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Analytical study on post-buckling and nonlinear free vibration analysis of FG beams resting on nonlinear elastic foundation under thermo-mechanical loadings using VIM

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Abstract. In this paper, nonlinear vibration and post-buckling analysis of beams made of functionally graded materials (FGMs) resting on nonlinear elastic foundation subjected to thermo-mechanical loading are studied. The thermo-mechanical material properties of the beams are assumed to be graded in the thickness direction according to a simple power law distribution in terms of the volume fractions of the constituents, and to be temperature-dependent. The assumption of a small strain, moderate deformation is used. Based on Euler-Bernoulli beam theory and von-Karman geometric nonlinearity, the integral partial differential equation of motion is derived. Then this PDE problem which has quadratic and cubic nonlinearities is simplified into an ODE problem by using the Galerkin method. Finally, the governing equation is solved analytically using the variational iteration method (VIM). Some new results for the nonlinear natural frequencies and buckling load of the FG beams such as the influences of thermal effect, the effect of vibration amplitude, elastic coefficients of foundation, axial force, end supports and material inhomogenity are presented for future references. Results show that the thermal loading has a significant effect on the vibration and post-buckling response of FG beams.

Keywords: functionally graded beams; thermal and axial loadings; nonlinear free vibration; post-buckling; Galerkin method; variational iteration method

1. Introduction

In microscopically inhomogeneous functionally graded materials (FGMs), material properties vary smoothly and continuously from one surface to the other by gradually changing the volume fraction of their constituent materials. Due to their unique advantages, these materials are able to withstand severe high-temperature environment while maintaining structural integrity. Therefore, they have received considerable attention in many industries, especially in high-temperature applications such as space shuttle, aircraft, and etc.

Noda (1991) and Tanigawa (1995) reported that the weakness of the fiber reinforced laminated

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composite materials, such as debonding, huge residual stress, locally largely plastic deformations, etc., can be avoided or reduced in FGMs. The considerable advantages offered by FGMs over conventional materials and the need of overcoming the technical challenges involving high temperature environments have prompted an increased use of FGM structures. FGMs were initially designed as thermal barrier materials for aerospace structures and fusion reactors where extremely high temperature and large thermal gradient exist. With the increasing demand, FGMs have been widely used in general structures. Hence, many FGM structures have been extensively studied, such as functionally graded (FG) beams, plates, shells, etc.

Furthermore, due to huge application of beams in different fields such as civil, marine and aerospace engineering, it is necessary to study their dynamic behavior at large amplitudes which is effectively nonlinear and therefore, is governed by nonlinear equations. Nonlinear free vibrations and buckling analysis of isotropic and composite beams have received a good amount of attention in the literature.

Fang and Wickert (1994) investigated the static deformation of micro-machined beams under in-plane compressive stresses in both the pre-buckling and post-buckling domains. They considered the mid-plane stretching and the imperfection of the beam. A closed-form analytical solution using a complete elliptic integral is obtained. Lacarbonara (1997) studied the thermal post-buckling and vibrations of imperfect fixed-fixed beams. A solution was assumed for the post-buckling and a relationship between the critical thermal load and the imperfection amplitude is obtained. Hatsunaga (2001) presented natural frequencies and buckling stresses of simply supported laminated composite beams taking into account the effects of the transverse shear and the rotary inertia. Nonlinear modal analysis approach based on invariant manifold method is utilized to obtain the nonlinear normal modes of a clamped–clamped beam for large amplitude displacements by Xie *et al*. (2002). Vaz and Solano (2003) investigated the buckling response of a geometrically nonlinear hinged-hinged slender elastic rod (elastica) subjected to a uniform temperature gradient. Guo and Zhong (2004) have investigated nonlinear vibrations of thin beams based on sextic cardinal spline functions, a spline-based differential quadrature method. Nonlinear normal modes of vibration for a hinged-hinged beam with fixed ends have been evaluated considering both the continuous system and finite element models by Carlos et al. (2004). Sapountzakis and Tsiatas (2007) investigated the flexural buckling of composite Euler-Bernoulli beams of arbitrary cross sections. The resulting boundary-value problems were solved using the boundary element method. Aydogdu (2007) investigated the thermal buckling of cross-ply laminated beams with different boundary conditions. Nayfeh and Emam (2008) obtained a closed-form solution for the post-buckling configurations of beams composed of isotropic materials with various boundary conditions. Jun *et al*. (2008) investigated the free vibration and buckling behaviors of axially loaded laminated composite beams having arbitrary lay-up using the dynamic stiffness method taking into account the influences of axial forces, Poisson effect, axial deformation, shear deformation, and rotary inertia.

In recent years, Pirbodaghi *et al*. (2009) have used the first-order approximation of the homotopy analysis method to investigate the nonlinear free vibration analysis of Euler-Bernoulli beam. Emam (2009) presented the static and dynamic response of geometrically imperfect composite beams. Results showed that the imperfection has a significant effect on the static and dynamic response of composite beams. Malekzadeh and Vosoughi (2009) studied nonlinear free vibration analysis of laminated composite thin beams on nonlinear elastic foundation (including shearing layer) with elastically restrained against rotation edges by Differential Quadrature (DQ) approach. Gupta *et al*. (2009) studied the nonlinear free vibration analysis of isotropic beams using simple iterative finite element formulation. An exact solution for the post-buckling of a symmetrically laminated composite beam with fixed-fixed, fixed-hinged, and hinged-hinged boundary conditions is presented by Emam and Nayfeh (2009). Recently, Gupta *et al*. (2010a, b) recently applied the concept of coupled displacement field (CDF) criteria to investigate the post-buckling behavior of isotropic and composite beams, respectively. Gunda *et al*. (2010) employed Rayleigh-Ritz method to study large amplitude vibration analysis of laminated composite beam with symmetric and asymmetric layup orientations.

More recently, the large amplitude vibration and post-buckling analysis of FG beams have attracted a huge number of research efforts. Ke *et al*. (2010) used the direct numerical integration method together with Runge-Kutta numerical technique to find the nonlinear vibration response of FGM beams with different end supports. Simsek (2010) studied the nonlinear forced vibration of the Timoshenko FG beams under action of moving harmonic load.

Ait Atmane *et al*. (2011) presented a theoretical investigation in free vibration of sigmoid FG beams with variable cross-section by using Bernoulli-Euler beam theory. Ma and Lee (2011) presented a further discussion of nonlinear mechanical behavior for FGM beams under in-plane thermal loading. Fallah and Aghdam (2011) presented simple analytical expressions for large amplitude free vibration and post-buckling analysis of FG beams resting on nonlinear elastic foundation subjected to axial force. Fallah *et al*. (2011) presented simple analytical expression for large amplitude thermo-mechanical free vibration analysis of symmetrically laminated composite beams. Recently, Bouremana *et al*. (2013) presented a new first shear deformation beam theory based on neutral surface position for FG beams. Bouderba *et al*. (2013) developed a simple trigonometric shear deformation theory to investigate thermo-mechanical behavior of simply supported FG plates resting on a Winkler–Pasternak elastic foundation. Kettaf *et al*. (2013) studied the thermal buckling behavior of FG sandwich plates using a new hyperbolic displacement model. Free vibration of FGM box beam was investigated by the formulation of an exact dynamic stiffness matrix on the basis of first-order shear deformation theory by Ziane *et al*. (2013). Yaghoobi and Torabi (2013a, b) presented analytical studies on large amplitude vibration and post-buckling of perfect and imperfect FG beams resting on non-linear elastic foundation, respectively.

The primary purpose of the present paper is to investigate the influences of thermal effect on nonlinear vibration and post-buckling analysis of beams made of FGMs resting on nonlinear elastic foundation subjected to axial force. Analytical expressions for nonlinear natural frequencies and buckling load of the FG beams are determined using the variational iteration method (VIM) given by He (1999).

2. Basic idea of variational iteration method

To illustrate the basic concept of the technique, we consider the following general differential equation

$$Lu + Nu = g(x) \tag{1}$$

where L is a linear operator, N a nonlinear operator, and $g(x)$ is the forcing term. According to variational iteration method, we can construct a correct functional as follows

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(Lu_n(t) + N\tilde{u}_n(t) - g(t))dt \tag{2}$$

where λ is a Lagrange multiplier, which can be identified optimally via variational iteration method. The subscripts n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation, that is, $\delta \tilde{u}_n = 0$; and (2) is called as a correct functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. In this method, it is required first to determine the Lagrange multiplier λ optimally. The successive approximation u_{n+1}, $n \geq 0$ of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function u_0, consequently, the solution is given by

$$u = \lim_{n \to \infty} u_n \tag{3}$$

3. Problem statement

Consider a straight FG beam of length L, width b and thickness h rests on an elastic nonlinear foundation and subjected to an axial force of magnitude $\overline{P}$ as shown in Fig. 1.

The beam is supported on an elastic foundation with cubic nonlinearity and shearing layer. In this study, material properties are considered to vary in accordance with the rule of mixtures as

$$P = P_M V_M + P_C V_C \tag{4}$$

where P and V are material property and volume fraction, respectively and subscripts M and C refer to the metal and ceramic constituents, respectively. The material properties of the form P that are highly temperature-dependent can be written as

$$P = P_0 \left(P_{-1} T^{-1} + 1 + P_1 T + P_2 T^2 + P_3 T^3 \right) \tag{5}$$

where P_0, P_{-1}, P_1, P_2 and P_3 are the coefficients of the temperature T and are unique to each constituent. In this paper, Poisson's ratio, v, is assumed to be a constant value of 0.28.

Simple power law distribution from pure metal at bottom face ($\bar{z} = -h/2$) to pure ceramic at the top face ($\bar{z} = +h/2$) in terms of the volume fractions of the constituents is assumed.

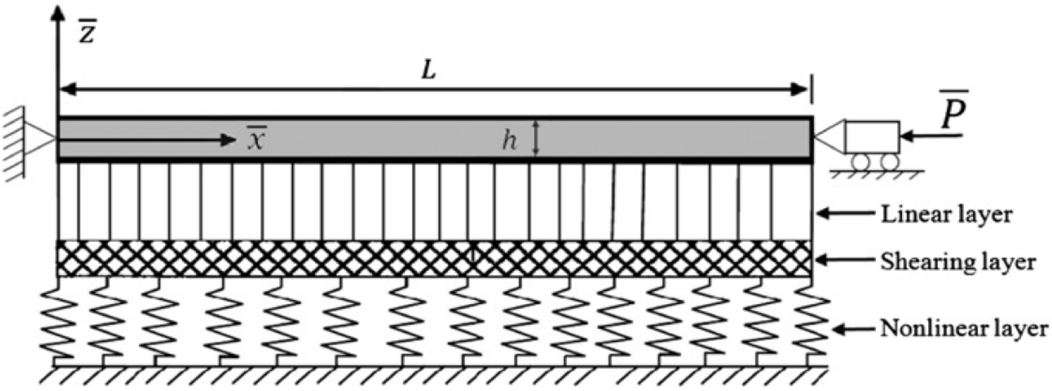


Fig. 1 Schematic of the FG beam with nonlinear foundation

$$V_C = \left(\frac{2\bar{z}+h}{2h}\right)^n \tag{6a}$$

$$V_M = 1 - V_C \tag{6b}$$

where n is the volume fraction exponent. The value of n equal to zero represents a fully ceramic beam. The mechanical and thermal properties of FGMs are determined from the volume fraction of the material constituents. We assume that the non-homogeneous material properties such as the modulus of elasticity (E) and the thermal expansion coefficient (α) and mass density (ρ) can be determined by substituting Eq. (6) into Eq. (4) as

$$E(\bar{z}) = E_M + (E_C - E_M)\left(\frac{2\bar{z}+h}{2h}\right)^n \tag{7a}$$

$$\alpha(\bar{z}) = \alpha_M + (\alpha_C - \alpha_M)\left(\frac{2\bar{z}+h}{2h}\right)^n \tag{7b}$$

$$\rho(\bar{z}) = \rho_M + (\rho_C - \rho_M)\left(\frac{2\bar{z}+h}{2h}\right)^n \tag{7c}$$

The force and moment resultants per unit length, based on classical theory of beams in a Cartesian coordinate system, can be written as

$$\begin{Bmatrix} N_{\bar{x}} \\ M_{\bar{x}} \end{Bmatrix} = b\begin{bmatrix} A_{11} & B_{11} \\ B_{11} & D_{11} \end{bmatrix}\begin{Bmatrix} \bar{u}_{,\bar{x}} + \frac{1}{2}\bar{w}_{,\bar{x}}^2 \\ \bar{w}_{,\bar{x}\bar{x}} \end{Bmatrix} + b\begin{Bmatrix} E_{11} \\ F_{11} \end{Bmatrix} \tag{8}$$

in which $\bar{w}$ and $\bar{u}$ are the transverse and axial displacements of the beam along the $\bar{z}$ and $\bar{x}$ directions, respectively. Where A_{11}, B_{11}, D_{11}, E_{11} and F_{11} are given as follows

$$(A_{11}, B_{11}, D_{11}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E(\bar{z})}{1-\nu^2}(1, \bar{z}, \bar{z}^2)d\bar{z}, \qquad (E_{11}, F_{11}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E(\bar{z})\alpha(\bar{z})T}{1-\nu^2}(1, \bar{z})d\bar{z} \tag{9}$$

After some mathematical simplifications, the governing equation of nonlinear free vibration of an FG beam in terms of transverse displacement can be written as

$$I_1\bar{w}_{,\bar{t}\bar{t}} + b\left(D_{11} - \frac{B_{11}^2}{A_{11}}\right)\bar{w}_{,\bar{x}\bar{x}\bar{x}\bar{x}} + \left[\bar{P} - bE_{11} - \frac{bA_{11}}{2L}\int_0^L \left(\bar{w}_{,\bar{x}}^2\right)d\bar{x} - \frac{bB_{11}}{L}\left(\bar{w}_{,\bar{x}}(L,\bar{t}) - \bar{w}_{,\bar{x}}(0,\bar{t})\right)\right]\bar{w}_{,\bar{x}\bar{x}} = F_{\bar{w}} \tag{10}$$

in which comma denotes derivative with respect to $\bar{x}$ or $\bar{t}$. Furthermore, I_1 and $F_{\bar{w}}$ are the inertia term and reaction of the elastic foundation on the beam which are defined as

$$I_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(\bar{z})d\bar{z} \tag{11}$$

$$F_{\overline{w}} = -\overline{k}_L \overline{w} - \overline{k}_{NL} \overline{w}^3 + \overline{k}_S \overline{w}_{,\overline{xx}} \tag{12}$$

where $\overline{k}_L$ and $\overline{k}_{NL}$ are linear and nonlinear elastic foundation coefficients, respectively and $\overline{k}_S$ is the coefficient of shear stiffness of the elastic foundation. The in-plane inertia and damping are assumed to be negligible and the distributed axial force is zero.

For the subsequent results to be general, we use the following non-dimensional variables are used

$$x = \frac{\overline{x}}{L}, \quad w = \frac{\overline{w}}{r}, \quad t = \overline{t}\sqrt{b\left(D_{11} - \frac{B_{11}^2}{A_{11}}\right)\Big/ I_1 L^4} \tag{13}$$

where $r = \sqrt{I/A}$ is the radius of gyration of the cross section. Using Eqs. (10) and (12) together with the dimensionless variables defined in Eq. (13), the dimensionless form of the governing equation becomes

$$w_{,tt} + w_{,xxxx} + \left[P - \Theta - \frac{1}{2}\Lambda\int_0^1 \left(w_{,x}^2\right)dx - \mathrm{B}\left(w_{,x}(1,t) - w_{,x}(0,t)\right)\right]w_{,xx} + k_L w + k_{NL} w^3 - k_S w_{,xx} = 0 \tag{14}$$

where

$$P = \frac{\overline{P}L^2}{b\left(D_{11} - \frac{B_{11}^2}{A_{11}}\right)}, \quad \Theta = \frac{E_{11}L^2}{\left(D_{11} - \frac{B_{11}^2}{A_{11}}\right)}, \quad \Lambda = \frac{A_{11}r^2}{\left(D_{11} - \frac{B_{11}^2}{A_{11}}\right)}, \quad B = \frac{B_{11}r}{\left(D_{11} - \frac{B_{11}^2}{A_{11}}\right)},$$
$$k_L = \frac{\overline{k}_L L^4}{b\left(D_{11} - \frac{B_{11}^2}{A_{111}}\right)}, \quad k_{NL} = \frac{\overline{k}_{NL} r^2 L^4}{b\left(D_{11} - \frac{B_{11}^2}{A_{111}}\right)}, \quad k_S = \frac{\overline{k}_S L^2}{b\left(D_{11} - \frac{B_{11}^2}{A_{11}}\right)} \tag{15}$$

Assuming $w(x, t) = V(t)\phi(x)$ where $V(t)$ is an unknown time-dependent function and $\phi(x)$ is the first eigenmode of the beam presented in Table 1 which must satisfy the kinematic boundary conditions. Applying the Galerkin method, the governing equation of motion is obtained as follows

$$\ddot{V}(t) + (\alpha_1 + P\alpha_p + \alpha_{k_L} + \alpha_{k_S} - \Theta\alpha_p)V(t) + \alpha_2 V^2(t) + (\alpha_{k_{NL}} + \alpha_3)V^3(t) = 0 \tag{16}$$

where

$$\alpha_1 = \frac{\int_0^1 \phi''''\phi dx}{\int_0^1 \phi^2 dx}, \quad \alpha_p = \frac{\int_0^1 \phi''\phi dx}{\int_0^1 \phi^2 dx}, \quad \alpha_2 = -B(\phi'(1) - \phi'(0))\alpha_p, \quad \alpha_{k_L} = k_L,$$
$$\alpha_{k_{NL}} = k_{NL}\frac{\int_0^1 \phi^4 dx}{\int_0^1 \phi^2 dx}, \quad \alpha_{k_S} = -k_S\alpha_p, \quad \alpha_3 = -\Lambda\int_0^1 \phi'^2 dx \tag{17}$$

It is worth mentioning that for isotropic and symmetrically laminated beams, the coefficient $V^2(t)$, α_2 is vanished i.e., the governing equation exhibits only cubic nonlinearity. However, the

Table 1 Trial functions for FG beam with various boundary condition

Boundary condition	$\phi(x)$	Value of q
Simply supported	$\sin\left(\frac{qx}{L}\right)$	π
Clamped-clamped	$\left(\cosh\left(\frac{qx}{L}\right)-\cos\left(\frac{qx}{L}\right)\right)-\frac{\cosh(q)-\cos(q)}{\sinh(q)-\sin(q)}\left(\sinh\left(\frac{qx}{L}\right)-\sin\left(\frac{qx}{L}\right)\right)$	4.730041

analysis of nonlinear vibrations for perfect or imperfect FG beams is thus significantly different from that of isotropic and symmetrically laminated beams, since the bending-stretching coupling (the constant B_{11}) induces the quadratic term $V^2(t)$.

The beam centroid is subjected to the following initial conditions

$$V(0)=a \tag{18a}$$

$$\frac{dV(0)}{dt}=0 \tag{18b}$$

where a denotes the non-dimensional maximum amplitude of oscillation with respect to the mid-surface. From Eq. (16), the post-buckling load-deflection relation of the FG beam can be obtained as

$$P_{NL}=-\frac{\alpha_1+\alpha_{k_L}+\alpha_{k_S}-\Theta\alpha_P+\alpha_2 V+\left(\alpha_{k_{NL}}+\alpha_3\right)V^2}{\alpha_p} \tag{19}$$

It should be noted that neglecting the contribution of V in Eq. (19), the linear buckling load can be determined as

$$P_L=-\frac{\alpha_1+\alpha_{k_L}+\alpha_{k_S}-\Theta\alpha_P}{\alpha_p} \tag{20}$$

4. Implementation of VIM

Eq. (16) can be simplified as

$$\ddot{V}+\beta_1 V+\beta_2 V^2+\beta_3 V^3=0 \tag{21}$$

where $\beta_1=\alpha_1+P\alpha_P+\alpha_{K_L}+\alpha_{K_S}-\Theta\alpha_P$, $\beta_2=\alpha_2$ and $\beta_3=\alpha_{K_{NL}}+\alpha_3$.

In order to solve Eq. (21) using VIM, we construct a correction functional, as follows

$$V_{n+1}(t)=V_n(t)+\int_0^t\lambda\left\{\frac{d^2V_n(\tau)}{d\tau^2}+\omega^2V_n(\tau)+\beta_1\tilde{V}_n(\tau)+\beta_2\tilde{V}_n^2(\tau)+\beta_3\tilde{V}_n^3(\tau)-\omega^2\tilde{V}_n(\tau)\right\}d\tau \tag{22}$$

Its stationary conditions can be obtained as follows

$$\lambda''(\tau)\big|_{\tau=t} + \omega^2 \lambda(\tau)\big|_{\tau=t} = 0 \tag{23a}$$

$$1 - \lambda'(\tau)\big|_{\tau=t} = 0 \tag{23b}$$

$$\lambda(\tau)\big|_{\tau=t} = 0 \tag{23c}$$

Thus the Lagrangian multiplier can therefore be identified as

$$\lambda = \frac{1}{\omega}\sin\omega(\tau - t) \tag{24}$$

As a result, we obtain the following iteration formula

$$\begin{aligned} V_{n+1}(t) = V_n(t) + \int_0^t \left(\frac{1}{\omega}\sin\omega(\tau - t)\right) \\ \times\left\{\frac{d^2V_n(\tau)}{d\tau^2} + \omega^2 V_n(\tau) + \beta_1\widetilde{V}_n(\tau) + \beta_2\widetilde{V}_n^2(\tau) + \beta_3\widetilde{V}_n^3(\tau) - \omega^2\widetilde{V}_n(\tau)\right\}d\tau \end{aligned} \tag{25}$$

From the initial conditions in Eq. (18), that we have it in point $t = 0$ an arbitrary initial approximation can be obtained

$$V_0(t) = a\cos(\omega\, t) \tag{26}$$

This initial approximation is a trial function and it is used to obtain a more accurate approximate solution of Eq. (16). Here ω, is the nonlinear frequency. Expanding the non-linear part, we have

$$N[V_0(t)] = \beta_1 a\cos(\omega\, t) + \beta_2(a\cos(\omega\, t))^2 + \beta_3(a\cos(\omega\, t))^3 - \omega^2 a\cos(\omega\, t) \tag{27}$$

Then

$$N[V_0(t)] = \left\{\left(-a\omega^2 + a\beta_1 + \frac{3}{4}\beta_3 a^3\right)\right\}\cos(\omega\, t) + \frac{1}{4}\beta_3 a^3\cos(3\omega\, t) + \frac{1}{2}\beta_2 a^2\cos(2\omega\, t) + \frac{1}{2}\beta_2 a^2 \tag{28}$$

In order to ensure that no secular terms appear in the next iteration, the coefficient of $\cos(\omega\, t)$ must vanish. Therefore

$$\omega = \sqrt{\beta_1 + \frac{3}{4}\beta_3 a^2}, \quad a \neq 0 \tag{29}$$

The nonlinear to the linear frequency ratio can be determined as

$$\frac{\omega_{NL}}{\omega_L} = \sqrt{1 + \frac{3}{4}\frac{\beta_3}{\beta_1}a^2} \tag{30}$$

Using the variational formula (25), we have

Table 2 Temperature dependent coefficients of the constituent materials of the FG beams

Material		P_0	P_{-1}	P_1	P_2	P_3
	E (Pa)	348.43e + 9	0.0	– 3.070e – 4	2.160e – 7	– 8.946e – 11
Si_3N_4	α (K^{-1})	5.8723e – 6	0.0	9.095e – 4	0.0	0.0
	ρ (kg/m^3)	2370.0	0.0	0.0	0.0	0.0
	E (Pa)	201.04e + 9	0.0	3.079e – 4	– 6.534e – 7	0.0
$SuS304$	α (K^{-1})	12.330e – 6	0.0	8.086e – 4	0.0	0.0
	ρ (kg/m^3)	8166.0	0.0	0.0	0.0	0.0

$$V_1(t) = a\cos(\omega\, t) + \int_0^t \left(\frac{1}{\omega}\sin\omega(\tau - t)\right) \left\{\frac{d^2\left(a\cos(\omega\,\tau)\right)}{d\tau^2} + \omega^2\left(a\cos(\omega\,\tau)\right) + \frac{1}{4}\beta_3 a^3\cos(3\omega\,\tau) + \frac{1}{2}\beta_2 a^2\cos(2\omega\,\tau) + \frac{1}{2}\beta_2 a^2\right\} d\tau \tag{31}$$

Then first-order approximate solution is obtained as

$$V_1(t) = \left(a + \frac{\beta_2 a^2}{3\omega^2} - \frac{\beta_3 a^3}{32\omega^2}\right)\cos(\omega\, t) + \left(\frac{\beta_2 a^2}{6\omega^2}\right)\cos(2\omega\, t) + \left(\frac{\beta_3 a^3}{32\omega^2}\right)\cos(3\omega\, t) - \frac{\beta_2 a^2}{2\omega^2} \tag{32}$$

Accordingly, inserting Eq. (32) into Eq. (19) the post-buckling load-deflection can be obtained.

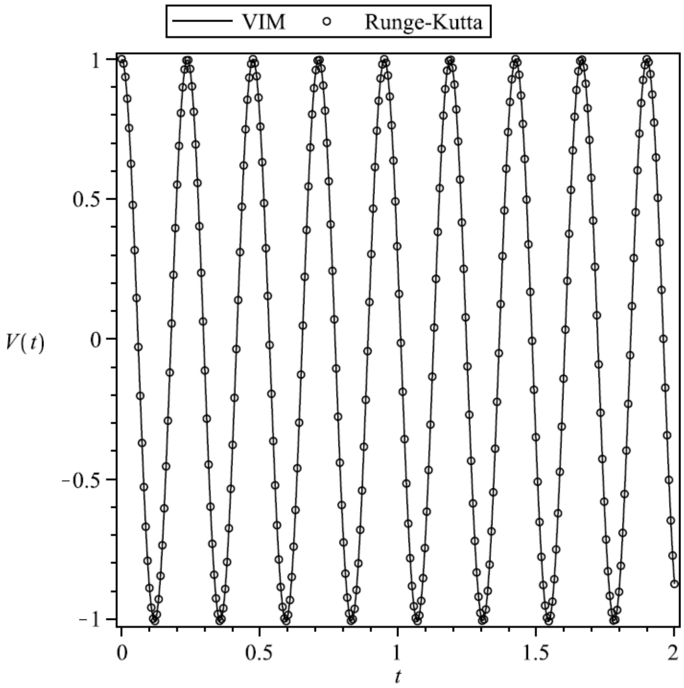
5. Results and discussion

In this section we present the results with VIM, described in the previous section for solving Eq. (16). The temperature coefficients corresponding to Si_3N_4 and $SuS304$ are listed in Table 2. Moreover, for all numerical results reported here, the following values of variables were used unless otherwise indicated by the graphs or tables.

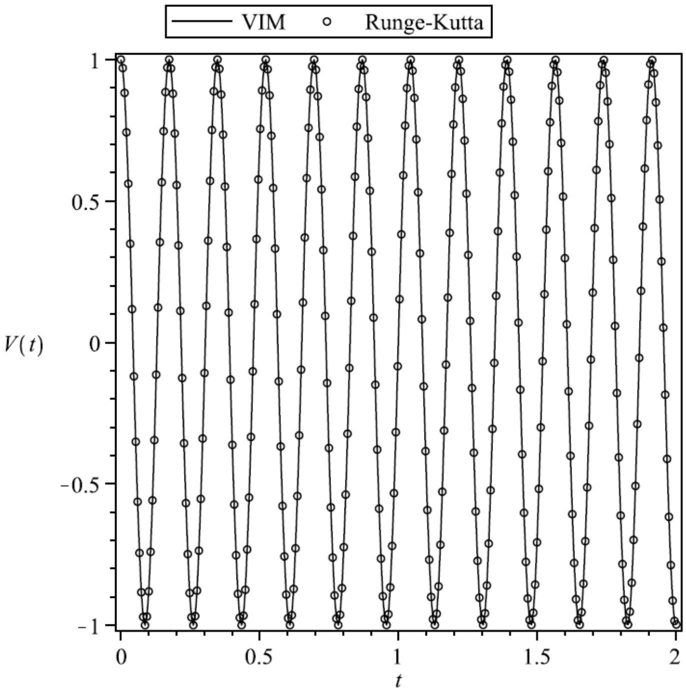
$$a = 1,\; n = 2,\; P = 1,\; k_L = 50,\; k_S = 50,\; k_{NL} = 50,\; L = 1(m),\; b = h = 0.1(m)$$

To test the validity and accuracy of the method used in this study, the variation of the non-dimensional amplitude versus time for a specific value of dimensionless maximum amplitude i.e., $a = 1$ obtained by VIM and well-established Runge-Kutta method are displayed in Fig. 2. This figure shows very good agreement between VIM and numerical solution.

When the beam's temperature difference is dropped i.e., $T = 0$ K the present analytical expression gives frequency ratio (ω_{NL} / ω_L) for FG beams without thermal loading. We did check our solution with published data for frequency ratio for both clamped-clamped (CC) and simply supported (SS) FG beams as shown in Table 3, further vouching for the accuracy of the VIM procedure.



(a)



(b)

Fig. 2 Variation of the non-dimensional amplitude versus *t*: (a) simply supported; (b) clamped-clamped

Table 3 Frequency ratio (ω_{NL} / ω_L) of FG beams without thermal load

	A	ω_{NL} / ω_L Present	Gunda *et al*. (2010)	Ke *et al*. (2010)	Fallah and Aghdam (2011)
SS	0	1.000	1.000	1.000	1.000
	0.5	1.009	1.006	1.007	1.006
	1	1.036	1.031	1.032	1.031
	1.5	1.079	1.072	1.072	1.072
	2	1.137	1.128	1.130	1.128
CC	0	1.000	1.000	1.000	1.000
	0.5	1.014	1.014	1.014	1.013
	1	1.053	1.053	1.053	1.053
	1.5	1.116	1.116	1.115	1.116
	2	1.198	1.198	1.198	1.198

After these verifications, we investigate the influences of thermal effect, foundation parameters, axial force, vibration amplitude, end supports, such as SS and CC, and material inhomogenity on the nonlinear free vibrations and post-buckling behaviors of FG beams. Figs. 3 and 4 demonstrate the effects of linear foundation parameter together with the temperature difference for SS and CC FG beams, respectively.

It can be seen from these figures that all beams exhibit typical hardening behavior i.e. the frequency and post-buckling ratio decreases as the linear foundation parameter is increased. In addition, these figures show that the frequency ratio decreases as the temperature difference of the beam i.e., T increases.

It can be observed from Figs. 5 and 6 that an increase in the value of shearing layer stiffness results in decreasing hardening characteristic of the beam i.e. decrease in the rate of k_s increase in the non-linear frequency and post-buckling strength. Nevertheless, an increase in the value of non-linear foundation parameter results in increasing in the non-linear frequency and post-buckling strength. This interesting behavior is shown in Figs. 7 and 8. Figs. 5-8 show that increasing in the temperature difference causes decrease in the both frequency and post-buckling ratio of the FG beams.

Moreover, the effect of axial force on the nonlinear natural frequency of SS and CC FG beams is presented in Figs. 9 and 10. Results in the figures reveal that as the value of P increases the frequency ratio also increases.

The effect of the dimensionless maximum amplitude on SS and CC beams, illustrated in Figs. 11 and 12, shows an increase in both the frequency and post-buckling ratio with the increase in a and decrease in T.

The influences of material inhomogenity in terms of volume fraction exponent on the frequency and post-buckling ratio are presented in Figs. 13 and 14. It is interesting to note from these figures that the frequency and post-buckling ratio of the beam initially increases, and then decays by increasing in the value of n. It can be clearly seen from these figures that a decrease in the value of temperature difference causes an increase in the values of both frequency and post-buckling ratio of the beam.

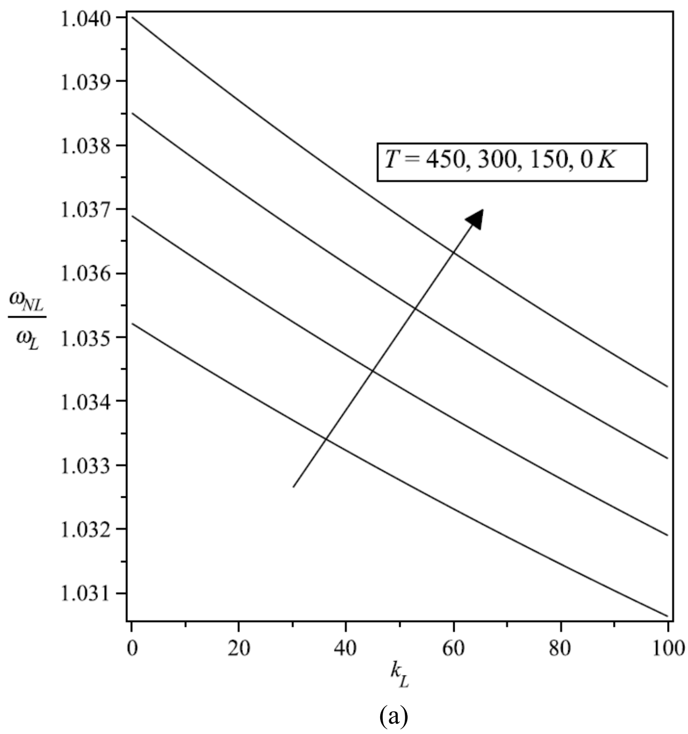
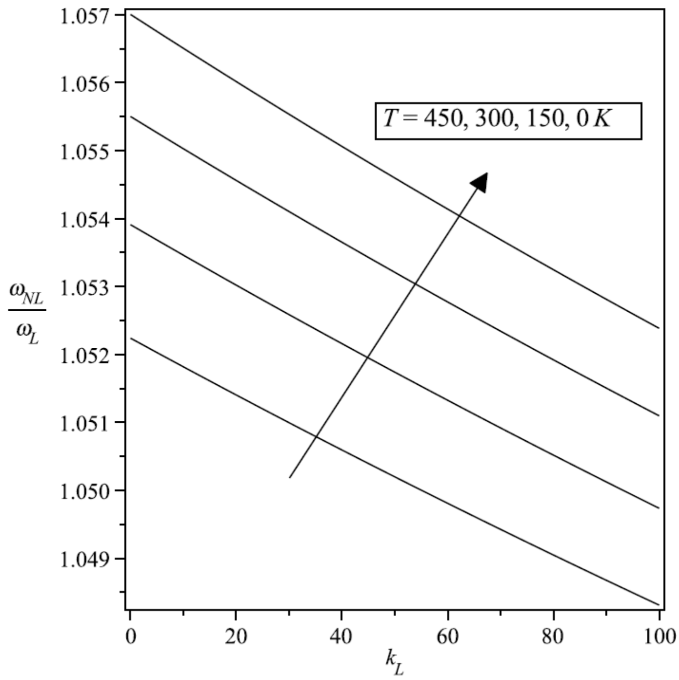
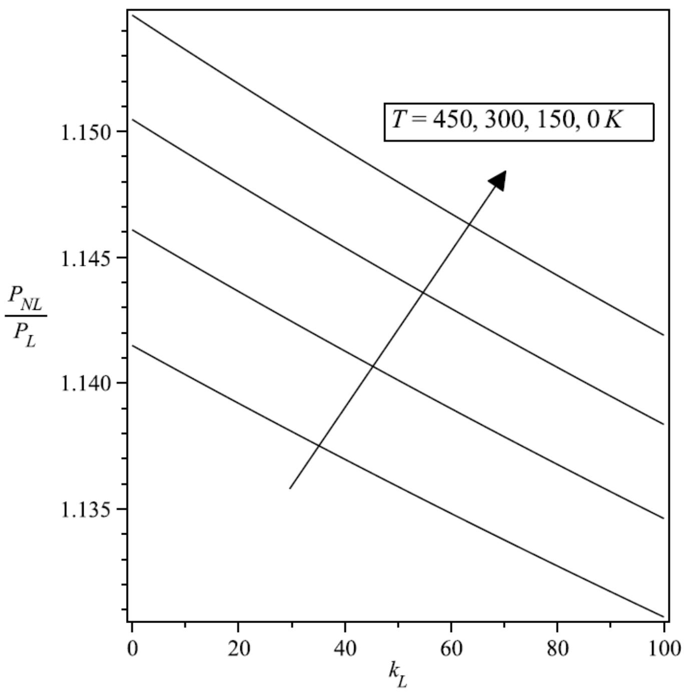


Fig. 3 Effect of the linear foundation stiffness on SS FG beam: (a) frequency ratio; and (b) buckling load ratio

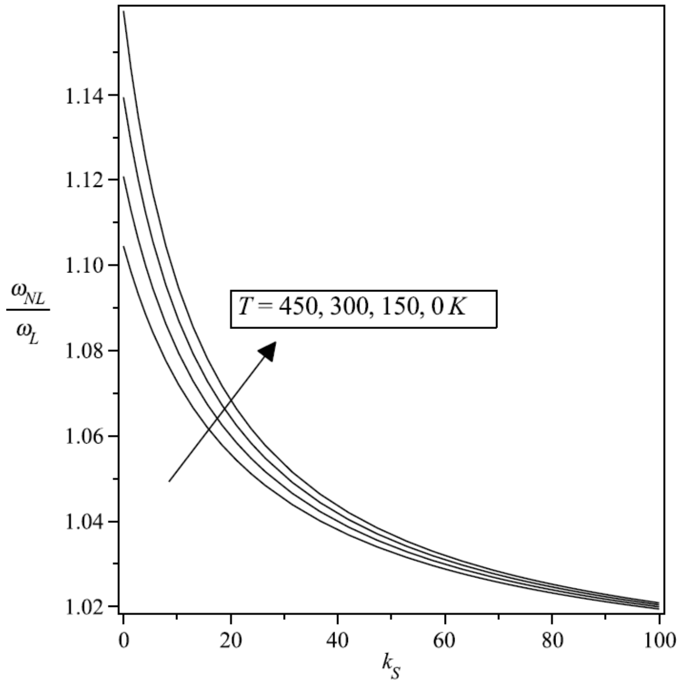


(a)

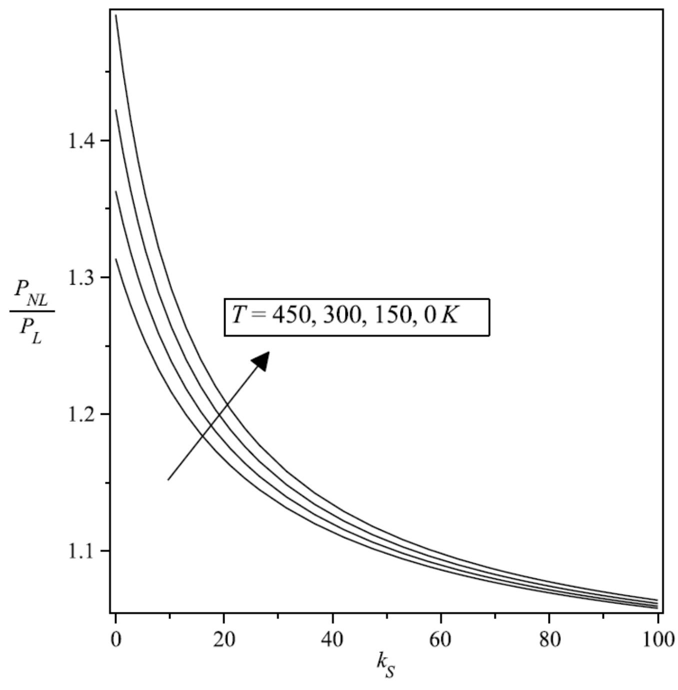


(b)

Fig. 4 Effect of the linear foundation stiffness on CC FG beam: (a) frequency ratio; and (b) buckling load ratio

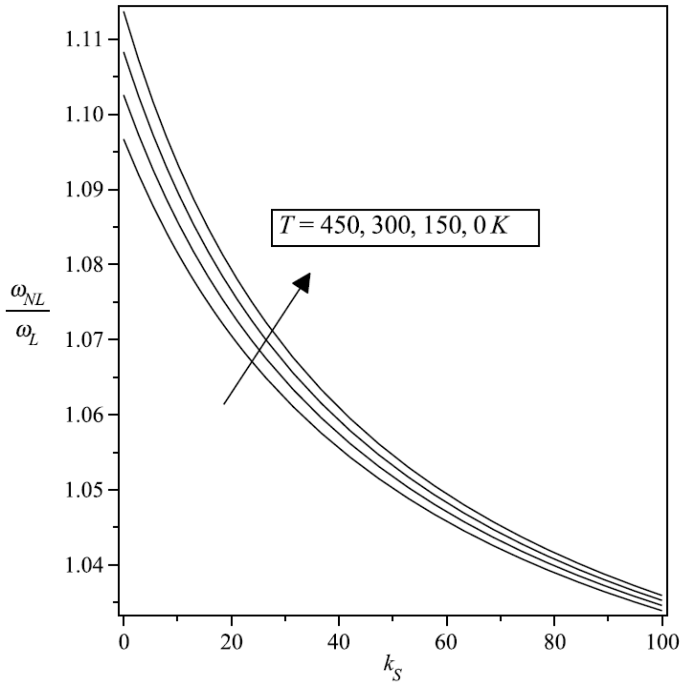


(a)

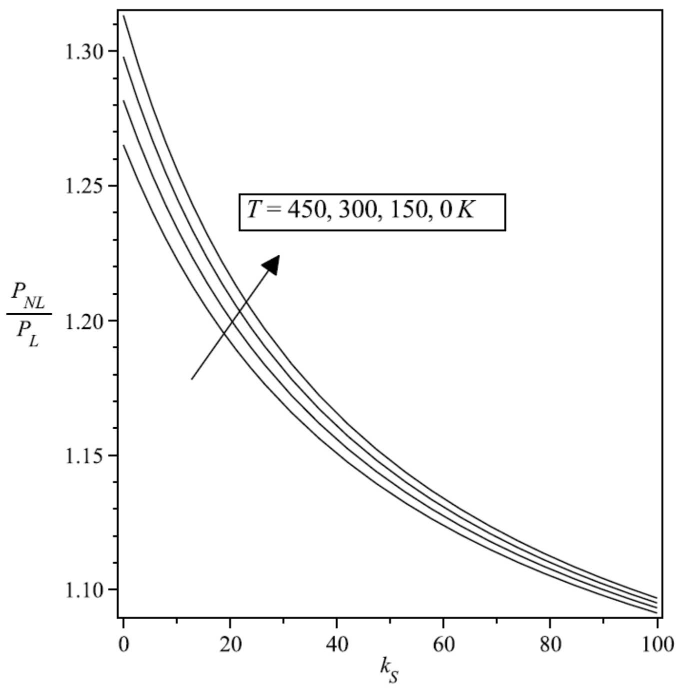


(b)

Fig. 5 Effect of the shearing layer stiffness on SS FG beam: (a) frequency ratio; and (b) buckling load ratio

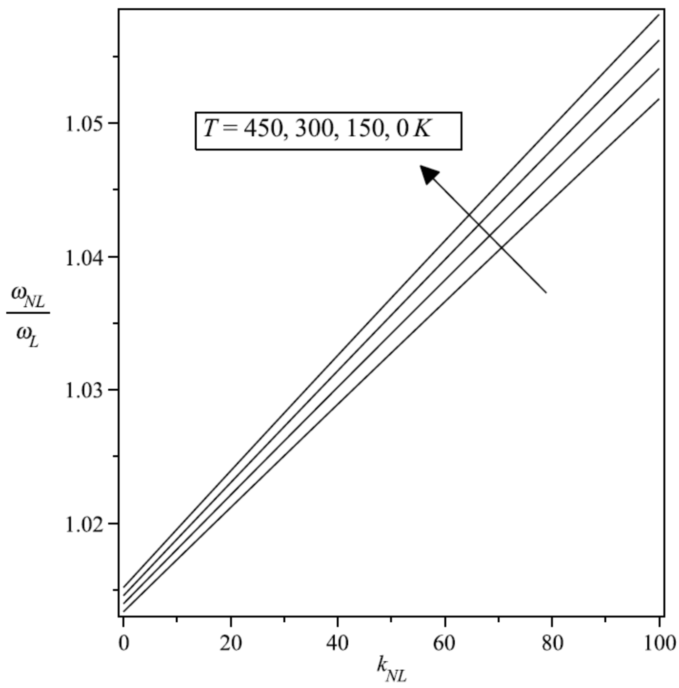


(a)

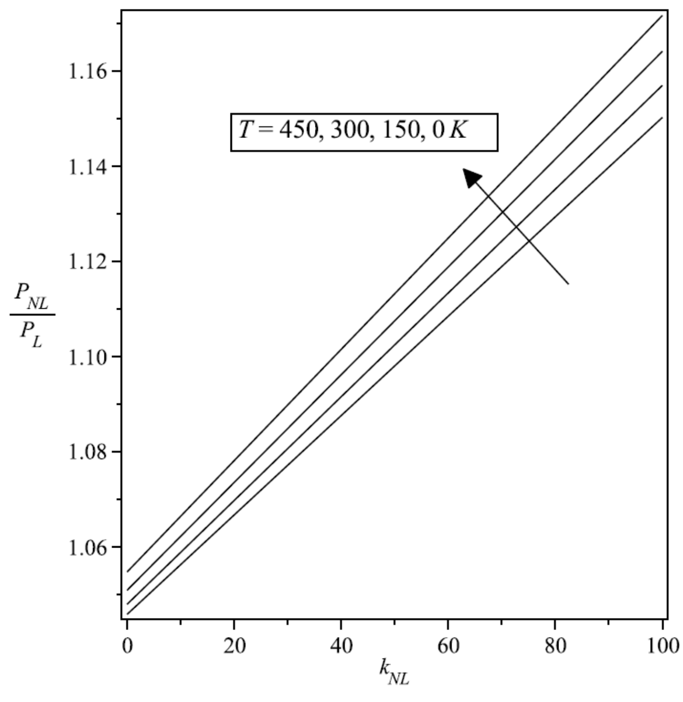


(b)

Fig. 6 Effect of the shearing layer stiffness on CC FG beam: (a) frequency ratio; and (b) buckling load ratio



(a)



(b)

Fig. 7 Effect of the nonlinear foundation stiffness on SS FG beam: (a) frequency ratio; and (b) buckling load ratio

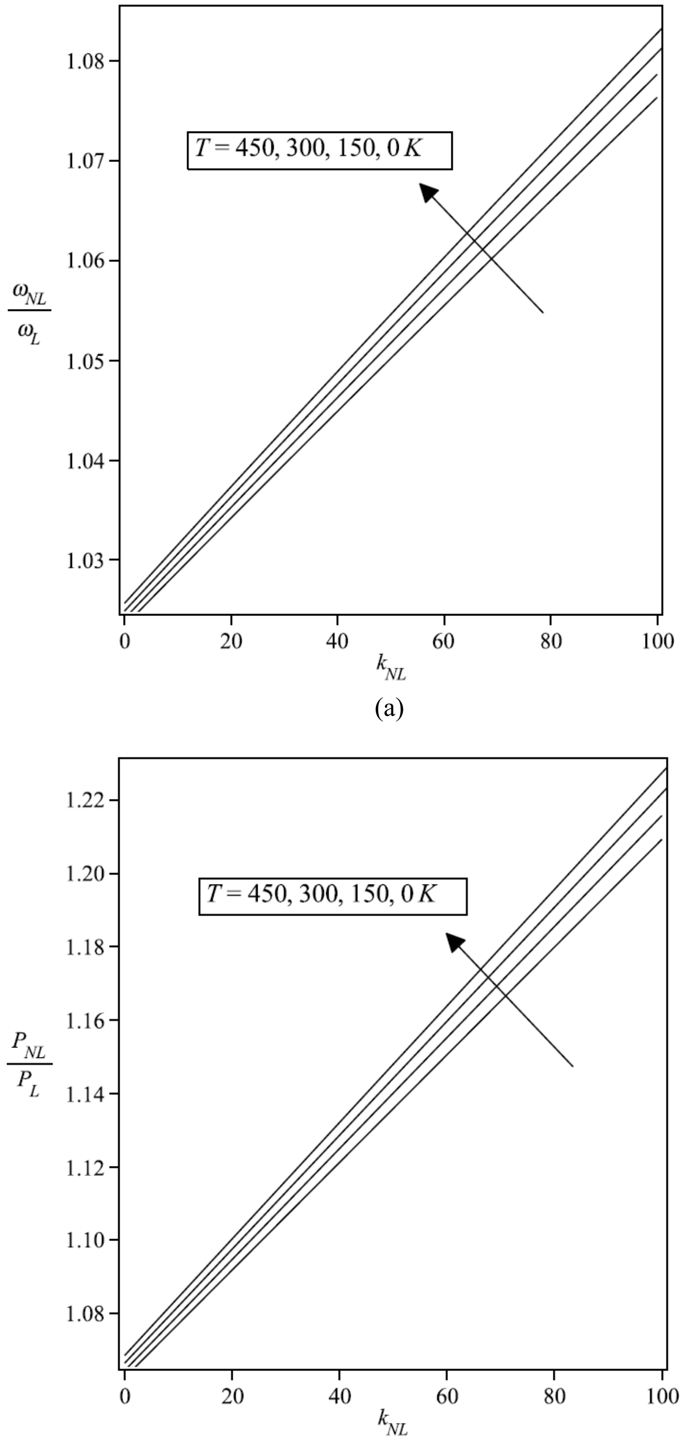


Fig. 8 Effect of the nonlinear foundation stiffness on CC FG beam: (a) frequency ratio; and (b) buckling load ratio

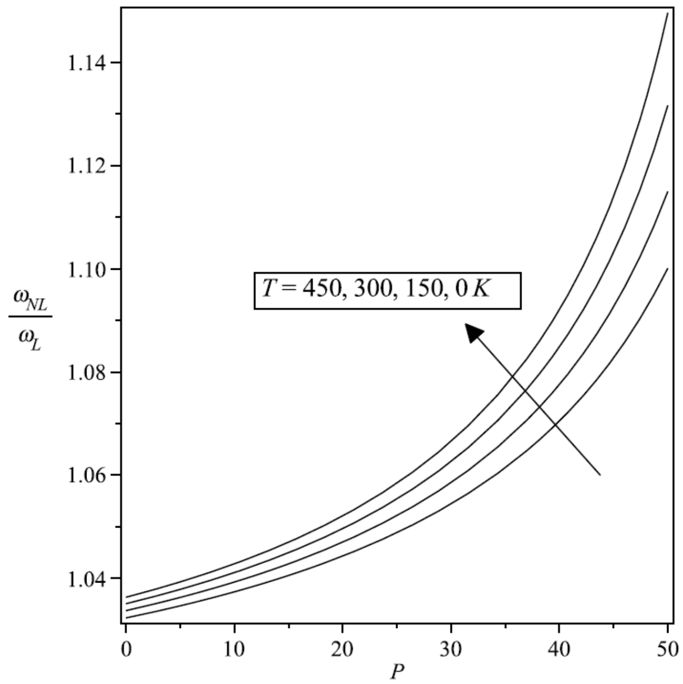


Fig. 9 Effect of the axial load on frequency ratio of SS FG beam

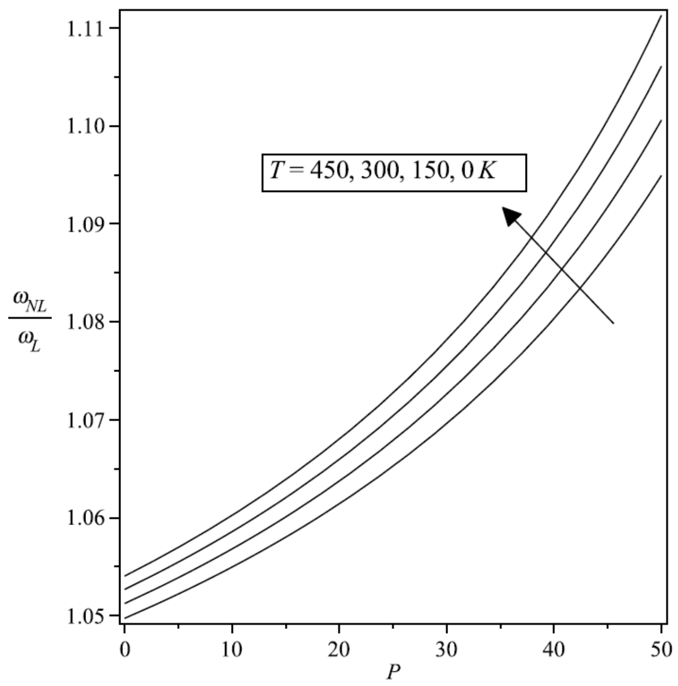


Fig. 10 Effect of the axial load on frequency ratio of CC FG beam

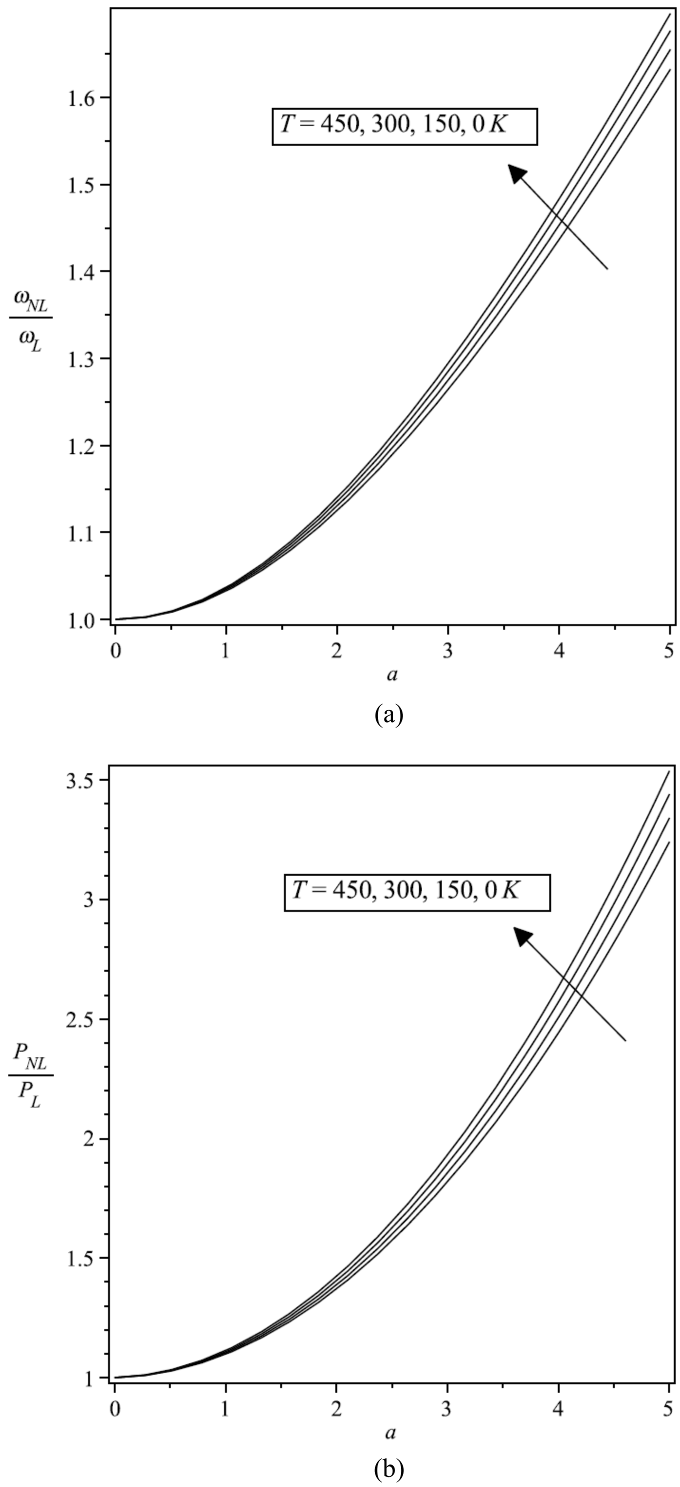
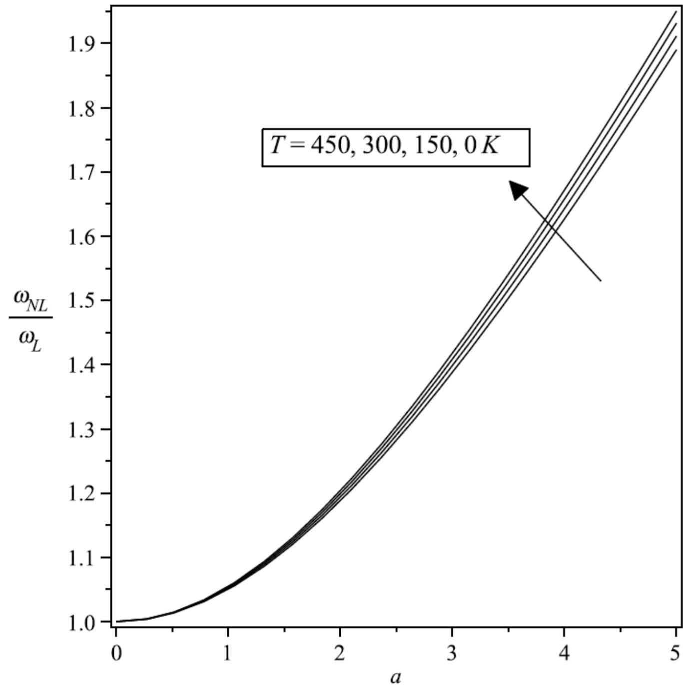
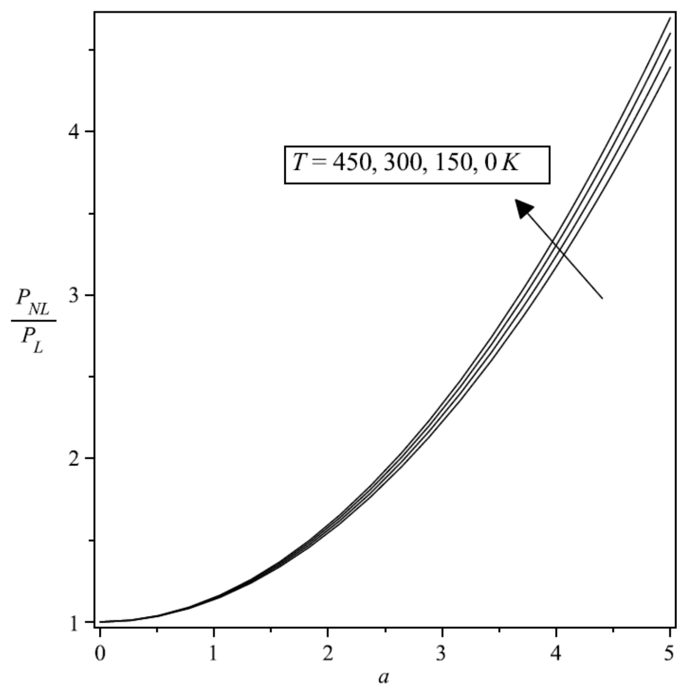


Fig. 11 Effect of the dimensionless maximum amplitude on SS FG beam: (a) frequency ratio; and (b) buckling load ratio



(a)



(b)

Fig. 12 Effect of the dimensionless maximum amplitude on CC FG beam: (a) frequency ratio; and (b) buckling load ratio

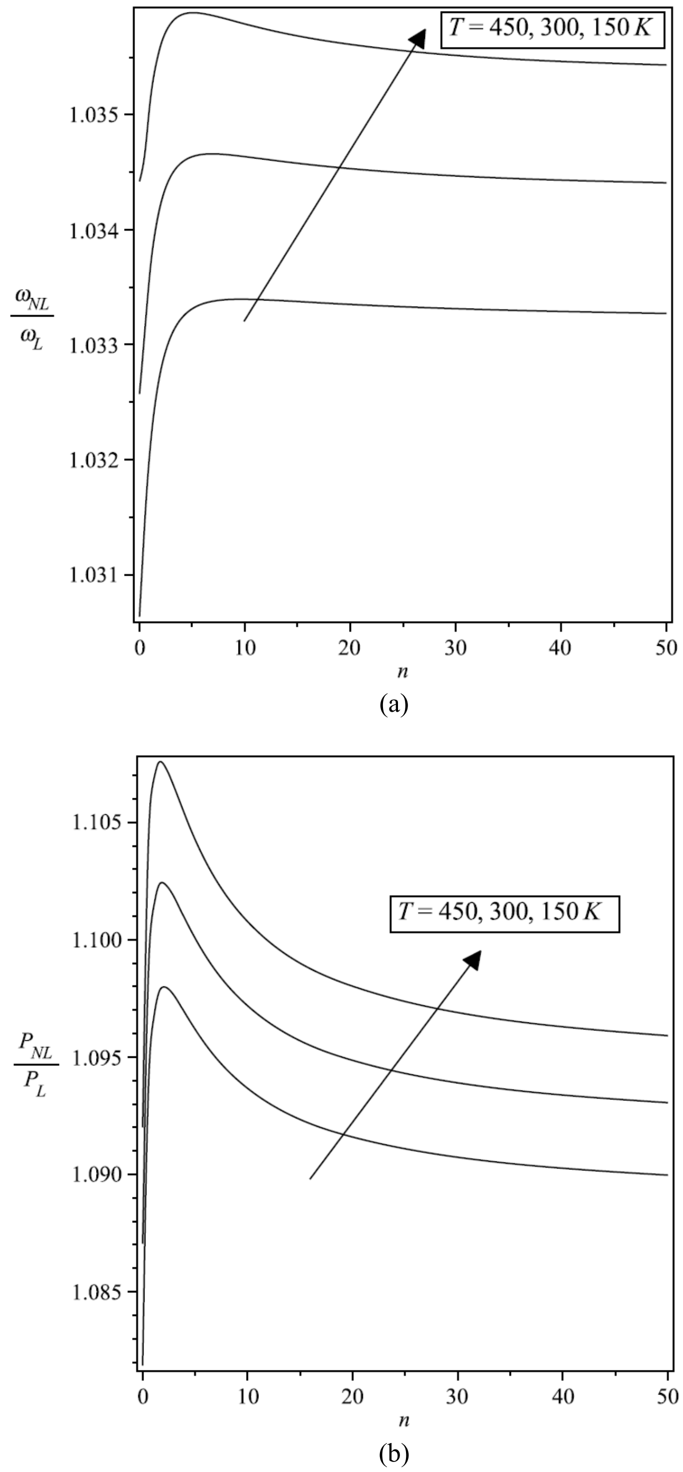
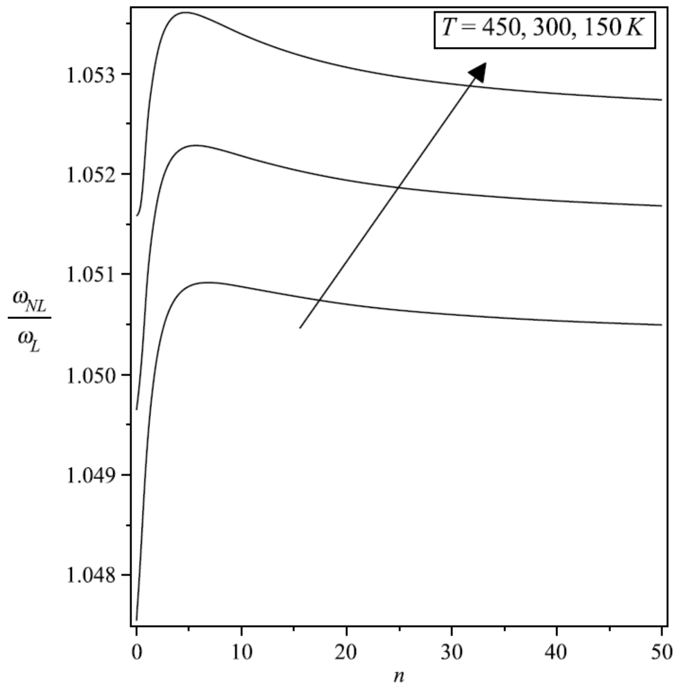
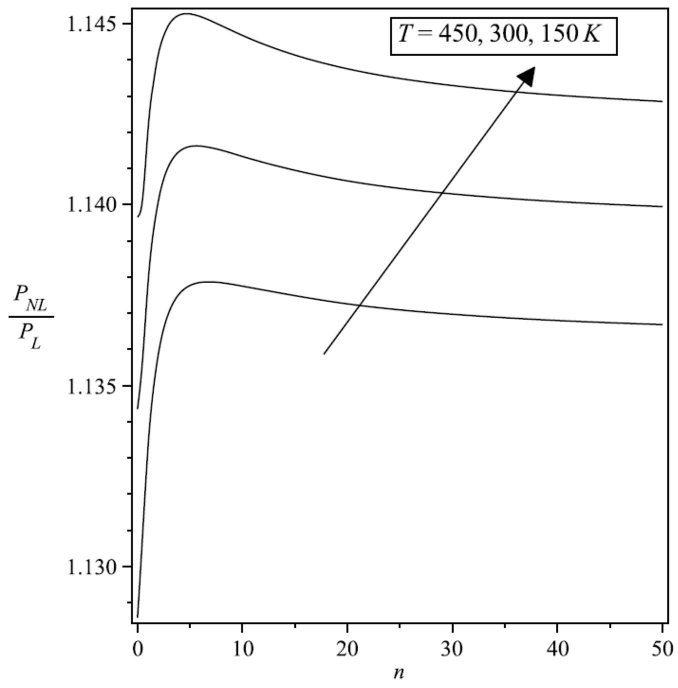


Fig. 13 Effect of the volume fraction exponent on SS FG beam: (a) frequency ratio; and (b) buckling load ratio



(a)



(b)

Fig. 14 Effect of the volume fraction exponent on CC FG beam: (a) frequency ratio; and (b) buckling load ratio

6. Conclusions

Large amplitude vibration and post-buckling behavior of FG beams rest on nonlinear elastic foundation subjected to thermo-mechanical loading with simply supported and clamped-clamped boundary conditions are investigated. This study is within the framework of Euler-Bernoulli beam theory and von-Karman type displacement-strain relationship. The convergence and accuracy of the method are investigated by comparing the results with those available from the literature and well-established Runge-Kutta numerical method. The effects of thermal loading, foundation parameters, axial force, vibration amplitude, end supports and material inhomogenity on the nonlinear dynamic behavior of the FG beams are discussed in detail. As a result, an increase in the values of linear and shear layers of foundation parameters decreases the frequency ratio of the FG beam. But, as the nonlinear foundation stiffness gets stronger the frequency ratio and post-buckling load are progressively increase. Moreover, the thermal loading was found to be significant when investigating the vibrations and post-buckling that take place in the vicinity of a deflected position.

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