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Study of complex nonlinear vibrations by means of accurate analytical approach

Mahmoud Bayat [*1], Iman Pakar [2] and Mahdi Bayat [1]

[1] *Department of Civil Engineering, College of Engineering, Mashhad Branch, Islamic Azad University, Mashhad, Iran*
[2] *Young Researchers and Elites Club, Mashhad Branch, Islamic Azad University, Mashhad, Iran*

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Abstract. In the current study, we consider a new class of analytical periodic solution for free nonlinear vibration of mechanical systems. Hamiltonian approach is applied to analyze nonlinear problems which occur in dynamics. The proposed method doesn't have the limitations of the classical methods and leads us to a high accurate solution by only one iteration. Two well known examples are studied to show the convenience and effectiveness of this approach. Runge-Kutta's algorithm is also applied and the results of it are compared with the Hamiltonian approach. High accuracy of the proposed approach reveals that the Hamiltonian approach can be very useful for other nonlinear practical problems in engineering.

Keywords: non-linear vibration; Hamiltonian approach; Runge-Kutta's algorithm

1. Introduction

Many engineering problems can be parted into linear or nonlinear according to the type of differential equations of motion. Nonlinear oscillators systems are used in many subjects of mechanical and civil engineering. In recent years many researchers have been focused on obtaining new approximations for nonlinear problems because of advantages of numerical methods. In fact, it is too difficult to find an exact solution for nonlinear governing equations.

Perturbation technique is one of the well-known analytical methods. They are not applicable for strongly nonlinear equations, and to eliminate the imperfections, novel techniques have been developed and are well documented in open literature, for instance, for instance; Homotopy perturbation method (Bayat *et al*. 2013a, 2014a), Hamiltonian approach (He 2010, Xu 2010, Bayat *et al*. 2014b, c, d, e, f, g, Bayat and Pakar 2013b), Energy balance method (Jamshidi and Ganji 2010, Mehdipour 2010), Variational iteration method (Dehghan and Tatari 2008), Amplitude frequency formulation (He 2008), Max-Min approach (Shen and Mo 2009, Zeng and Lee 2009), Variational approach (He 2007, Bayat and Pakar 2012, Bayat *et al*. 2012, Bayat and Pakar 2013a, Shahidi *et al*. 2011, Pakar and Bayat 2013), and the other analytical and numerical (Bayat and Abdollahzade 2011, Pakar *et al*. 2011, 2014a, b, Xu and Zhang 2009, Alicia *et al*. 2010, Kuo and Lo 2009, Wu 2011, Odibat *et al*. 2008, Liu *et al*. 2013, Rajasekaran 2013, Akgoz and Civalek

∗Corresponding author, Researcher, E-mail: mbayat14@yahoo.com

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2013).

In this paper, we introduce the solution procedure of Hamiltonian approach and then we apply the method for two strong nonlinear problems. We have presented some comparisons between analytical and numerical solutions to show the accuracy of this new approach. It has been indicated that the numerical results of other methods are trigger same conclusion; while Hamiltonian approach is much easier, more convenient and more efficient than other approaches

2. Basic idea of Hamiltonian approach

Previously, He (2002) had introduced the Energy Balance method based on collocation and the Hamiltonian. This approach is very simple but strongly depends upon the chosen location point. Recently, He (2010) has proposed the Hamiltonian approach to overcome the shortcomings of the energy balance method. This approach is a kind of energy method with a vast application in conservative oscillatory systems. In order to clarify this approach, consider the following general oscillator

$$\ddot{\theta} + f(\theta,\dot{\theta},\ddot{\theta}) = 0 \tag{1}$$

With initial conditions

$$\theta(0) = A, \quad \dot{\theta}(0) = 0. \tag{2}$$

Oscillatory systems contain two important physical parameters, i.e., the frequency ω and the amplitude of oscillation A. It is easy to establish a variational principle for Eq. (1), which reads

$$J(\theta) = \int_0^{T/4} \left\{ -\frac{1}{2}\dot{\theta}^2 + F(\theta) \right\} dt \tag{3}$$

where T is period of the nonlinear oscillator, $\partial F / \partial \theta = f$.

In Eq. (3), $\frac{1}{2}\dot{\theta}^2$ is kinetic energy and $F(\theta)$ potential energy, so the Eq. (3) is the least Lagrangian action, from which we can immediately obtain its Hamiltonian, which reads

$$H(\theta) = \frac{1}{2}\dot{\theta}^2 + F(\theta) = \text{constant} \tag{4}$$

From Eq. (4), we have

$$\frac{\partial H}{\partial A} = 0 \tag{5}$$

Introducing a new function, $\overline{H}(\theta)$, defined as

$$\overline{H}(u) = \int_0^{T/4} \left\{ \frac{1}{2}\dot{\theta}^2 + F(\theta) \right\} dt = \frac{1}{4} TH \tag{6}$$

Eq. (5) is, then, equivalent to the following one

$$\frac{\partial}{\partial A}\left(\frac{\partial \overline{H}}{\partial T}\right)=0 \tag{7}$$

or

$$\frac{\partial}{\partial A}\left(\frac{\partial \overline{H}}{\partial (1/\omega)}\right)=0 \tag{8}$$

From Eq. (8) we can obtain approximate frequency–amplitude relationship of a nonlinear oscillator.

3. Basic idea of Runge-Kutta's method

For the numerical approach to verify the analytic solution, the fourth RK (Runge-Kutta) method has been used. This iterative algorithm is written in the form of the following formulation

$$\dot{\theta}=f(t,\theta), \quad \theta(t_0)=\theta_0 \tag{9}$$

θ is an unknown function of time t which we would like to approximate. Then RK4 method is given for this problem as below

$$\begin{aligned} \theta_{n+1}&=\theta_n+\frac{1}{6}h(k_1+2k_2+2k_3+k_4),\\ t_{n+1}&=t_n+h. \end{aligned} \tag{10}$$

for $n = 0, 1, 2, 3, \ldots$, using

$$\begin{aligned} k_1&=f\left(t_n,\ \theta_n\right),\\ k_2&=f\left(t_n+\frac{1}{2}h,\ \theta_n+\frac{1}{2}hk_1\right),\\ k_3&=f\left(t_n+\frac{1}{2}h,\ \theta_n+\frac{1}{2}hk_3\right),\\ k_4&=f\left(t_n+h,\ \theta_n+hk_3\right). \end{aligned} \tag{11}$$

where θ_{n+1} is the RK4 approximation of $\theta(t_{n+1})$. The fourth-order Runge-Kutta method requires four evaluations of the right hand side per step h.

4. Applications

In order to assess the advantages and the accuracy of the Hamiltonian approach, we will consider the following examples:

4.1 Example 1

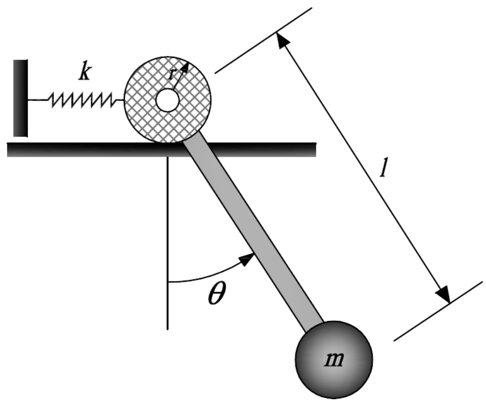


Fig. 1 Pendulum attached to rolling wheels that are restrained by a spring

An example of a single degree of freedom conservative system has been considered that is described by an equation as follows. A rigid rod is rigidly attached to the axle as shown in Fig. 1. The wheels roll without slip as the pendulum swings back and forth. The wheel is restrained by a spring which is fixed to a wall on the other side .Only the ball on the end of the pendulum has appreciable mass and it may be considered a particle. The equation governing would be (Nayfeh 1993)

$$m(l^2+r^2-2rl\cos(\theta))\ddot{\theta}+mrl\sin(\theta)\dot{\theta}^2+mgl\sin(\theta)+kr^2\theta=0 \tag{12}$$

With initial conditions

$$\theta(0)=A, \quad \dot{\theta}(0)=0. \tag{13}$$

Here, by using the Taylor's series expansion for $\cos(\theta(t))$, $\sin(\theta(t))$ and by some manipulation in Eq. (12) we can re-write Eq. (12) in the following form

$$\left(\alpha+\beta-2\delta\left(1-\frac{1}{2}\theta^2+\frac{1}{24}\theta^4\right)\right)\ddot{\theta}+\delta\sin\left(\theta-\frac{1}{6}\theta\right)\dot{\theta}^2+\lambda\sin\left(\theta-\frac{1}{6}\theta\right)+\mu\theta \tag{14}$$

where

$$\alpha=ml^2,\ \beta=mr^2,\ \delta=mrl,\ \lambda=mgl,\ \mu=kr^2 \tag{15}$$

The Hamiltonian of Eq. (14) is constructed as

$$H=-\frac{1}{2}(\alpha+\beta-2\delta)\dot{\theta}^2-\left(\delta\theta^2-\frac{1}{12}\delta\theta^4\right)\dot{\theta}^2+\frac{1}{2}(\mu+\lambda)\theta^2-\frac{1}{24}\lambda\theta^4 \tag{16}$$

Integrating Eq. (16) with respect to t from 0 to $T/4$, we have

$$\overline{H}=\int_0^{T/4}\left(-\frac{1}{2}(\alpha+\beta-2\delta)\dot{\theta}^2-\left(\delta\theta^2-\frac{1}{12}\delta\theta^4\right)\dot{\theta}^2+\frac{1}{2}(\mu+\lambda)\theta^2-\frac{1}{24}\lambda\theta^4\right)dt \tag{17}$$

Assume that the solution can be expressed as

$$\theta(t) = A\cos(\omega t) \tag{18}$$

Substituting Eq. (18) into Eq. (17), we obtain

$$\begin{aligned}
\overline{H} &= \int_0^{T/4} \left(\begin{array}{l} -\frac{1}{2}(\alpha+\beta-2\delta)A^2\omega^2\sin^2(\omega t) \\ -\left(\delta A^2\cos(\omega t)-\frac{1}{12}\delta A^4\cos^4(\omega t)\right)A^2\omega^2\sin^2(\omega t) \\ +\frac{1}{2}(\mu+\lambda)A^2\cos^2 t-\frac{1}{24}\lambda A^4\cos^4(\omega t) \end{array} \right) dt \\
&= \int_0^{\pi/2} \left(\begin{array}{l} -\frac{1}{2}(\alpha+\beta-2\delta)A^2\omega\sin^2 t-\left(\delta A^2\cos^2 t-\frac{1}{12}\delta A^4\cos^4 t\right)A^2\omega\sin^2 t \\ +\frac{1}{2\omega}(\mu+\lambda)A^2\cos^2 t-\frac{1}{24\omega}\lambda A^4\cos^4 t \end{array} \right) dt \\
&= -\frac{1}{8}\omega\alpha\pi A^2-\frac{1}{8}\omega\beta\pi A^2+\frac{1}{4}\omega\delta\pi A^2-\frac{1}{32}\omega\delta\pi A^4+\frac{1}{768}\omega\delta\pi A^6 \\
&\quad +\frac{1}{8}\frac{\pi}{\omega}\lambda A^2+\frac{1}{8}\frac{\pi}{\omega}\mu A^2-\frac{1}{128}\frac{\pi}{\omega}\lambda A^4
\end{aligned} \tag{19}$$

$$\begin{aligned}
\frac{\partial}{\partial A}\left(\frac{\partial \overline{H}}{\partial(1/\omega)}\right) &= -\frac{1}{4}\alpha\pi\omega^2 A-\frac{1}{4}\beta\pi\omega^2 A+\frac{1}{2}\delta\pi\omega^2 A-\frac{1}{8}\delta\pi\omega^2 A^3 \\
&\quad +\frac{1}{128}\delta\pi\omega^2 A^5+\frac{1}{4}\lambda\pi A+\frac{1}{4}\mu\pi A-\frac{1}{32}\lambda\pi A^3 \\
&= 0
\end{aligned} \tag{20}$$

Solving the above equation, an approximate frequency as a function of amplitude equal to

$$\omega = \frac{2\sqrt{-8\lambda-8\mu+\lambda A^2}}{\sqrt{-32\alpha-32\beta+64\delta-16\delta A^2+\delta A^4}} \tag{21}$$

By substituting Eq. (15) in to Eq. (21) we have

$$\omega_{HA} = \frac{2\sqrt{(A^2-8)m\,\mathrm{lg}-8kr^2}}{\sqrt{-32ml^2-32mr^2+(A^2-8)^2 mrl}} \tag{22}$$

According to Eqs. (18) and (22), we can obtain the following approximate solution

$$\theta(t) = A\cos\left(\frac{2\sqrt{(A^2-8)mlg - 8kr^2}}{\sqrt{-32ml^2 - 32mr^2 + (A^2-8)^2 mrl}}t\right) \tag{23}$$

4.2 Example 2

In this problem we have a rigid frame (Fig. 2) which is forced to rotate at the fixed rate Ω. While the frame rotates, the simple pendulum oscillates**.**The governing equation is

$$\ddot{\theta} + (1-\beta\cos\theta)\sin\theta = 0, \quad \theta(0) = A, \quad \dot{\theta}(0) = 0 \tag{24}$$

where

$$\beta = \frac{\Omega^2 r}{g} < 1$$

by using the Taylor's series expansion for $\cos(\theta(t))$, $\sin(\theta(t))$ and applying them in Eq. (24) we can re-write Eq. (24) in the following form

$$\ddot{\theta} + \left(1-\beta\left(1-\frac{1}{2}\theta^2+\frac{1}{24}\theta^4\right)\right)\left(\theta-\frac{1}{6}\theta^3\right) = 0, \quad \theta(0) = A, \quad \dot{\theta}(0) = 0 \tag{25}$$

The Hamiltonian of Eq. (25) is constructed as

$$H = -\frac{1}{2}\dot{\theta}^2 + \frac{1}{2}(1-\beta)\theta^2 + \left(\frac{1}{6}\beta - \frac{1}{24}\right)\theta^4 - \frac{1}{48}\beta\theta^6 + \frac{1}{1152}\beta\theta^8 \tag{26}$$

Integrating Eq. (26) with respect to t from 0 to $T/4$, we have

$$\overline{H} = \int_0^{T/4}\left(-\frac{1}{2}\dot{\theta}^2 + \frac{1}{2}(1-\beta)\theta^2 + \left(\frac{1}{6}\beta - \frac{1}{24}\right)\theta^4 - \frac{1}{48}\beta\theta^6 + \frac{1}{1152}\beta\theta^8\right)dt \tag{27}$$

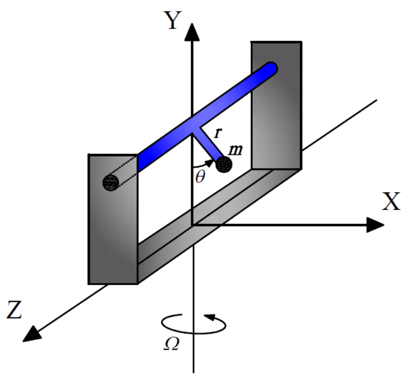


Fig. 2 Rotating frame connected to a pendulum

Table 1 Comparison of time history response of Hamiltonian approach with Runge-Kutta (example 1)

Time	Case 1			Time	Case 2		
	HA	RK4	Error		HA	RK4	Error
0	0.5236	0.5236	0.0000	0	0.7854	0.7854	0.0000
0.05	0.4974	0.4995	0.0043	0.1	0.5934	0.6203	0.0433
0.1	0.4213	0.4281	0.0157	0.2	0.1114	0.1230	0.0938
0.15	0.3031	0.3126	0.0304	0.3	-0.4251	-0.4651	0.0860
0.2	0.1545	0.1613	0.0419	0.4	-0.7538	-0.7610	0.0095
0.25	-0.0096	-0.0105	0.0860	0.5	-0.7141	-0.7239	0.0136
0.3	-0.1727	-0.1808	0.0451	0.6	-0.3253	-0.3509	0.0730
0.35	-0.3185	-0.3286	0.0306	0.7	0.2224	0.2604	0.1458
0.4	-0.4324	-0.4391	0.0151	0.8	0.6615	0.6871	0.0372
0.45	-0.5030	-0.5049	0.0037	0.9	0.7772	0.7771	0.0001
0.5	-0.5232	-0.5233	0.0000	1	0.5130	0.5359	0.0427
0.55	-0.4911	-0.4934	0.0048	1.1	-0.0019	-0.0209	0.9080
0.6	-0.4097	-0.4165	0.0162	1.2	-0.5159	-0.5621	0.0822
0.65	-0.2873	-0.2961	0.0299	1.3	-0.7777	-0.7812	0.0044
0.7	-0.1361	-0.1414	0.0373	1.4	-0.6594	-0.6696	0.0153
0.75	0.0287	0.0314	0.0859	1.5	-0.2188	-0.2218	0.0137
0.8	0.1907	0.2001	0.0471	1.6	0.3288	0.3855	0.1472
0.85	0.3335	0.3440	0.0304	1.7	0.7157	0.7363	0.0281
0.9	0.4429	0.4494	0.0143	1.8	0.7527	0.7520	0.0009
0.95	0.5080	0.5097	0.0032	1.9	0.4218	0.4338	0.0277
1	0.5222	0.5222	0.0000	2	-0.1152	-0.1636	0.2956

*Case 1: $m = 10$, $l = 1.5$, $r = 0.5$, $g = 10$, $A = \pi/6$, $k = 1200$
*Case 2: $m = 5$, $l = 0.8$, $r = 0.3$, $g = 10$, $A = \pi/4$, $k = 500$

Table 2 Comparison of time history response of Hamiltonian approach with Runge-Kutta (example 2)

Time	Case 1			Time	Case 2		
	HA	RK4	Error		HA	RK4	Error
0	1.0472	1.0472	0.0000	0	0.5236	0.5236	0.0000
0.5	0.9706	0.9682	0.0024	1	0.4762	0.4703	0.0125
1	0.7519	0.7452	0.0090	2	0.3426	0.3293	0.0403
1.5	0.4233	0.4163	0.0167	3	0.1469	0.1394	0.0537
2	0.0326	0.0324	0.0081	4	-0.0754	-0.0666	0.1319
2.5	-0.3627	-0.3556	0.0200	5	-0.2840	-0.2653	0.0705
3	-0.7051	-0.6972	0.0113	6	-0.4412	-0.4283	0.0302
3.5	-0.9442	-0.9407	0.0037	7	-0.5185	-0.5166	0.0036

Table 2 Continued

Time	Case 1			Time	Case 2		
	HA	RK4	Error		HA	RK4	Error
4	-1.0452	-1.0450	0.0002	8	-0.5019	-0.5011	0.0016
4.5	-0.9932	-0.9918	0.0014	9	-0.3944	-0.3872	0.0187
5	-0.7959	-0.7903	0.0070	10	-0.2155	-0.2103	0.0250
5.5	-0.4821	-0.4755	0.0139	11	0.0024	-0.0071	1.3435
6	-0.0978	-0.0970	0.0079	12	0.2199	0.1969	0.1169
6.5	0.3008	0.2937	0.0242	13	0.3976	0.3767	0.0555
7	0.6554	0.6465	0.0139	14	0.5032	0.4961	0.0143
7.5	0.9141	0.9094	0.0052	15	0.5178	0.5190	0.0024
8	1.0391	1.0386	0.0004	16	0.4386	0.4371	0.0034

*Case 1: $A = \pi/3, \beta = 0.5$
*Case 2: $A = \pi/6, \beta = 0.9$

Assume that the solution can be expressed as

$$\theta(t) = A\cos(\omega t) \tag{28}$$

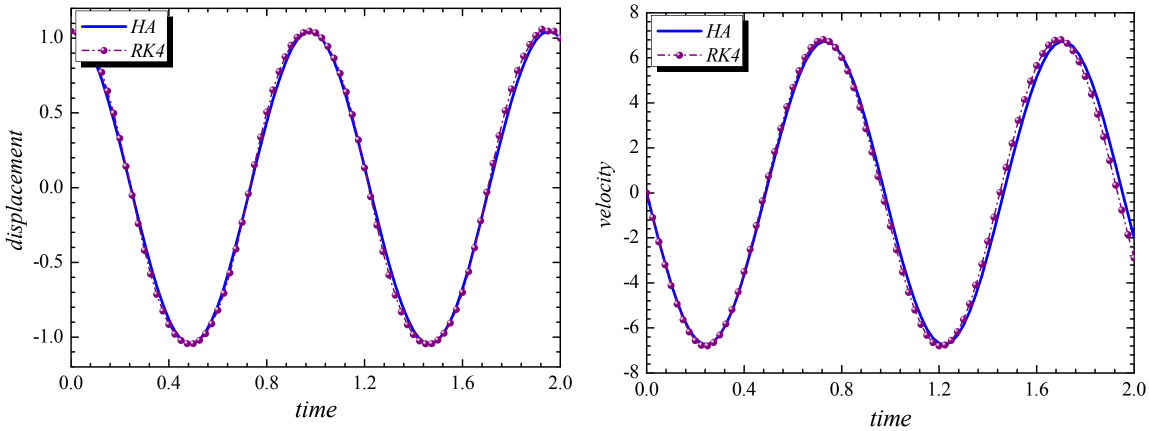
Substituting Eq. (28) into Eq. (27), we obtain

$$\begin{aligned}
\overline{H} &= \int_0^{T/4} \begin{pmatrix} -\frac{1}{2}A^2\omega^2\sin^2(\omega t) + \frac{1}{2}(\omega t) + \left(\frac{1}{6}\beta - \frac{1}{24}\right)A^4\cos^4(\omega t) \\ -\frac{1}{48}\beta A^6\cos^6(\omega t) + \frac{1}{1152}\beta A^8\cos^8(\omega t) \end{pmatrix} dt \\
&= \int_0^{\pi/2} \begin{pmatrix} -\frac{1}{2}A^2\omega\sin^2 t + \frac{1}{2\omega}(1-\beta)A^2\cos^2 t + \left(\frac{1}{6}\beta - \frac{1}{24}\right)\frac{A^4}{\omega}\cos^4 t \\ -\frac{1}{48\omega}\beta A^6\cos^6 t + \frac{1}{1152\omega}\beta A^8\cos^8 t \end{pmatrix} dt \\
&= -\frac{1}{8}\omega\pi A^2 - \frac{1}{8}\frac{\pi}{\omega}\beta A^2 + \frac{1}{8}\frac{\pi}{\omega}A^2 - \frac{1}{128}\frac{\pi}{\omega}A^4 + \frac{1}{32}\frac{\pi}{\omega}\beta A^4 - \frac{5}{1536}\frac{\pi}{\omega}\beta A^6 + \frac{35}{294912}\frac{\pi}{\omega}\beta A^8
\end{aligned} \tag{29}$$

Setting

$$\begin{aligned}
\frac{\partial}{\partial A}\left(\frac{\partial \overline{H}}{\partial(1/\omega)}\right) = &-\frac{1}{4}\pi\omega^2 A - \frac{1}{4}\beta\pi A + \frac{1}{4}\pi A + \frac{1}{8}\beta A^3 \\
&-\frac{1}{32}\pi A^3 - \frac{5}{256}\beta\pi A^5 + \frac{35}{36864}\pi\beta A^7
\end{aligned} \tag{30}$$

Solving the above equation, an approximate frequency as a function of amplitude equal to



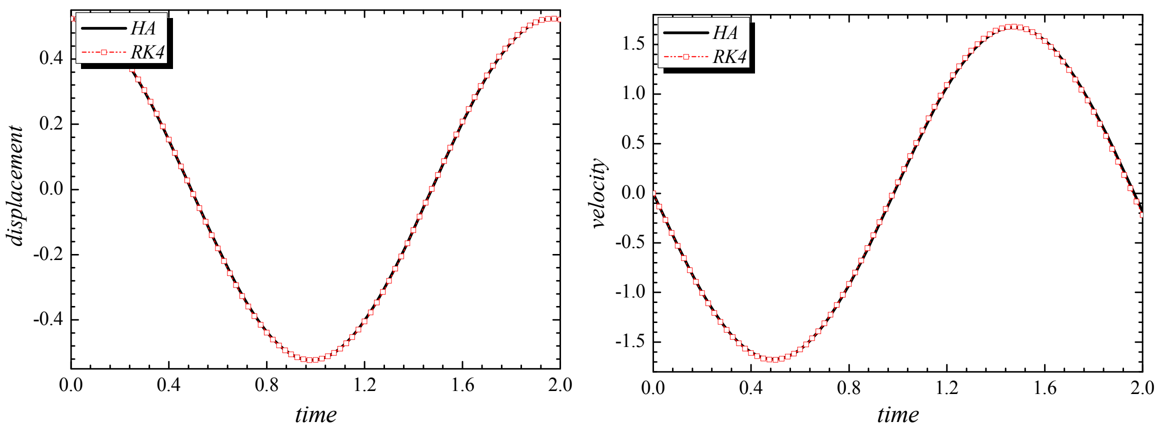
(a) (b)

Fig. 3 Comparison of analytical solution with the RK4 solution for $A = \pi/3$, $g = 10$, $m = 5$, $l = 1$, $r = 0.3$, $k = 1000$: (a) time history response of displacement; (b) time history response of velocity

$$\omega_{HA} = \sqrt{1-\beta-\frac{1}{8}A^2+\frac{1}{2}A^2\beta-\frac{5}{64}\beta A^4+\frac{35}{9216}\beta A^6} \tag{31}$$

Hence, the approximate solution can be readily obtained

$$\theta(t) = A\cos\left(\sqrt{1-\beta-\frac{1}{8}A^2+\frac{1}{2}A^2\beta-\frac{5}{64}\beta A^4+\frac{35}{9216}\beta A^6}\ t\right) \tag{32}$$



(a) (b)

Fig. 4 Comparison of analytical solution with the RK4 solution for $A = \pi/6$, $g = 10$, $m = 10$, $l = 1.5$, $r = 0.2$, $k = 800$: (a) time history response of displacement; (b) time history response of velocity

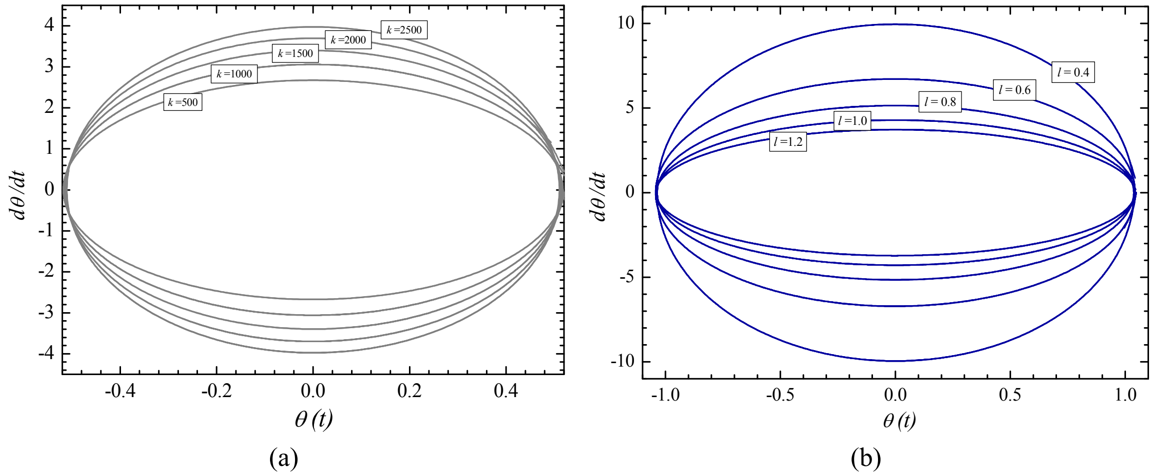


Fig. 5 Effect of spring stiffness and length of pendulum on Phase plan for cases: (a) $A = \pi/3$, $g = 10$, $m = 5$, $l = 1$, $r = 0.3$, $k = 1000$; (b) $A = \pi/6$, $g = 10$, $m = 10$, $l = 1.5$, $r = 0.2$, $k = 800$

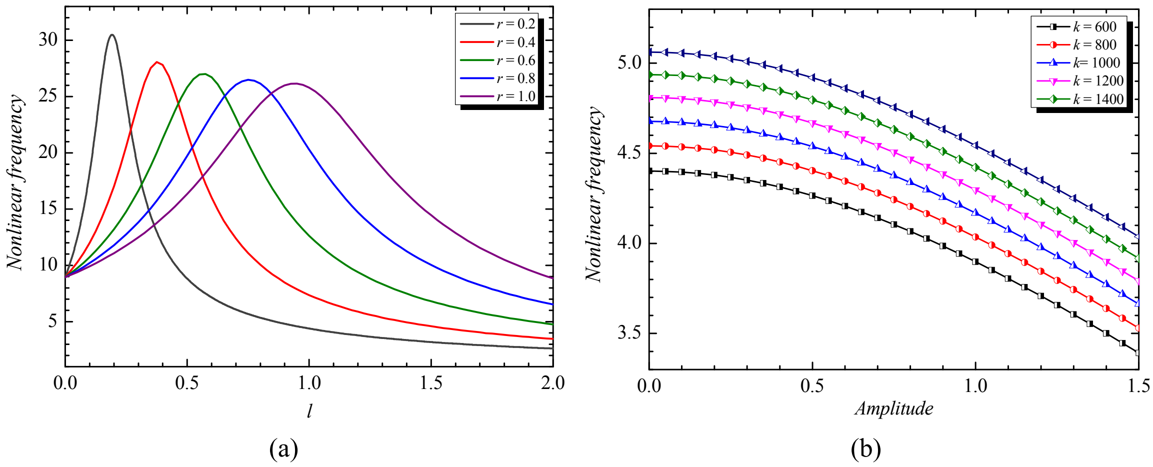


Fig. 6 (a) Effect of radius on nonlinear frequency base on length for $A = \pi/6$, $g = 10$, $m = 10$, $k = 800$; (b) Effect of spring stiffness on nonlinear frequency base on amplitude for $l = 1$, $r = 0.2$, $g = 10$, $m = 10$

5. Results and discussions

In this section to verify the results of Hamiltonian approach we have prepared tables for two different cases for examples 1 and 2. Tables 1 and 2 show the comparisons of Hamiltonian approach and Runge-Kutta's algorithm.

Figs. 3, 4 and 8, 9 show the displacement and velocity time history of the problems. The motions of the problems are periodic and it is a function of initial conditions.

The effect of spring stiffness and length of pendulum on Phase plan for cases are shown in Fig. 5 for example 1. Fig. 6 is considered the effect of radius on nonlinear frequency base on length and

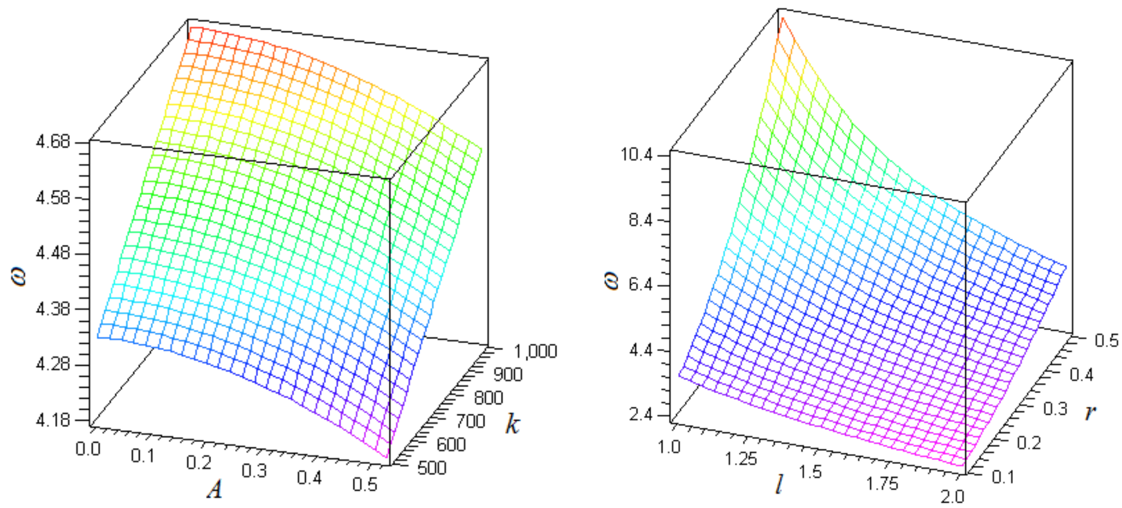


Fig. 7 3D plot, effect of various parameter of system on nonlinear frequency

also the effect of spring stiffness on nonlinear frequency base on amplitude. Fig. 7 is sensitive analysis on the nonlinear frequency shows the effect of various parameters of the system.

Fig. 10 is the effect of various parameter of β on the Phase plane curve and the nonlinear frequency of the system in example 2. Fig. 11 is a sensitive analysis which shows the effect of β and amplitude on nonlinear frequency of system. It has been obviously seen that one of the most advantages of analytical methods is to see the effect of important parameters on the response of the systems. We can easily find out that the nonlinear frequency of the systems have been effected by which parameters. The new proposed method has a good agreement with the numerical methods and it is valid for large amplitudes and whole domain.

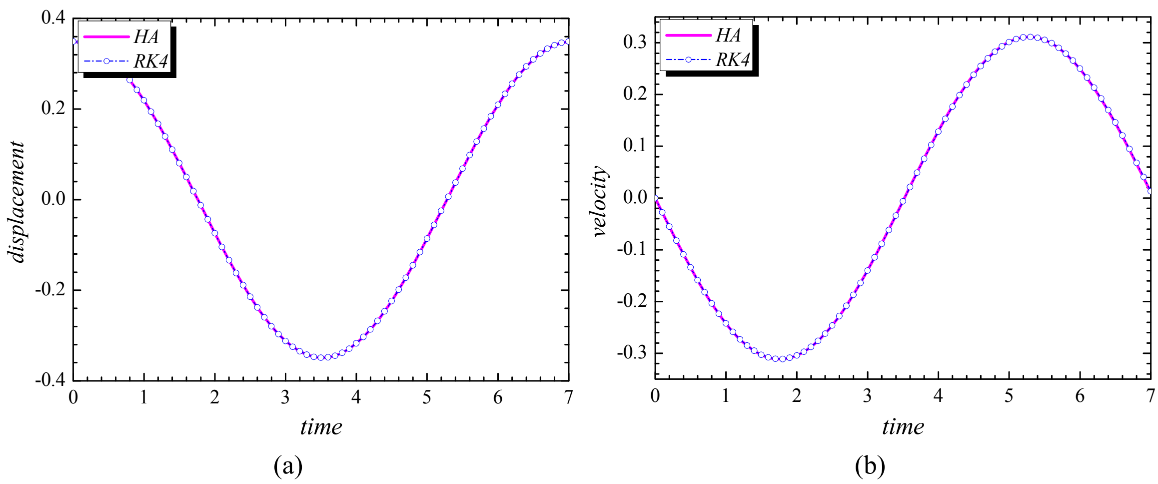


Fig. 8 Comparison of analytical solution with the RK4 solution for $A = \pi/9$, $\beta = 0.2$: (a) time history response of displacement; (b) time history response of velocity

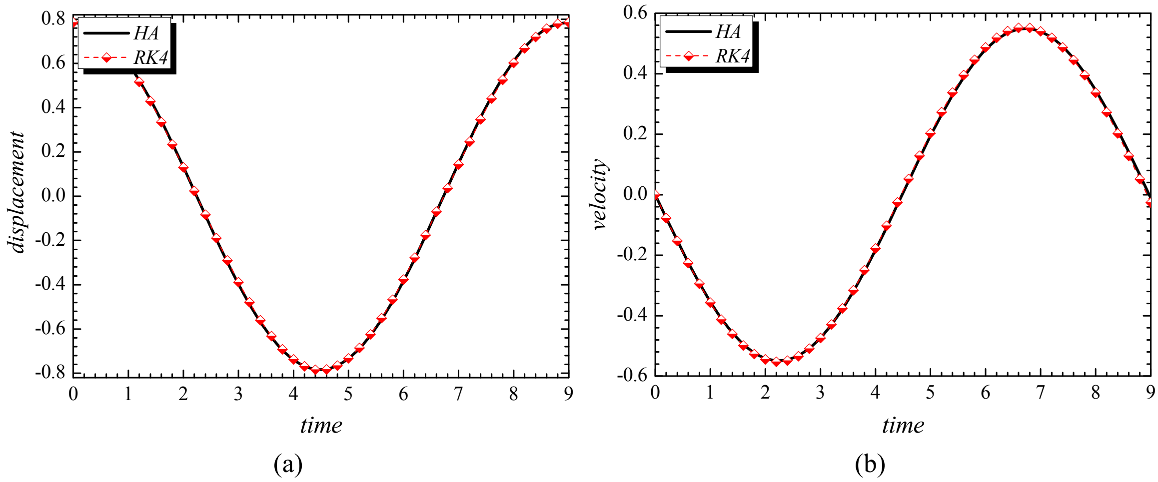


Fig. 9 Comparison of analytical solution with the RK4 solution for $A = \pi/4$, $\beta = 0.6$: (a) time history response of displacement; (b) time history response of velocity

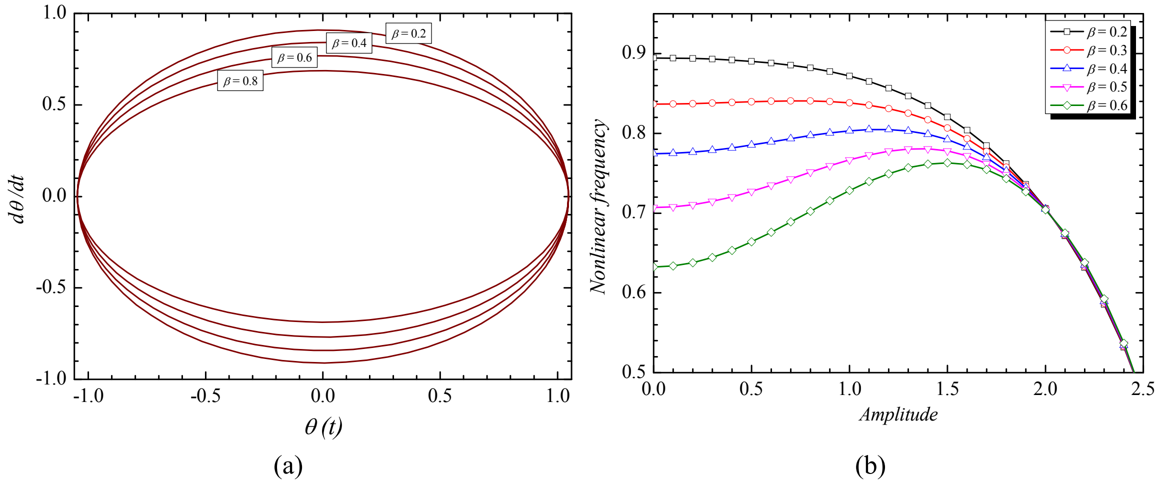


Fig. 10 Effect of various parameter of β on: (a) phase plane; (b) nonlinear frequency

6. Conclusions

It has been applied a simple and accurate analytical method called Hamiltonian approach for two strong nonlinear problems. The results were compared with the numerical solutions using Runge-Kuttas algorithm. The Hamiltonian approach has a very good agreement with the numerical one. Some patterns were also presented to show the accuracy of this new approach. The Hamiltonian approach does not require any linearization or small perturbation, and adequately accurate to both linear and nonlinear problems. This new approach can be easily extended to nonlinear problems to see the effects of important parameters.

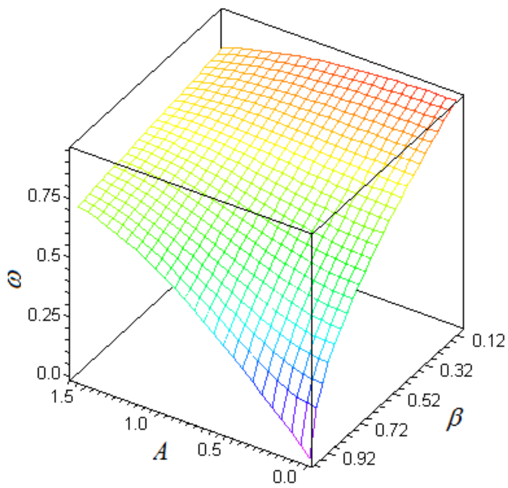


Fig. 11 3D plot, effect of β and amplitude on nonlinear frequency of system

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