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A trigonometric four variable plate theory for free vibration of rectangular composite plates with patch mass

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Abstract. The novelty of this paper is the use of trigonometric four variable plate theory for free vibration analysis of laminated rectangular plate supporting a localized patch mass. By dividing the transverse displacement into bending and shear parts, the number of unknowns and governing equations of the present theory is reduced, and hence, makes it simple to use. The Hamilton's Principle, using trigonometric shear deformation theory, is applied to simply support rectangular plates. Numerical examples are presented to show the effects of geometrical parameters such as aspect ratio of the plate, size and location of the patch mass on natural frequencies of laminated composite plates. It can be concluded that the proposed theory is accurate and simple in solving the free vibration behavior of laminated rectangular plate supporting a localized patch mass.

Keywords: laminated plates; free vibration; four variable plate theory; patch mass

1. Introduction

The analysis of free vibration of rectangular plates has been an active research subject due to its relevance to civil, mechanical and aeronautical engineering and extensive research works have been accumulated in the last 50 years (Timoshenko 1955, Leissa 1969, Szilard 1974, Reddy 1999). These structural components with localized patch mass are often encountered in engineering practices such as slabs and cladding panels in building structures, bridge and ship decks.

Laminated composite plates are widely used in the aerospace, automotive, marine, civil and other structural applications because of advantageous features such as high ratio of stiffness and strength to weight and low maintenance cost. In company with the increase in the application of laminates in engineering structures, a variety of laminated theories have been developed. The classical laminate plate theory (CLPT), which neglects the transverse shear effects, provides reasonable results for thin plates. However, the CLPT underpredicts deflections and overpredicts frequencies as well as buckling loads with moderately thick plates. Many shear deformation theories account for transverse shear effects have been developed to overcome the deficiencies of

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the CLPT. The first-order shear deformation theories (FSDTs) based on Reissner (1945) and Mindlin (1951) account for the transverse shear effects by the way of linear variation of in-plane displacements through the thickness. Since FSDT violates equilibrium conditions at the top and bottom faces of the plate, shear correction factors are required to rectify the unrealistic variation of the shear strain/stress through the thickness. Different higher order theories were proposed in order to satisfy the plate boundary conditions. Ambartsumian (1958), proposed a transverse shear stress function in order to explain plate deformation. A similar method was used later by Soldatos and Timarci (1993), for dynamic analysis of laminated shells. Later some new functions were proposed by Reddy (1984), Touratier (1991), Karama *et al*. (2003) and Soldatos (1992). A two variable refined plate theory (RPT) using only two unknown functions was developed by Shimpi (2002) for isotropic plates, and was extended by Shimpi and Patel (2006ab) for orthotropic plates. Alibeigloo *et al*. (2008) studied the vibration response of anti-symmetric rectangular plates with distributed patch mass using third order shear deformation theory (TSDT) and obtained the first natural frequency of the plate considering the size and location of the distributed mass on the top surface of the plate. Alibeigloo and Kari (2009) also investigated the forced vibration behavior of anti-symmetric laminated rectangular plates with distributed patch mass. Kim *et al*. (2009) employed the two variable refined plate theory for laminated composite plates which are under the action of the transverse and in-plane forces and obtained the stiffness and mass matrices using Hamilton principle. Alibakhshi (2012) used the two variable refined plate theory for free vibration of laminated rectangular plate supporting a localized patch mass. Tounsi *et al*. (2013) studied the bending response of functionally graded sandwich plates using a new four variable refined plate theory under both thermal and thermomechanical loading conditions. Bouderba *et al*. (2013) used a refined plate theory to investigate the thermomechanical bending response of functionally graded plates resting on Winkler–Pasternak elastic foundations. Recently, Nedri *et al*. (2014) developed a four variable refined plate theory for free vibration response of laminated composite plates resting on elastic foundations. The theory accounts for hyperbolic distribution of the transverse shear strains, and satisfies the zero traction boundary conditions on the surfaces of the plate without using shear correction factor.

In this paper the free vibration of a simply supported laminated composite plate with distributed patch mass is investigated using a trigonometric four variable plate theory. The present theory has only four unknowns and four governing equations, but it satisfies the stress-free boundary conditions on the top and bottom surfaces of the plate without requiring any shear correction factors. The displacement field of the proposed theory is chosen based on a constant transverse displacement and sinusoidal variation of in-plane displacements through the thickness. The partition of the transverse displacement into the bending and shear parts leads to a reduction in the number of unknowns and governing equations, hence makes the theory simple to use. The effect of various parameters such as position of the patch mass and the aspect ratio of the plate on free vibration are also studied in the present work. The current study is relevant to aero-structures.

2. Theoretical formulation

2.1 Steel and composite structures

A rectangular plate with length, width and thickness equal to a, b and h respectively is considered. The plate supports a distributed patch mass, M, with dimensions equal to c and d in the $\boldsymbol{x}$ and $\boldsymbol{y}$-direction, respectively which is located in arbitrary position (x_m, y_m) in Fig. 1. The mass is

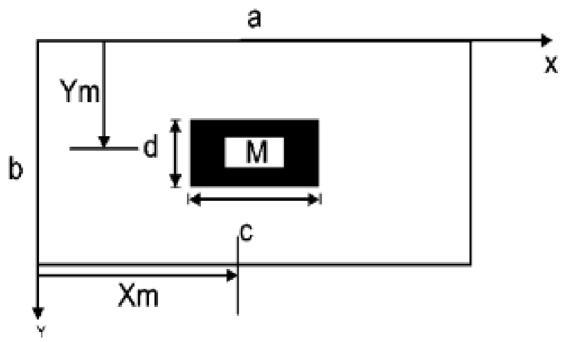

Fig. 1 Plate with distributed patch mass

considered to be placed on the upper surface of the plate and it is assumed that the mass does not prevent any bending of the plate segment on which it is. The global Cartesian coordinate system is chosen with the origin at the corner and on the middle plane of the plate, $z = 0$. Therefore, the domain of plate is defined as $0 \le x \le a$, $0 \le y \le b$ and $-h \le z \le h/2$.

The displacement field of the present theory is chosen based on the following assumptions: (1) the in-plane and transverse displacements are partitioned into bending and shear components; (2) the bending parts of the in-plane displacements are similar to those given by CLPT; and (3) the shear parts of the in-plane displacements give rise to the trigonometric variations of shear strains and hence to shear stresses through the thickness of the plate in such a way that the shear stresses vanish on the top and bottom surfaces of the plate. Based on these assumptions, the following displacement field can be obtained

$$
\begin{aligned}
u(x,y,z) &= u_0(x,y) - z\frac{\partial w_b}{\partial x} - f(z)\frac{\partial w_s}{\partial x} \\
v(x,y,z) &= v_0(x,y) - z\frac{\partial w_b}{\partial y} - f(z)\frac{\partial w_s}{\partial y} \\
w(x,y,z) &= w_b(x,y) + w_s(x,y)
\end{aligned}
\tag{1}
$$

with

$$
f(z) = z - \frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right) \tag{2}
$$

where u_0 and v_0 denote the displacements along the x and y coordinate directions of a point on the midplane of the plate; w_b and w_s are the bending and shear components of the transverse displacement, respectively.

The strain-displacement equations of linear elasticity are (Reddy 1979)

$$
\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z\begin{Bmatrix} k_x^b \\ k_y^b \\ k_{xy}^b \end{Bmatrix} + f(z)\begin{Bmatrix} k_x^s \\ k_y^s \\ k_{xy}^s \end{Bmatrix}, \quad \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = g(z)\begin{Bmatrix} \gamma_{yz}^s \\ \gamma_{xz}^s \end{Bmatrix} \tag{3}
$$

where

$$\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial y} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{Bmatrix}, \quad \begin{Bmatrix} k_x^b \\ k_y^b \\ k_{xy}^b \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\partial^2 w_b}{\partial x^2} \\ -\dfrac{\partial^2 w_b}{\partial y^2} \\ -2\dfrac{\partial^2 w_b}{\partial x \partial y} \end{Bmatrix}, \quad \begin{Bmatrix} k_x^s \\ k_y^s \\ k_{xy}^s \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\partial^2 w_s}{\partial x^2} \\ -\dfrac{\partial^2 w_s}{\partial y^2} \\ -2\dfrac{\partial^2 w_s}{\partial x \partial y} \end{Bmatrix}, \quad \begin{Bmatrix} \gamma_{yz}^s \\ \gamma_{xz}^s \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial w_s}{\partial y} \\ \dfrac{\partial w_s}{\partial x} \end{Bmatrix} \tag{4a}$$

and

$$g(z) = 1 - \frac{df(z)}{dz} = \cos\left(\frac{\pi z}{h}\right) \tag{4b}$$

The constitutive relations for any layer in the (x, y) system are given by

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{44} & \overline{Q}_{45} \\ \overline{Q}_{45} & \overline{Q}_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \tag{5}$$

where $\overline{Q}_{ij}$ are the plane-stress reduced stiffness components of the layer material in the laminate coordinate system.

The strain energy of the plate is

$$U_P = \frac{1}{2} \int_0^a \int_0^b \int_{-h/2}^{h/2} \sigma_{ij} \varepsilon_{ij} \, dx dy dz \tag{6}$$

Substituting Eqs. (3) and (5) into Eq. (6) and integrating through the thickness of the plate, the strain energy of the plate can be rewritten as

$$\begin{aligned} U_P = \frac{1}{2} \int_0^a \int_0^b \Big[& N_x \delta \varepsilon_x^0 + N_y \delta \varepsilon_y^0 + N_{xy} \delta \varepsilon_{xy}^0 + M_x^b \delta k_x^b + M_y^b \delta k_y^b + M_{xy}^b \delta k_{xy}^b + M_x^s \delta k_x^s \\ & + M_y^s \delta k_y^s + M_{xy}^s \delta k_{xy}^s + S_{yz}^s \delta \gamma_{yz}^s + S_{xz}^s \delta \gamma_{xz}^s \Big] dx dy \end{aligned} \tag{7}$$

The stress resultants N, M, and S are defined by

$$\begin{Bmatrix} N_x, & N_y, & N_{xy} \\ M_x^b, & M_y^b, & M_{xy}^b \\ M_x^s, & M_y^s, & M_{xy}^s \end{Bmatrix} = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy}) \begin{Bmatrix} 1 \\ z \\ f(z) \end{Bmatrix} dz, \tag{8a}$$

$$\left(S_{xz}^s, S_{yz}^s\right) = \int_{-h/2}^{h/2} \left(\tau_{xz}, \tau_{yz}\right) g(z) dz. \tag{8b}$$

Using Eq. (5) in Eq. (8), the stress resultants of the laminated plate can be related to the total strains by

$$\begin{Bmatrix} N \\ M^b \\ M^s \end{Bmatrix} = \begin{bmatrix} A & B & B^s \\ B & D & D^s \\ B^s & D^s & H^s \end{bmatrix} \begin{Bmatrix} \varepsilon \\ k^b \\ k^s \end{Bmatrix}, \quad S = A^s \gamma \tag{9}$$

where

$$N = \{N_x, N_y, N_{xy}\}^{Tr}, \quad M^b = \{M_x^b, M_y^b, M_{xy}^b\}^{Tr}, \quad M^s = \{M_x^s, M_y^s, M_{xy}^s\}^{Tr}, \tag{10a}$$

$$\varepsilon = \{\varepsilon_x^0, \varepsilon_y^0, \gamma_{xy}^0\}^{Tr}, \quad k^b = \{k_x^b, k_y^b, k_{xy}^b\}^{Tr}, \quad k^s = \{k_x^s, k_y^s, k_{xy}^s\}^{Tr}, \tag{10b}$$

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix}, \quad D = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}, \tag{10c}$$

$$B^s = \begin{bmatrix} B_{11}^s & B_{12}^s & B_{16}^s \\ B_{12}^s & B_{22}^s & B_{26}^s \\ B_{16}^s & B_{26}^s & B_{66}^s \end{bmatrix}, \quad D^s = \begin{bmatrix} D_{11}^s & D_{12}^s & D_{16}^s \\ D_{12}^s & D_{22}^s & D_{26}^s \\ D_{16}^s & D_{26}^s & D_{66}^s \end{bmatrix}, \quad H^s = \begin{bmatrix} H_{11}^s & H_{12}^s & H_{16}^s \\ H_{12}^s & H_{22}^s & H_{26}^s \\ H_{16}^s & H_{26}^s & H_{66}^s \end{bmatrix}, \tag{10d}$$

$$S = \{S_{yz}^s, S_{xz}^s\}^{Tr}, \quad \gamma = \{\gamma_{yz}, \gamma_{xz}\}^{Tr}, \quad A^s = \begin{bmatrix} A_{44}^s & A_{45}^s \\ A_{45}^s & A_{55}^s \end{bmatrix}, \tag{10e}$$

where A_{ij}, B_{ij}, etc., are the plate stiffness, defined by

$$\left(A_{ij}, B_{ij}, D_{ij}, B_{ij}^s, D_{ij}^s, H_{ij}^s\right) = \int_{-h/2}^{h/2} \overline{Q}_{ij} \left(1, z, z^2, f(z), z\, f(z), [f(z)]^2\right) dz, \quad i, j = 1, 2, 6, \tag{11a}$$

and

$$A_{ij}^s = \int_{-h/2}^{h/2} \overline{Q}_{ij} [g(z)]^2 dz, \quad i, j = 4,5 \tag{11b}$$

The total kinetic energy is the summation of the kinetic energy of the plate and the kinetic energy of the uniformly distributed patch mass with dimensions c and d acting on the top surface of the plate.

$$T = T_P + T_M \tag{12}$$

The kinetic energy of plate is defined as

$$T_P = \frac{1}{2} \int_{-h/2}^{h/2} \int_{A_p} \rho \left[(\dot{u}_0)^2 + (\dot{v}_0)^2 + (\dot{w})^2 \right] dA_p dz, \tag{13}$$

where dot-superscript convention indicates the differentiation with respect to the time variable t; and ρ and A_p are the material density and the area of the plate, respectively.

Substituting Eq. (1) into Eq. (13) gives

$$\begin{aligned}T_P = \int_{A_P} \{&\delta u_0\left(I_1\ddot{u}_0 - I_2\ddot{w}_{b,x} - I_4\ddot{w}_{s,x}\right) + \delta v_0\left(I_1\ddot{v}_0 - I_2\ddot{w}_{b,y} - I_4\ddot{w}_{s,y}\right)\\ &+ \delta w_b\left(I_1\left(\ddot{w}_b + \ddot{w}_s\right) + I_2\left(\ddot{u}_{0,x} + \ddot{v}_{0,y}\right) - I_3\left(\ddot{w}_{b,xx} + \ddot{w}_{b,yy}\right) - I_5\left(\ddot{w}_{s,xx} + \ddot{w}_{s,yy}\right)\right)\\ &+ \delta w_s\left(I_1\left(\ddot{w}_b + \ddot{w}_s\right) + I_4\left(\ddot{u}_{0,x} + \ddot{v}_{0,y}\right) - I_5\left(\ddot{w}_{b,xx} + \ddot{w}_{b,yy}\right) - I_6\left(\ddot{w}_{s,xx} + \ddot{w}_{s,yy}\right)\right)\}dA_P\end{aligned} \tag{14}$$

(I_1, I_2, I_3, I_4, I_5, I_6) are mass inertias defined as

$$\left(I_1, I_2, I_3, I_4, I_5, I_6\right) = \int_{-h/2}^{h/2} \rho\left(1, z, z^2, f(z), z\,f(z), [f(z)]^2\right)dz \tag{15}$$

Using the same procedure described by Alibeigloo *et al.* (2008), the kinetic energy of distributed patch mass located on the top surface $(z = -h/2)$ of the plate is defined after considering the displacement field of Eq. (1) as

$$T_M = \int_{A_M} \gamma_M \left[\left(\dot{u}_0 + \frac{h}{2}\dot{w}_{b,x} + \left(\frac{h}{2} - \frac{h}{\pi}\right)\dot{w}_{s,x}\right)^2 + \left(\dot{v}_0 + \frac{h}{2}\dot{w}_{b,y} + \left(\frac{h}{2} - \frac{h}{\pi}\right)\dot{w}_{s,y}\right)^2 + \left(\dot{w}_b + \dot{w}_s\right)^2\right]dxdy \tag{16}$$

where A_M and γ_M is the area and the distributed patch mass per unit area of the patch mass, respectively.

It can be verified that the first variation of the Lagrangian, $L = T - V$, (i.e., Hamilton's Principle) leads to the equation of motion. Here V denotes the total potential energy (i.e., the sum of the strain energy and the energy due to applied loads) of the plate. Since primary interest here is in the free vibration analysis, the potential energy due to the applied loads is zero (Meirovitch 2001).

Two different types of laminated plates are considered, cross-ply $[0/90]_n$, and angle-ply $[\theta/-\theta]$. For the cross-ply one, the SS-1 boundary conditions are (Reddy, 1979)

$$u_0,\ w_b,\ w_s,\ \frac{\partial w_b}{\partial x},\ \frac{\partial w_s}{\partial x},\ N_y,\ M_y^b,\ M_y^s,\qquad \text{are equal to zero},\ \ (y = 0, b) \tag{17a}$$

$$v_0,\ w_b,\ w_s,\ \frac{\partial w_b}{\partial y},\ \frac{\partial w_s}{\partial y},\ N_x,\ M_x^b,\ M_x^s,\qquad \text{are equal to zero},\ \ (x = 0, a) \tag{17b}$$

The boundary conditions in Eq. (17) are satisfied by the following expansions

$$\begin{Bmatrix} u_0 \\ v_0 \\ w_b \\ w_s \end{Bmatrix} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \begin{Bmatrix} U_{mn}\cos(\lambda\, x)\sin(\mu\, y)\, e^{i\,\omega\, t} \\ V_{mn}\sin(\lambda\, x)\cos(\mu\, y)\, e^{i\,\omega\, t} \\ W_{bmn}\sin(\lambda\, x)\sin(\mu\, y)\, e^{i\,\omega\, t} \\ W_{smn}\sin(\lambda\, x)\sin(\mu\, y)\, e^{i\,\omega\, t} \end{Bmatrix} \tag{18}$$

where U_{mn}, V_{mn}, W_{bmn}, and W_{smn} are arbitrary parameters to be determined, ω is the eigenfrequency associated with (m, n)th eigenmode, and $\lambda = m\pi/a$ and $\mu = n\pi/b$.

And for the angle-ply case, the SS-2 boundary conditions and related displacements are (Reddy 1979)

$$v_0,\ w_b,\ w_s,\ \frac{\partial w_b}{\partial x},\ \frac{\partial w_s}{\partial x},\ N_{xy},\ M_y^b,\ M_y^s, \qquad \text{are equal to zero,}\ \ (y=0,b) \tag{19a}$$

$$u_0,\ w_b,\ w_s,\ \frac{\partial w_b}{\partial y},\ \frac{\partial w_s}{\partial y},\ N_{xy},\ M_x^b,\ M_x^s, \qquad \text{are equal to zero,}\ \ (y=0,a) \tag{19b}$$

The boundary conditions in Eq. (19) are satisfied by the following expansions

$$\begin{Bmatrix} u_0 \\ v_0 \\ w_b \\ w_s \end{Bmatrix} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \begin{Bmatrix} U_{mn}\sin(\lambda\, x)\cos(\mu\, y)\, e^{i\,\omega\, t} \\ V_{mn}\cos(\lambda\, x)\sin(\mu\, y)\, e^{i\,\omega\, t} \\ W_{bmn}\sin(\lambda\, x)\sin(\mu\, y)\, e^{i\,\omega\, t} \\ W_{smn}\sin(\lambda\, x)\sin(\mu\, y)\, e^{i\,\omega\, t} \end{Bmatrix} \tag{20}$$

Substitution of displacements into the first variation of the Lagrangian, the solutions can be obtained from the following equations

$$\left([K]-\omega^2\left\lfloor\overline{M}\right\rfloor\right)\{\Delta\}=\{0\}, \tag{21}$$

where $[K]$, $\left\lfloor\overline{M}\right\rfloor$, ω and Δ are the stiffness, mass matrices, natural frequency and the vector of unknown coefficients, respectively.

3. Numerical results

Three sets of dimensionless material properties are considered

• MAT1

$$E_1/E_2 = 25,\ \ G_{23}/E_2 = 0.2,\ \ G_{13}/E_2 = G_{12}/E_2 = 0.5,\ \ \nu_{12} = 0.5$$

• MQT2

$$E_1/E_2 = 40,\ \ G_{23}/E_2 = 0.5,\ \ G_{13}/E_2 = G_{12}/E_2 = 0.6,\ \ \nu_{12} = 0.5$$

• MAT3

$$E_1/E_2 = 40,\ \ G_{23}/E_2 = 0.6,\ \ G_{13}/E_2 = G_{12}/E_2 = 0.5,\ \ \nu_{12} = 0.5$$

The following non-dimensional fundamental frequency is used

$$\Omega = \omega\frac{a^2}{h}\sqrt{\frac{\rho}{E_2}}$$

By setting the $\gamma_M = 0$ in Eq. (21) the problem reduces to the free vibration of an unloaded plate. The results are then compared with those of Noor (1973), Kant and Swaminathan (2001) and

Reddy (1979) in Table 1. It can be seen that the results obtained by TSDT (Reddy 1979) are almost identical. This table shows the influence of the orthotropy on the natural frequencies and as expected, it indicated that with increasing the number of layer, the stiffness of the plate will be increased, and consequently the natural frequencies increase. The variation of natural frequencies of a four-layer angle- ply composite plate $[45/\text{-}45]_2$ with respect to thickness ratio (a/h) and aspect ratio (a/b) is presented in Table 2. The natural frequencies obtained using the present theory, are compared with those predicted using various models such as Bert and Chen (1978) and Shankara and Iyengar (1996). From Table 2, it can be observed that increasing the aspect ratio can cause to decrease the stiffness of the plate, to decease the natural frequencies, and consequently to increase the nondimensional natural frequencies. The first five non-dimensional natural frequencies obtained using the present theory are compared with those predicted by Singh *et al*. (2001) in Table 3 for two types of materials and a good agreement is observed.

The influence of the aspect ratio and thickness ratio on non-dimensional fundamental frequencies for two angle- ply composite plates $[45/\text{-}45]_2$ and $[30/\text{-}30]_2$ with patch mass is presented in Table 4. The plate with distributed patch mass acted on it in a square area at the center of the plate with dimension ratios of $c/a = d/b = 0.4$ is considered here with mass ratio $M / M_p = 0.5$ (M_p is plate's mass). It can be observed that the influence of the aspect ratio on non-dimensional fundamental frequencies for the plate with patch mass is less than for the plate without patch mass.

Table 1 Non-dimensional fundamental frequencies of antisymmetric square plate (MAT2) for various values of orthotropy ratio with $a/h = 5$

No. of layers	Source	E_1/E_2				
		3	10	20	30	40
$[0/90]_1$	Noor (1973)	6.2578	6.9845	7.6745	8.1763	8.5625
	Model-1 (Kant and Swaminathan 2001)	6.2336	6.9741	7.7140	8.2775	8.7272
	Model-2 (Kant and Swaminathan 2001)	6.1566	6.9363	7.6883	8.2570	8.7097
	Reddy (1979)	6.2169	6.9887	7.8210	8.5050	9.0871
	Present	6.2188	6.9964	7.8379	8.5316	9.1236
$[0/90]_2$	Noor (1973)	6.5455	8.1445	9.4055	10.1650	10.6798
	Model-1 (Kant and Swaminathan 2001)	6.5146	8.1482	9.4675	10.2733	10.8221
	Model-2 (Kant and Swaminathan 2001)	6.4319	8.1010	9.4338	10.2463	10.7993
	Reddy (1979)	6.5008	8.1954	9.6265	10.5348	11.1716
	Present	6.5012	8.1929	9.6205	10.5268	11.1628
$[0/90]_3$	Noor (1973)	6.6100	8.4143	9.8398	10.6958	11.2728
	Model-1 (Kant and Swaminathan 2001)	6.5711	8.3852	9.8346	10.7113	11.3051
	Model-2 (Kant and Swaminathan 2001)	6.4873	8.3372	9.8012	10.6853	11.2838
	Reddy (1979)	6.5552	8.4041	9.9175	10.8542	11.5007
	Present	6.5567	8.4065	9.9210	10.8603	11.5102
$[0/90]_5$	Noor (1973)	6.6458	8.5625	10.0843	11.0027	11.6245
	Model-1 (Kant and Swaminathan 2001)	6.6458	8.5163	10.0438	10.9699	11.5993
	Model-2 (Kant and Swaminathan 2001)	6.5177	8.4680	10.0107	10.9445	11.5789
	Reddy (1979)	6.5842	8.5126	10.0674	11.0197	11.6730
	Present	6.5854	8.5156	10.0740	11.0309	11.6893

Table 2 Non-dimensional fundamental frequencies for $[45/\text{-}45]_2$ laminate (MAT3) with various a/b and a/h ratios

a/h	Source	a/b						
		0.2	0.6	0.8	1	1.2	1.6	2
10	Reddy (1979)	8.7240	12.9650	15.7120	18.6090	21.5670	27.7360	34.2470
	(Bert and Chen 1978)	8.6640	12.8200	15.5400	18.4600	21.5100	27.9500	34.8700
	(Shankara and Iyengar 1996)	8.5557	12.5588	15.1802	17.9735	20.8797	26.9916	33.5534
	Present	8.6036	12.6364	15.2016	17.9670	20.9070	27.2616	34.1652
20	Reddy (1979)	9.4750	14.8960	18.5570	22.5840	26.8570	36.2490	46.7890
	(Bert and Chen 1978)	9.3000	14.4500	17.9700	21.8700	26.1200	35.5600	46.2600
	(Shankara and Iyengar 1996)	9.3011	14.3856	17.8458	21.6808	25.8363	35.0421	45.4096
	Present	9.2817	14.3865	17.8267	21.6501	25.8207	35.1650	45.7985
30	Reddy (1979)	9.6670	15.3850	19.3040	23.6760	28.3810	38.9400	51.1320
	(Bert and Chen 1978)	9.4360	14.8400	18.5600	22.7400	27.3500	37.8200	49.9800
	(Shankara and Iyengar 1996)	9.4880	14.8427	18.5390	22.6911	27.2555	37.5907	49.5474
	Present	9.4270	14.8027	18.4890	22.6326	27.2010	37.6046	49.7155
40	Reddy (1979)	9.7590	15.8530	19.6040	24.1180	29.0030	40.0710	53.0120
	(Bert and Chen 1978)	9.4850	14.9800	18.7800	23.0800	27.8300	38.7200	51.5200
	(Shankara and Iyengar 1996)	9.5724	15.0248	18.8134	23.0940	27.8286	38.6523	51.3324
	Present	9.4796	14.9576	18.7398	23.0110	27.7417	38.5929	51.3575
50	Reddy (1979)	9.8160	15.6890	19.7590	24.3430	29.3210	40.6530	53.9890
	(Bert and Chen 1978)	9.5070	15.0400	18.8900	23.2400	28.0600	39.1700	52.2900
	(Shankara and Iyengar 1996)	9.6216	15.1177	18.9510	23.2956	28.1168	39.1932	52.2539
	Present	9.5043	15.0311	18.8595	23.1930	28.0036	39.0788	52.1777

Table 3 Non-dimensional fundamental frequencies for square plates [0/90] with $a/h = 10$

Mode	Source	MAT1	MAT2
1	(Singh *et al.* 2001)	8.98083	10.56565
	Present	8.98691	10.58110
2	(Singh *et al.* 2001)	21.93360	26.30276
	Present	22.16046	26.59112
3	(Singh *et al.* 2001)	30.34590	36.34791
	Present	30.46608	36.49444
4	(Singh *et al.* 2001)	39.98180	48.70060
	Present	40.59950	49.51371
5	(Singh *et al.* 2001)	45.59216	55.14367
	Present	46.07020	55.88987

Fig. 2 presents the effect of the orthotropy (E_1 / E_2) on the frequency parameter. It can be observed that the increase of E_1 / E_2 ratio leads to an increase of non-dimensional fundamental frequencies and this is due to the increase of the stiffness of the square plate.

Table 4 Non-dimensional fundamental frequencies (MAT3)

a/b	$[45/-45]_2$					$[30/-30]_2$				
	a/h					a/h				
	10	20	30	40	50	10	20	30	40	50
0.2	5.2041	5.7389	5.8607	5.9055	5.9267	6.6404	7.6532	7.9043	7.9990	8.0442
0.4	6.2602	7.0677	7.2607	7.3327	7.3668	7.2302	8.4201	8.7217	8.8362	8.8910
0.6	7.6323	8.8758	9.1911	9.3109	9.3682	8.1068	9.5726	9.9554	10.1022	10.1727
0.8	9.1617	10.9777	11.4675	11.6574	11.7490	9.1830	11.0065	11.4988	11.6896	11.7817
1	10.7946	13.3009	14.0183	14.3022	14.4404	10.3965	12.6481	13.2767	13.5233	13.6429
1.2	12.5130	15.8199	16.8205	17.2248	17.4235	11.7067	14.4511	15.2433	15.5579	15.7114
1.4	14.3075	18.5244	19.8688	20.4235	20.6988	13.0887	16.3899	17.3748	17.7710	17.9653
1.6	16.1699	21.4077	23.1624	23.9017	24.2724	14.5286	18.4531	19.6629	20.1560	20.3994
1.8	18.0925	24.4627	26.7000	27.6629	28.1506	16.0167	20.6341	22.1047	22.7123	23.0141
2	20.0671	27.6813	30.4787	31.7088	32.3385	17.5437	22.9247	24.6950	25.4369	25.8078

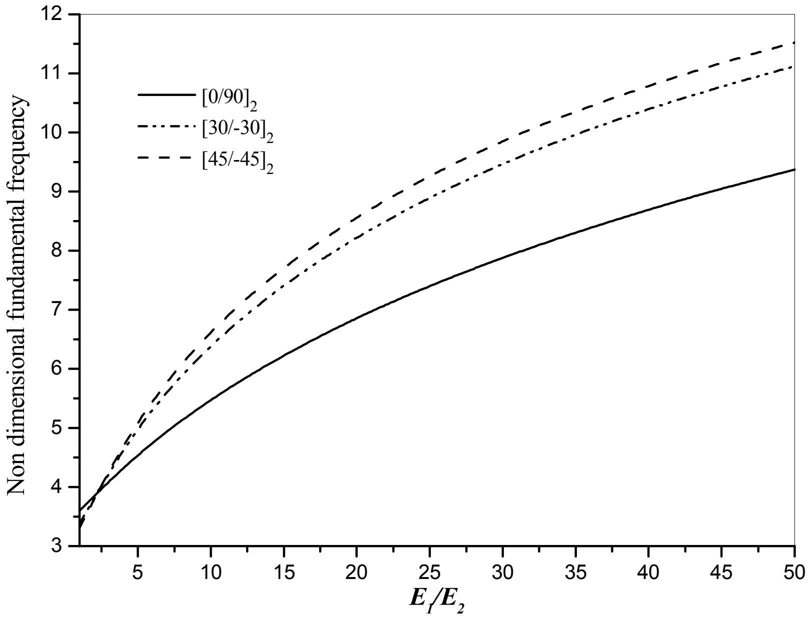

Fig. 2 Influence of orthotropy on non-dimensional fundamental frequency (Ω) of square plate, $a/h = 10$, $M/M_p = 0.5$, Material 3, $c/a = d/b = 0.4$

Fig. 3 shows the effect of thickness ratio on non-dimensional fundamental frequencies and it can be observed that the nondimensional fundamental frequency increases with the length to thickness ratio (a/h). However at $a/h \geq 20$, the nondimensional fundamental frequency reaches to a constant value and the plate behaves as a thin plate.

Fig. 4 shows the effect of the mass ratio (with $c/a = d/b = 0.4$) on non-dimensional fundamental frequencies of a square plate. According to this figure, by increasing the amount of

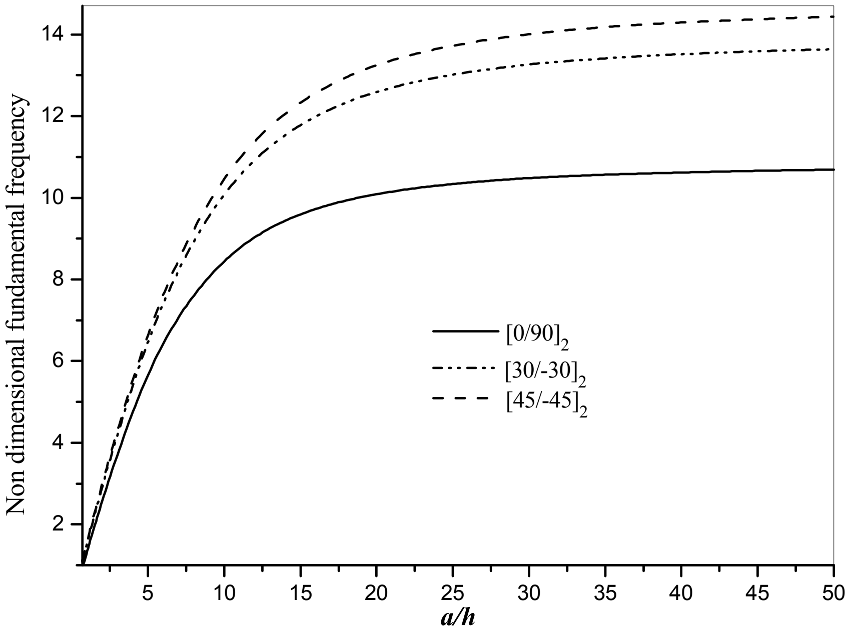

Fig. 3 Influence of length to thickness ratio on non-dimensional fundamental frequency (Ω) of square plate, $M / M_p = 0.5$, Material 3, $c / a = d / b = 0.4$

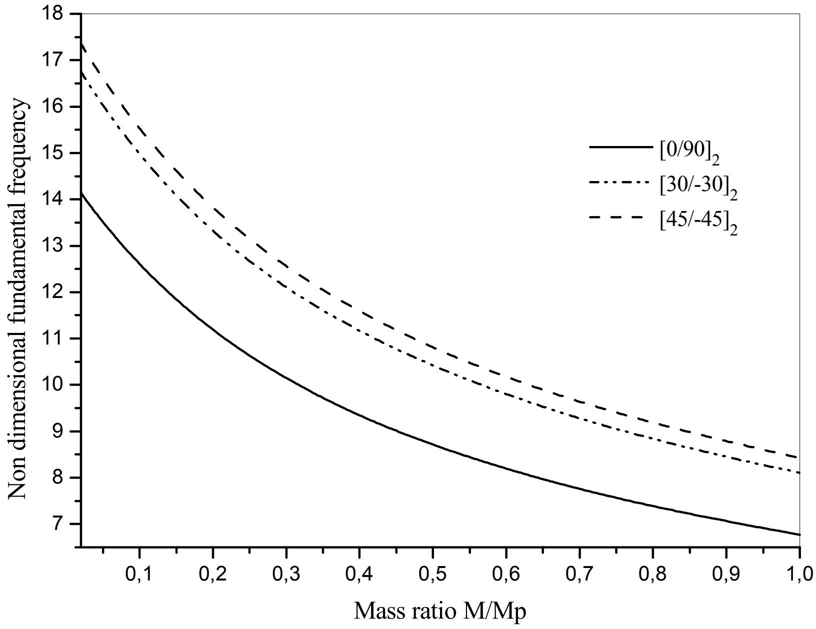

Fig. 4 The effect of mass ratio on non-dimensional fundamental frequency (Ω) for square plate, $a / h = 10$ a/h, MAT3

local mass, the stiffness of the plate decreases and causes to decrease the non-dimensional frequencies.

4. Conclusions

In this work, a trigonometric four variable plate theory for vibration study of laminated composite isotropic and anisotropic plates with patch mass was developed. The theory accounts for the shear deformation effects without requiring a shear correction factor. By dividing the transverse displacement into bending and shear components, the number of unknowns and governing equations of the present theory is reduced to four as against five in the FSDT and common higher-order shear deformation theories. The Hamilton's Principle, using the present trigonometric four variable theory, was applied to rectangular plates carrying a distributed patch mass, to determine the effect of the distributed patch mass on the fundamental frequencies of the plates. The effects of mass ratio, aspect ratio, and location of distributed patch mass on the plate's behaviour have been investigated. We note that the present approach can be extended to study the thermal buckling of laminated orthotropic plates (Moradi and Mansouri, 2012), nonlinear vibration of rectangular plates (Rashidi et al., 2012) or plates made of functionally graded materials (Yaghoobi and Yaghoobi, 2013).

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