.A. and Mahmoud, F.F. (2013), "Static analysis of nanobeams using nonlocal FEM", *J. Mech. Sci. Technol.*, **27**(7), 2035-2041.

Aydogdu, M. (2009), "A general nonlocal beam theory: Its application to nanobeam bending, buckling and vibration", *Physica E.*, **41**(9), 1651-1655.

Budynas, R.G. and Nisbett, J.K. (2010), *Shigley's Mechanical Engineering Design*, McGraw-Hill, New York, NY, USA.
Chakraborty, A., Gopalakrishnan, S. and Reddy, J.N. (2003), "A new beam finite element for the analysis of functionally graded materials", *Int. J. Mech. Sci.*, **45**(3), 519-539.
Challamel, N. and Wang, C.M. (2008), "The small length scale effect for a non-local cantilever beam- a paradox solved", *Nanotechnology*, **19**(34), 345703.
Eltaher, M.A., Emam, S.A. and Mahmoud, F.F. (2012), "Free vibration analysis of functionally graded size-dependent nanobeams", *Appl. Math. Comput.*, **218**(14), 7406-7420.
Eltaher, M.A., Alshorbagy, A.E. and Mahmoud, F.F. (2013a), "Determination of neutral axis position and its effect on natural frequencies of functionally graded macro or nanobeams", *Compos. Struct.*, **99**, 193-201.
Eltaher, M.A., Emam, S.A. and Mahmoud, F.F. (2013b), "Static and stability analysis of nonlocal functionally graded nanobeams", *Compos. Struct.*, **96**, 82-88.
Eringen, A.C. (1972a), "Nonlocal polar elastic continua", *Int. J. Eng. Sci.*, **10**(1), 1-16.
Eringen, A.C. (1972b), "On nonlocal elasticity", *Int. J. Eng. Sci.*, **10**(3), 233-248.
Eringen, A.C. (1983), "On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves", *J. Appl. Phys.*, **54**(9), 4703-4710.
Ghannadpour, S.A.M., Mohammadi, B. and Fazilati, J. (2013), "Bending, buckling and vibration problems of nonlocal Euler beams using Ritz method", *Compos. Struct.*, **96**, 584-589.
Hein, H. and Feklistova, L. (2011), "Free vibrations of non-uniform and axially functionally graded beams using Haar wavelets", *Eng. Struct.*, **33**(12), 3696-3701.
Huang, Y. and Li, X.F. (2010), "A new approach for free vibration of axially functionally graded beams with non-uniform cross-section", *J. Sound. Vib.*, **329**(11), 2291-2303.
Kang, Y.A. and Li, X.F. (2009), "Bending of functionally graded cantilever beam with power-law non-linearity subjected to an end force", *Int. J. Non-Linear. Mech.*, **44**(6), 696-703.
Li, X.F. (2008), "A unified approach for analyzing static and dynamic behaviors of functionally graded Timoshenko and Euler–Bernoulli beams", *J. Sound. Vib.*, **318**(4-5), 1210-1229.
Li, Y., Zhang, H. and Zhang, N. (2005), "Stress analysis of functionally graded beam using effective principle axes", *Int. J. Mech. Mater. Des.*, **2**(3-4), 157-164.
Li, X.F., Wang, B.L. and Han, J.C. (2010), "A higher-order theory for static and dynamic analyses of functionally graded beams", *Arch. Appl. Mech.*, **80**(10), 1197-1212.
Nakamura, T., Wang, T. and Sampath, S. (2000), "Determination of properties of graded materials by inverse analysis and instrumented indentation", *Acta. Mater.*, **48**(17), 4293-4306.
Nie, G.J., Zhong, Z. and Chen, S. (2013), "Analytical solution for a functionally graded beam with arbitrary graded material properties", *Compos. Pt. B-Eng.*, **44**(1), 274-282.
Peddieson, J., Buchanan, G.R. and McNitt, R.P. (2003), "Application of nonlocal continuum models to nanotechnology", *Int. J. Eng. Sci.*, **41**(3-5), 305-312.
Phadikar, J.K. and Pradhan, S.C. (2010), "Variational formulation and finite element analysis for nonlocal elastic nanobeams and nanoplates", *Comput. Mater. Sci.*, **49**(3), 492-499.
Pradhan, S.C. and Murmu, T. (2010), "Application of nonlocal elasticity and DQM in the flapwise bending vibration of a rotating nanocantilever", *Physica E*, **42**(7), 1944-1949.
Rajasekaran, S. (2013), "Free vibration of centrifugally stiffened axially functionally graded tapered Timoshenko beams using differential transformation and quadrature methods", *Appl. Math. Model.*, **37**(6), 4440-4463.
Reddy, J.N. (1999), *Theory and Analysis of Elastic Plates*, Taylor & Francis, Philadelphia, PA, USA
Reddy, J.N. (2002), *Energy Principles and Variational Methods in Applied Mechanics*, John Wiley & Sons, New York, NY, USA.
Reddy, J.N. (2007), "Nonlocal theories for bending, buckling and vibration of beams", *Int. J. Eng. Sci.*, **45**(2-8), 288-307.
Reddy, J.N. (2010), "Nonlocal nonlinear formulations for bending of classical and shear deformation theories of beams and plates", *Int. J. Eng. Sci.*, **48**(11), 1507-1518.
Sankar, B.V. (2001), "An elasticity solution for functionally graded beams", *Compos. Sci. Technol.*, **61**(5),

689-696.
Shahba, A. and Rajasekaran, S. (2012), "Free vibration and stability of tapered Euler–Bernoulli beams made of axially functionally graded materials", *Appl. Math. Model.*, **36**(7), 3094-3111.
Shahba, A., Attarnejad, R., Marvi, M.T. and Hajilar, S. (2011), "Free vibration and stability analysis of axially functionally graded tapered Timoshenko beams with classical and non-classical boundary conditions", *Compos. Pt. B-Eng.*, **42**(4), 801-808.
Shahba, A., Attarnejad, R. and Hajilar, S. (2013), "A mechanical-based solution for axially functionally graded tapered Euler-Bernoulli beams", *Mech. Adv. Mater. Struct.*, **20**(8), 696-707.
Şimşek, M. (2012), "Nonlocal effects in the free longitudinal vibration of axially functionally graded tapered nanorods", *Comput. Mater. Sci.*, **61**, 257-265.
Şimşek, M. and Yurtcu, H.H. (2013), "Analytical solutions for bending and buckling of functionally graded nanobeams based on the nonlocal Timoshenko beam theory", *Compos. Struct.*, **97**, 378-386.
Taeprasartsit, S. (2012), "A buckling analysis of perfect and imperfect dunctionally graded columns", *J. Mater. Des. Appl.*, **226**(1), 16-33.
Uymaz, B. (2013), "Forced vibration analysis of functionally graded beams using nonlocal elasticity", *Compos. Struct.*, **105**, 227-239.
Wang, L. and Hu, H. (2005), "Flexural wave propagation in single-walled carbon nanotubes", *Phys. Rev. B.*, **71**, 195412.
Wang, C.M., Zhang, Y.Y. and He, X.Q. (2007), "Vibration of nonlocal Timoshenko beams", *Nanotechnology*, **18**(10), 105401.
Ying, A. K., Li, X. F. (2009), "Bending of functionally graded cantilever beam with power-law non-linearity subjected to an end force", *Int. J. Non-Linear. Mech.*, **44**(6), 696-703.
Zhang, Y.Y., Wang, C.M. and Challamel, N. (2010), "Bending, buckling, and vibration of micro nanobeams by hybrid nonlocal beam model", *J. Eng. Mech.*, **136**(5), 562-574.
Zhong, Z. and Yu, T. (2007), "Analytical solution of a cantilever functionally graded beam", *Compos. Sci. Technol.*, **67**(3-4), 481-488.
Zhu, H. and Sankar, B.V. (2004), "A combined Fourier series–Galerkin method for the analysis of functionally graded beams", *J. Appl. Mech.*, **71**(3), 421-424.

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## Appendix A

The explicit solutions of T-FG beams

(1) Hinged-Roller (HR) T-FG beam

$$u_o = \frac{\Omega_1 q L^3}{12}\left(2\xi^3 - 3\xi^2 - \frac{12\mu}{L^2}\xi\right) \tag{A-1}$$

$$w_o = \frac{\Omega_2 q L^4}{24}\left\{\xi^4 - 2\xi^3 - \frac{12\mu}{L^2}\xi^2 + \left(1 + \frac{12\mu}{L^2}\right)\xi\right\} \tag{A-2}$$

BCs

$$u_o(0) = 0\,; \quad w_o(0) = 0; \quad M_x(0) = 0 \tag{A-3}$$

$$w_o(1) = 0\,; \quad N_x(1) = 0; \quad M_x(1) = 0 \tag{A-4}$$

(2) Hinged-Hinged (HH) T-FG beam

$$u_o = \frac{\Omega_1 q L^3}{12}\left(2\xi^3 - 3\xi^2 + \xi\right) \tag{A-5}$$

$$w_o = \frac{\Omega_2 q L^4}{24}\left[\xi^4 - 2\xi^3 + \left\{\frac{B_{11}^{T\,2}}{A_{11}^T D_{11}^T} + \frac{12\mu}{L^2}\left(\frac{B_{11}^{T\,2}}{A_{11}^T D_{11}^T} - 1\right)\right\}\xi^2 - \left(1 + \frac{12\mu}{L^2}\right)\left(\frac{B_{11}^{T\,2}}{A_{11}^T D_{11}^T} - 1\right)\xi\right] \quad \text{A-6)}$$

BCs

$$u_o(0) = 0\,; \quad w_o(0) = 0; \quad M_x(0) = 0 \tag{A-7}$$

$$u_o(1) = 0\,; \quad w_o(1) = 0; \quad M_x(1) = 0 \tag{A-8}$$

(3) Clamped-Free (CF) T-FG beam

$$u_o = \frac{\Omega_1 q L^3}{6}\left\{\xi^3 - 3\xi^2 + \left(3 - \frac{6\mu}{L^2}\right)\xi\right\} \tag{A-9}$$

$$w_o = \frac{\Omega_2 q L^4}{24}\left\{\xi^4 - 4\xi^3 + \left(6 - \frac{12\mu}{L^2}\right)\xi^2\right\} \tag{A-10}$$

BCs

$$u_o(0) = 0\,; \quad w_o(0) = 0; \quad \frac{dw_o(0)}{d\xi} = 0 \tag{A-11}$$

$$N_x(1) = 0\,; \quad M_x(1) = 0; \quad \frac{dM_x(1)}{d\xi} = 0 \tag{A-12}$$

(4) Clamped-Roller (CR) T-FG beam

$$u_o = \frac{\Omega_1 q L^3}{48}\left\{8\xi^3 - \left(15 + \frac{36\mu}{L^2}\right)\xi^2 + \left(6 + \frac{24\mu}{L^2}\right)\xi\right\} \tag{A-13}$$

$$w_o = \frac{\Omega_2 q L^4}{48}\left\{2\xi^4 - \left(5 + \frac{12\mu}{L^2}\right)\xi^3 + \left(3 + \frac{12\mu}{L^2}\right)\xi^2\right\} \tag{A-14}$$

BCs

$$u_o(0) = 0\,; \quad w_o(0) = 0; \quad \frac{dw_o(0)}{d\xi} = 0 \tag{A-15}$$

$$w_o(1) = 0\,; \quad N_x(1) = 0; \quad M_x(1) = 0 \tag{A-16}$$

(5) Clamped-Hinged (CH) T-FG beam

$$u_o = \frac{\Omega_1 q L^3}{6}\left\{\xi^3 - 3\Lambda_1\xi^2 - (1 - 3\Lambda_1)\xi\right\} \tag{A-17}$$

$$w_o = \frac{\Omega_2 q L^4}{24}\left\{\xi^4 - 4\Lambda_1\xi^3 - (1 - 4\Omega_1)\xi^2\right\} \tag{A-18}$$

where

$$\Lambda_1 = \frac{\left(\frac{5}{6}A_{11}^T D_{11}^T - \frac{2}{3}B_{11}^{T\,2}\right) + \frac{2\mu}{L^2}\left(A_{11}^T D_{11}^T - B_{11}^{T\,2}\right)}{\frac{4}{3}A_{11}^T D_{11}^T - B_{11}^{T\,2}} \tag{A-19}$$

BCs

$$u_o(0) = 0\,; \quad w_o(0) = 0; \quad \frac{dw_o(0)}{d\xi} = 0 \tag{A-20}$$

$$u_o(1) = 0\,; \quad w_o(1) = 0; \quad M_x(1) = 0 \tag{A-21}$$

(6) Clamped-Clamped T-FG beam

$$u_o = \frac{\Omega_1 q L^3}{12}\left(2\xi^3 - 3\xi^2 + \xi\right) \tag{A-22}$$

$$w_o = \frac{\Omega_2 q L^4}{24}\left(\xi^4 - 2\xi^3 + \xi^2\right) \tag{A-23}$$

BCs

$$u_o(0) = 0\,; \quad w_o(0) = 0; \quad \frac{dw_o(0)}{d\xi} = 0 \tag{A-24}$$

$$u_o(1)=0\ ;\quad w_o(1)=0;\qquad \frac{dw_o(1)}{d\xi}=0 \tag{A-25}$$

## Appendix B

The explicit solutions of A-FG beams

(1) Hinged-Roller (HR) and Hinged-Hinged (HH)A-FG beams

$$w_o=\Phi\sum_{n=0}^{\infty}\beta^n\left\{\frac{\xi^{nk+4}}{(nk+3)(nk+4)}-\frac{\xi^{nk+3}}{(nk+2)(nk+3)}-\frac{2\mu}{L^2}\frac{\xi^{nk+2}}{(nk+1)(nk+2)}\right\}+\Lambda_2\xi \tag{B-1}$$

where

$$\Lambda_2=-\Phi\sum_{n=0}^{\infty}\beta^n\left\{\frac{1}{(nk+3)(nk+4)}-\frac{1}{(nk+2)(nk+3)}-\frac{2\mu}{L^2}\frac{1}{(nk+1)(nk+2)}\right\} \tag{B-2}$$

(2) Clamped-Free (CF) A-FG beam

$$w_o=\Phi\sum_{n=0}^{\infty}\beta^n\left\{\frac{\xi^{nk+4}}{(nk+3)(nk+4)}-2\frac{\xi^{nk+3}}{(nk+2)(nk+3)}+\left(1-\frac{2\mu}{L^2}\right)\frac{\xi^{nk+2}}{(nk+1)(nk+2)}\right\} \tag{B-3}$$

(3) Clamped-Roller (CR) and Clamped-Hinged (CH) A-FG beams

$$w_o=\Phi\sum_{n=0}^{\infty}\beta^n\left\{\frac{\xi^{nk+4}}{(nk+3)(nk+4)}-\Lambda_2\frac{\xi^{nk+3}}{(nk+2)(nk+3)}-\left(1-\Lambda_3+\frac{2\mu}{L^2}\right)\frac{\xi^{nk+2}}{(nk+1)(nk+2)}\right\} \tag{B-4}$$

where

$$\Lambda_3=\frac{\sum_{n=0}^{\infty}\beta^n\left\{\left(1+\frac{2\mu}{L^2}\right)\frac{1}{(nk+1)(nk+2)}-\frac{1}{(nk+3)(nk+4)}\right\}}{2\sum_{n=0}^{\infty}\beta^n\left\{\frac{1}{(nk+1)(nk+2)(nk+3)}\right\}} \tag{B-5}$$

(4) Clamped-Clamped (CC) A-FG beam

$$w_o=\Phi\sum_{n=0}^{\infty}\beta^n\left\{\frac{\xi^{nk+4}}{(nk+3)(nk+4)}-\Lambda_4\frac{\xi^{nk+3}}{(nk+2)(nk+3)}-\Lambda_5\frac{\xi^{nk+2}}{(nk+1)(nk+2)}\right\} \tag{B-6}$$

where

$$\Lambda_4=\frac{\sum_{n=0}^{\infty}\frac{\beta^n}{nk+3}\sum_{n=0}^{\infty}\frac{\beta^n}{(nk+1)(nk+2)}-\sum_{n=0}^{\infty}\frac{\beta^n}{nk+1}\sum_{n=0}^{\infty}\frac{\beta^n}{(nk+3)(nk+4)}}{\sum_{n=0}^{\infty}\frac{\beta^n}{nk+2}\sum_{n=0}^{\infty}\frac{\beta^n}{(nk+1)(nk+2)}-\sum_{n=0}^{\infty}\frac{\beta^n}{nk+1}\sum_{n=0}^{\infty}\frac{\beta^n}{(nk+2)(nk+3)}} \tag{B-7}$$

$$\Lambda_5 = \frac{\sum_{n=0}^{\infty} \frac{\beta^n}{nk+3} - \Lambda_4 \sum_{n=0}^{\infty} \frac{\beta^n}{nk+2}}{\sum_{n=0}^{\infty} \frac{\beta^n}{nk+1}} \tag{B-8}$$