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Rayleigh-Ritz procedure for determination of the critical load of tapered columns

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Abstract. EC3 provides several methodologies for the stability verification of members and frames. However, when dealing with the verification of non-uniform members in general, with tapered cross-section, irregular distribution of restraints, non-linear axis, castellated, etc., several difficulties are noted. Because there are yet no guidelines to overcome any of these issues, safety verification is conservative. In recent research from the authors of this paper, an Ayrton-Perry based procedure was proposed for the flexural buckling verification of web-tapered columns. However, in order to apply this procedure, Linear Buckling Analysis (LBA) of the tapered column must be performed for determination of the critical load. Because tapered members should lead to efficient structural solutions, it is therefore of major importance to provide simple and accurate formula for determination of the critical axial force of tapered columns. In this paper, firstly, the fourth order differential equation for non-uniform columns is derived. For the particular case of simply supported web-tapered columns subject to in-plane buckling, the Rayleigh-Ritz method is applied. Finally, and followed by a numerical parametric study, a formula for determination of the critical axial force of simply supported linearly web-tapered columns buckling in plane is proposed leading to differences up to 8% relatively to the LBA model.

Keywords: stability; Eurocode 3; tapered columns; FEM; steel structures; Rayleigh-Ritz

1. Introduction

EC3 provides several methodologies for the stability verification of members and frames. The stability of uniform members in EC3-1-1 (CEN, 2005) is checked by the application of clauses 6.3.1 – stability of columns; clause 6.3.2 – stability of beams and clause 6.3.3 – interaction formulae for beam-columns.

Regarding the stability of a non-uniform member, clauses 6.3.1 to 6.3.3 do not apply. The evaluation of the buckling resistance of such members lies outside the range of application of the interaction formulae of EC3-1-1 and raises some new problems to be solved. For those cases, verification should be performed according to clause 6.3.4 (general method) (Simões da Silva et. al, 2010a). Alternatively, the strength capacity may also be checked by a numerical analysis that accounts for geometrical and/or material imperfections and material and/or geometrical nonlinearities, henceforth denoted as GMNIA. However, for any of these methodologies, several

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difficulties are noted for the verification of a non-uniform member (Marques *et al*. 2012, Simões da Silva *et al*. 2010a).

Tapered steel members are commonly used over prismatic members because of their structural efficiency: by optimizing cross section utilization, significant material can be saved. However, if proper rules and guidance are not developed for these types of members, safety verification will lead to an over prediction of the material to be used. As a result, in Marques *et al*. (2012) a proposal was made for the stability verification of web-tapered columns subject to flexural buckling in-plane. Here, an analytical formulation for web-tapered steel columns subject to flexural buckling based on an Ayrton-Perry formulation was derived, this way making it possible to maintain consistency with EC3-1-1 flexural buckling verification procedure, clause 6.3.1, by extending it with adequate modifications. Columns with fork conditions, subjected to constant axial force were treated and adequate modification factors were calibrated against numerical analyses – LBA both for definition of the imperfections and slenderness determination, and GMNIA – Geometrical and Material Non-linear Analysis with Imperfections – for determination of the buckling load. In order to apply this procedure, however, it is necessary to know the critical axial force of the tapered column. Because this procedure was calibrated considering the numerical critical load from the LBA model, and because nowadays it is fairly simple to perform a linear buckling analysis with commercial software, the critical load may always be determined by performing a LBA analysis. However, it is of practical interest to provide simple formula that will attain a similar level of accuracy as the numerical analysis.

There are many available formulae and studues in the literature concerning different types of tapering, mainly regarding elastic stability formula (e.g., Ermopoulos 1997, Hirt and Crisinel 2001, Lee *et al*. 1972, Petersen 1993, Saffari *et al*. 2008, Serna *et al*. 2011, Yossif 2008, Li 2008, Maiorana and Pellegrino 2011) although some studies have been made on the inelastic stability verification (Baptista and Muzeau 1998, Raftoyiannis and Ermopoulos 2005, Naumes 2009). A description of these methods is given in more detail in Marques *et al*. 2012. In addition, in AISC

Table 1 Determination of the in-plane critical axial force from the literature

Source	Description
Hirt and Crisinel (2001)	Expression for equivalent moment of inertia for the tapered column, I_{eq}, depending on the type of web variation. Suitable for I-shaped cross sections. $N_{cr} = \frac{\pi^2 EI_{y,eq}}{L^2}, \quad I_{y,eq} = CI_{y,\max}$
[*H&C*]	$C = 0.08 + 0.92r, \quad r = \sqrt{I_{y,\min} / I_{y,\max}}$
Lee *et al*. (1972) Galambos (1998)	Expression for a modification factor of the tapered member length, *g*, i.e., calculation of the equivalent length of a prismatic column with the smallest cross section which leads to the same critical load. Suitable for I-shaped cross sections. $N_{cr} = \frac{\pi^2 EI_{y,\min}}{L_{eq}^{\ 2}}, \quad L_{eq} = g \cdot L$
[*L&al.*]	$g = 1 - 0.375\gamma + 0.08\gamma^2(1 - 0.0775\gamma), \quad \gamma = h_{\max} / h_{\min} - 1$

(Kaehler *et al*. 2010) the treatment of non-prismatic columns is based on the definition of an equivalent prismatic member which shall have the same critical load and the same first order resistance. Such member is then to be verified considering the rules for prismatic columns. Focusing now on the determination of the elastic critical load, for example, in Hirt and Crisinel (2001) an expression is presented for determination of the equivalent inertia of tapered columns, I_{eq}, with I-shaped cross sections, depending on the type of web variation (see Table 1). In Lee *et al*. (1972) (see also Galambos 1998), an expression is presented for a modification factor g of the tapered member length. The critical load is then calculated based on the smallest cross section (Table 1). In Petersen (1993), design charts for extraction of a factor β to be applied to the critical load of a column with the same length and the smallest cross section are available for different boundary conditions and cross section shapes. Also, Ermopoulos (1997) presents the non-linear equilibrium equations of non-uniform members in frames under compression for non-sway and sway mode. Equivalent length factors are calibrated for both cases based and presented in forms of tables and graphs similar to the ones presented in Annex E of ENV1993-1-1 (1992).

In this paper, based on the Rayleigh-Ritz energy method, a formula for determination of the in-plane critical axial force of simply supported linearly web-tapered axial force is provided leading to maximum differences of 8% (on the safe side) relatively to the numerical analysis. For this, in a first step, the fourth order differential equation for non-uniform columns subject to arbitrary axial loading and boundary conditions is presented and further simplified for the case of simply supported columns subject to constant axial force. The total potential energy is also presented for application of the energy method to be applied. In a second step, the numerical model and parametric study are presented as these will be necessary for calibration of an adequate displacement function to be considered in the Rayleigh-Ritz method. The method is then applied for the tapered column case and a formula is developed based on the results obtained. The developed formula is finally compared to the formulae of Table 1 showing an improved level of error when compared to the numerical (benchmark) LBA analysis.

2. Theoretical background for non-prismatic columns

2.1 Fourth order differential equation

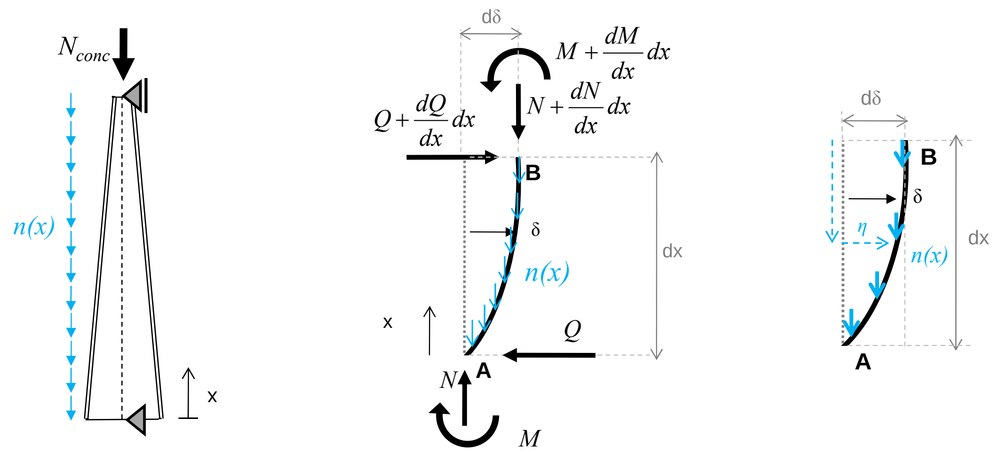
Fig. 1 illustrates the equilibrium of a column segment for arbitrary boundary conditions in its deformed configuration:

Considering the axial force as $N(x) = N_{conc} + \int_x^L n(\xi)d\xi$, neglecting second order terms and taking equilibrium of moments relatively to node B, gives

$$N(x).dy + Qdx - \left(M + \frac{dM}{dx}dx\right) + M - \underbrace{\int_0^{dx} n(\xi)\eta d\xi}_{\approx 0} = 0 \quad \rightarrow \quad Q = \frac{dM}{dx} - N(x)\frac{dy}{dx} \tag{1}$$

The equilibrium of horizontal forces gives

$$Q = Q + \frac{dQ}{dx}dx \quad \rightarrow \quad \frac{dQ}{dx} = 0 \tag{2}$$



(a) Non-uniform column (simply supported)

(b) Equilibrium of forces

(c) Detail regarding distributed force

Fig. 1 Equilibrium of a column segment

Eq. (1) can be differentiated one time

$$\frac{dQ}{dx}=0=\frac{d^2M}{dx^2}-\frac{d}{dx}\left(N(x)\frac{d\delta}{dx}\right) \tag{3}$$

Substituting the internal moment given by $M(x) = -EI(x)\dfrac{d^2\delta}{dx^2}$ in Eq. (3) leads to the differential equation Eq. (4)

$$E\frac{d^2}{dx^2}\left(I(x)\frac{d^2\delta}{dx^2}\right)+\frac{d}{dx}\left(N(x)\frac{d\delta}{dx}\right)=0 \tag{4}$$

or

$$E\big(I(x)\cdot\delta''\big)''+\big(N(x)\cdot\delta'\big)'=0 \tag{5}$$

The solution of this equation leads to the elastic critical load, see Eq. (6), in which α_{cr} is the critical load multiplier.

$$\begin{cases} N(x)=\alpha_{cr}N_{Ed}(x) \\ n(x)=\alpha_{cr}n_{Ed}(x) \\ \delta(x)=\delta_{cr}(x) \end{cases} \tag{6}$$

$N_{Ed}(x)$ is the applied axial force and α_{cr} is the critical load multiplier, and $\delta_{cr}(x)$ is the critical eigenmode.

2.2 Simply supported column

2.2.1 Differential equation

Simply supported columns with constant axial force are treated throughout this paper. The differential equation given by Eq. (4) is then simplified by

$$EI(x)\cdot\delta'' + N\cdot\delta = 0 \tag{7}$$

in which the following boundary conditions are considered for equilibrium

$$\begin{aligned} x = 0 \rightarrow &\begin{cases}\delta''_{cr} = 0\\ \delta_{cr} = 0\end{cases}\\ x = L \rightarrow &\begin{cases}\delta''_{cr} = 0\\ \delta_{cr} = 0\end{cases}\end{aligned} \tag{8}$$

2.2.2 Total potential energy

The total potential energy of the member is given by the sum of the strain U and potential V energy. Only the potential energy due to bending is considered here.

For a simple supported column the strain energy U_b due to bending is given by

$$U_b = \frac{1}{2}\int_V \sigma\varepsilon\, dV = \frac{1}{2}\int_V \left(\frac{My}{EI}\right)\left(\frac{My}{I}\right) dV = \frac{1}{2}\int_0^L \frac{1}{E}\left(\frac{M}{I}\right)^2 \underbrace{\left(\int_A y^2\, dA\right)}_{I} dx = \frac{1}{2}\int_0^L \frac{M^2}{EI}\, dx \tag{9}$$

Or, because $M=EI\delta_{cr}''$

$$U_b = \frac{1}{2}\int_0^L EI\left(\delta''_{cr}\right)^2 dx \tag{10}$$

And the potential energy V_b due to bending may be calculated by the work done on the system by the external forces

$$V_b = -N\Delta = -\int_0^L N\underbrace{\left(\frac{1}{2}\left(\delta'_{cr}\right)^2\right)}_{d\Delta} dx = -\frac{1}{2}\int_0^L N\left(\delta'_{cr}\right)^2 dx \tag{11}$$

3. Numerical model

A finite element model was implemented using the commercial finite element package Abaqus (2010), version 6.10. Four-node linear shell elements (S4) with six degrees of freedom per node and finite strain formulation were used.

For the material nonlinearity, an elastic-plastic constitutive law based on the Von Mises yield criterion is adopted.

A load stepping routine is used in which the increment size follows from accuracy and convergence criteria. Within each increment, the equilibrium equations are solved by means of the Newton-Raphson iteration.

S235 steel grade was considered with a modulus of elasticity of 210 GPa and a Poisson´s ratio of 0.3.

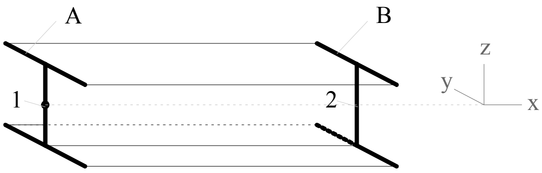


Fig. 2 Support conditions

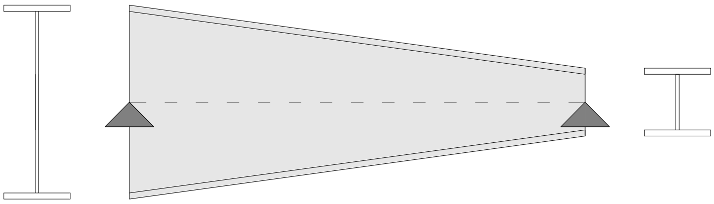


Fig. 3 Tapered member with horizontal centroid axis

Table 2 Parametric study

Taper ratio γ_h	Reference cross-section (i.e., with h_{min}, at $x = x_{min}$)	Reference column slenderness $\overline{\lambda}(x_{min}) = \sqrt{\dfrac{N_R(x_{min})/N_{Ed}}{\alpha_{cr}}}$	Fabrication procedure
1 ... 6	IPE 200	0 ... 3	Welded Hot-rolled (0.5 f_y)
	HEB 300		
	100 × 10 ($h = b = 100$ mm; $t_f = t_w = 10$ mm)		

The boundary conditions for a simply supported single span member with end fork conditions are implemented in the shell model as shown in Fig. 2. The following restraints are imposed: (ii) vertical (δ_y) and transverse (δ_z) displacements and rotation about xx axis (ϕ_x) are prevented at nodes 1 and 2. In addition, longitudinal displacement (δ_x) is prevented in node 1. Cross-sections A and B are modeled to remain straight.

For the in-plane behavior: δ_y is restrained at bottom and top of the web. In addition, cross-sections are modeled to remain straight against local displacements in the web.

Finally, regarding the tapered member, the web was considered to vary symmetrically to its centroid axis, according to Fig. 3.

4. Parametric study

More than 250 numerical LBA simulations with shell elements were carried to provide data for application of the energy method and for calibration of necessary parameters in Section 0. Table 2 summarizes the parametric study.

5. Application of the Rayleigh-Ritz method for the calculation of the elastic critical load

5.1 Rayleigh-Ritz method applied to the tapered column

The exact solution for the equilibrium equation is only possible to obtain for the most simple structures. For the case of tapered members, due to the variation of the second moment of area along the length, the solution of δ in Eq. (7) is not explicit and therefore, approximate or numerical methods are required to obtain the solution. Rayleigh-Ritz Method is presented and considered here. If an adequate displacement function δ_{cr} (Eq. (12)) satisfying the geometric boundary conditions is considered to approximate the real displacement, the structural system is reduced to a system with finite degrees of freedom (Chen and Lui 1987).

$$\delta_{cr} = a \cdot f \tag{12}$$

The total potential energy of the member is given by the sum of the strain U and potential V energy. Note that these are approximate, once the displacement function is also an approximation.

Considering the principle of stationary total potential energy, the solution for the critical load is obtained by solving Eq. (13), see e.g., (Chen and Lui 1987) for more details.

$$\frac{\partial(U+V)}{\partial a} = 0 \tag{13}$$

Considering Eqs. (10) and (11), Eq. (13) finally becomes

$$\frac{\partial(U+V)}{\partial a} = 0 \rightarrow \frac{\partial\left(\frac{1}{2}\int_0^L \left(EI(\delta''_{cr})^2 - N(\delta'_{cr})^2\right)dx\right)}{\partial a} = 0 \tag{14}$$

5.2 Adjustment of the displacement function

The displacement function δ_{cr} to be considered in Eq. (14) needs to satisfy the boundary conditions (Eq. (8)). For a tapered column buckling in-plane with the smallest cross-section $h = b = 100$ mm and $t_f = t_w = 10$ mm (denoted as $100x10$), the following function was adjusted based on the critical mode displacement obtained by the LBA analyses.

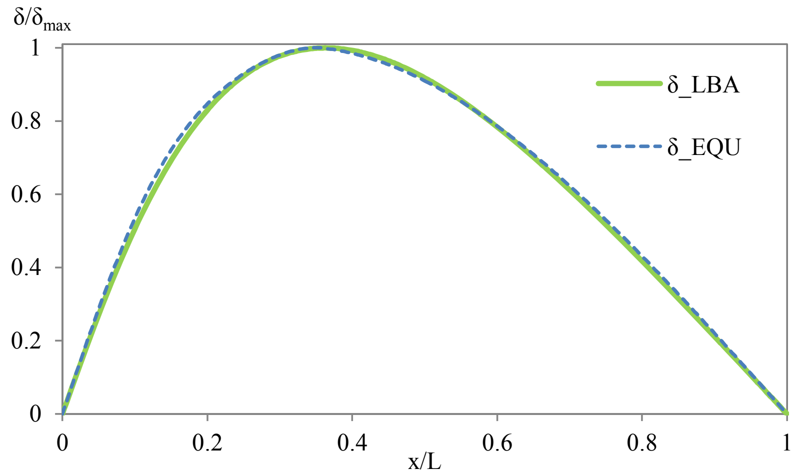


Fig. 4 Displacement function for the in-plane critical mode of a web-tapered column (100 × 10; $\gamma_h = 5$)

$$\delta_{cr} = a \times \begin{cases} \cos\left(\dfrac{\pi\left(x - x_{cr,\max}\right)}{2x_{cr,\max}}\right) - 0.058 \cdot \ln\gamma_h \sin\left(\dfrac{\pi\left(x - x_{cr,\max}\right)}{x_{cr,\max}}\right) & ;\quad 0 \le x \le x_{cr,\max} \\ + 0.024 \cdot \ln\gamma_h \sin\left(\dfrac{2\pi\left(x - x_{cr,\max}\right)}{x_{cr,\max}}\right) & \\ \cos\left(\dfrac{\pi\left(x - x_{cr,\max}\right)}{2\left(L - x_{cr,\max}\right)}\right) - 0.029 \cdot \ln\gamma_h \sin\left(\dfrac{\pi\left(x - x_{cr,\max}\right)}{L - x_{cr,\max}}\right) & ;\quad x_{cr,\max} \le x \le L \end{cases} \tag{15}$$

In Eq. (15), $x_{cr,\max}$ is the location corresponding to the maximum deflection and $\gamma_h = h_{\max} / h_{\min}$ is the taper ratio regarding the maximum and minimum depth. $x_{cr,\max}$ may be given by

$$x_{cr,\max} = 0.5 \cdot \gamma_h^{-0.208} \cdot L \tag{16}$$

The fitted function for δ_{cr} given by Eq. (15) (*δ_EQU*) is illustrated in Fig. 4 and compared to the eigenmode deflection (*δ_LBA*). A small error is obtained and Eq. (15) will be considered for application of the Rayleigh-Ritz method.

6. Results

Consider the cross section *100x10* for a range of taper ratios γ_h between 1 and 6. The solution of Eq. (15) is given in terms of $N_{cr,Tap} = a/L^2$ and can be represented as a function of the critical load of the smallest section, $N_{cr,\min}$.

$$N_{cr,Tap} = A \cdot N_{cr,\min} \rightarrow A = \frac{a}{\pi^2 EI_{y,\min}} \tag{17}$$

Based on the values of *a* obtained by the Rayleigh-Ritz analysis, an expression is now given for *A*.

$$\begin{aligned} &N_{cr,Tap} = A \cdot N_{cr,\min} \rightarrow A = \gamma_I^{\,0.56}\left(1 - 0.04 \cdot \tan^{-1}(\gamma_I - 1)\right) \\ &\gamma_I = I_{y.\max} / I_{y.\min} \end{aligned} \tag{18}$$

Eq. (18) was calibrated to give results on the safe side as it can be observed in Fig. 5. *EQU_RR* represents the results of A given by the Rayleigh-Ritz method, Eq. (17), whereas *EQU_Adjusted* represents Eq. (18). As it may be noticed by observing Fig. 5, the Rayleigh-Ritz Method leads to slightly unconservative results relatively to the LBA analysis. This happens because, from the moment the member is forced to displace according to a certain function (other than its natural displacement shape) the member becomes constrained and a higher load than the load that leads to the minimum (real) energy is obtained. As a result, the better the displacement function, the lower the unconservatism. It can be seen that even for the highest taper ratio this error is still small, which confirms that the displacement function is well adjusted. Nevertheless, for calibration of Eq. (18), a weighing factor relatively to the result of the Rayleigh-Ritz method was included in order to provide safe results.

Note that the taper ratio chosen for calculation of the critical load in Eq. (18) is represented in terms of the ratio between the maximum and minimum inertia, i.e., $\gamma_I = I_{\max} / I_{\min}$. This is the best parameter to characterize the elastic flexural buckling behavior of the tapered column. When analyzing other sections, e.g., a *HEB300* (smallest cross-section) that present the same γ_I, a very good agreement is noticed in the function for δ_{cr} and also in the function that characterizes the second moment of area along the column. As a result, the expression of Eq. (18) may be used for any section. For the member with a smallest cross section 100*x*10, $\gamma_h = 1.9$ and for the *HEB300*, $\gamma_h = 2$. Both members present $\gamma_I = 4.62$.

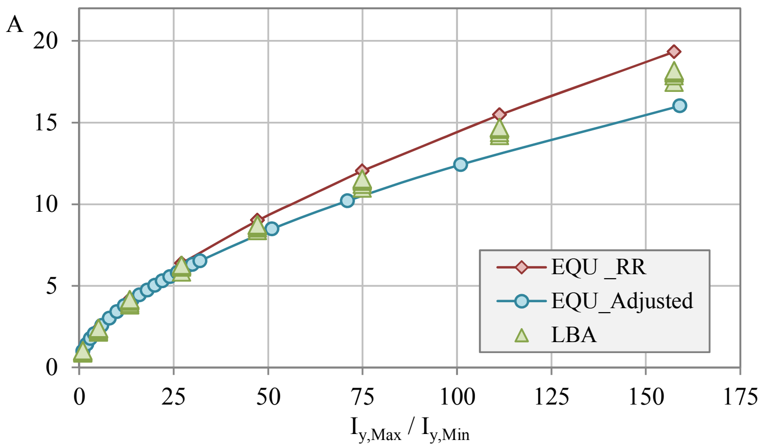


Fig. 5 Calibration of factor A

In addition, the above-defined expression may be considered with not much increase in error on cross sections with varying flange buckling out-of-plane. The inertia of the flanges buckling out-of-plane can be compared to the inertia of the web buckling in-plane. The analyzed member is composed of a smallest cross section 100x10 with $\gamma_b = b_{\max} / b_{\min} = 1.67$ (and accordingly, a $\gamma_I = 4.62$, in which for this case I_y is replaced by I_z). The same however cannot be considered for flange-tapered columns buckling in-plane, as the inertia varies linearly. A similar Rayleigh-Ritz procedure could be adopted for the latter, it is however not the scope of this study.

Fig. 6 illustrates the moment of inertia (I_z or I_y) variation and Table 3 compares the analyzed cases with a Linear Buckling Analysis. Lengths of the columns were chosen in order to lead to similar (numerical) slenderness $\bar{\lambda} = \sqrt{N_{pl,\min} / N_{cr,tap}^{LBA}}$. The critical displacement δ_{cr} is not illustrated as results practically match.

Finally, for a range of cross-sections with varying γ_h (or γ_I) the error is analyzed in Fig. 7. For comparison, the procedures given in Table 1 are also shown. Note that, because the taper ratio γ_h is an intuitive parameter to describe the tapered member, presentation of results relatively to that parameter γ_h is kept. The difference is given by Eq. (19), such that a positive difference illustrates a safe evaluation of N_{cr} by the given method. Maximum differences of 8% (on the safe side) are

Table 3 Analysis of the critical load obtained by Eq. (18)

Ref. section	γ_h	γ_b	Buckling mode	N_{cr}^{LBA} [kN]	$N_{pl,\min}$ [kN] (S235)	$\bar{\lambda}_y$	$N_{cr,\min}$ [kN]	γ_I	A	$N_{cr,\text{tap}}$ [kN]	Diff (%)
100 × 10	1.9	-	In	248.5	658	1.63	110.6			246.9	0.64
HEB300	2.0	-	In	1242.6	3356.27	1.64	551.1	4.62	2.23	1231.0	0.94
100 × 10	-	1.67	Out	252.4	658	1.61	114.4			255.4	-1.22

*For HEB300 the fillet radius is not considered

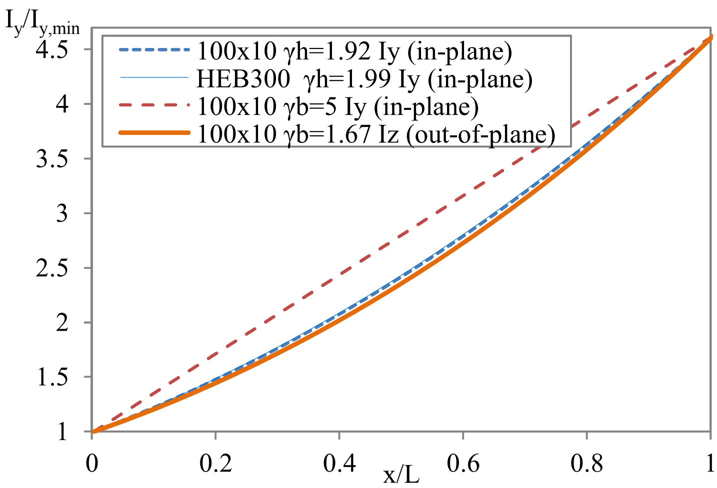
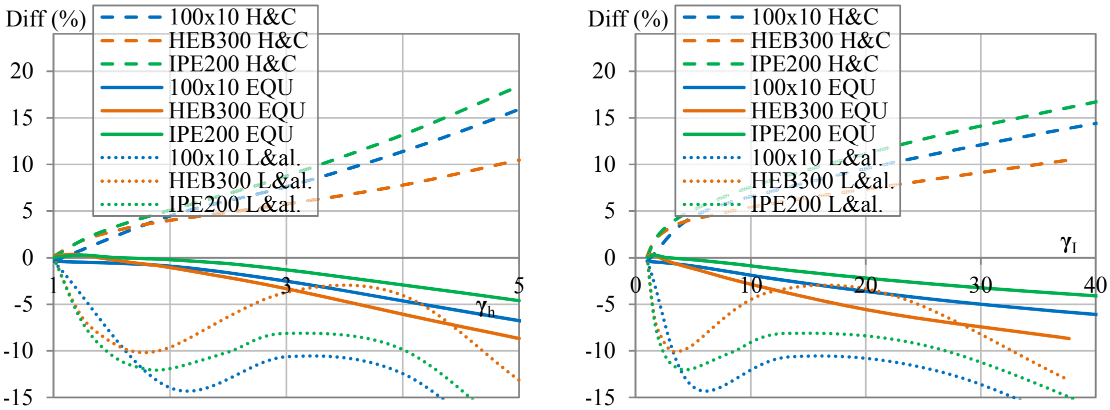


Fig. 6 Variation of inertia along the member for distinct sections with the same $\gamma_I = I_{\max} / I_{\min}$



(a) According to γ_h (b) According to γ_I

Fig. 7 Analysis of the error given by the proposed expression for $N_{cr,\mathrm{tap}}$

noted. It is measured relatively to the columns with higher slenderness, i.e., for which the numerical analysis does not present the effect of shear. For the low slenderness range this effect is higher and decreases asymptotically to the correct critical load – this can be observed for the well-known solution of a simply supported column with prismatic cross-section (Euler load).

$$Diff\ (\%) = 100 \times \left(1 - \frac{N_{cr,tap}^{Method}}{N_{cr,tap}^{LBA}} \right) \tag{19}$$

Finally, note that a given error in the critical load relatively to the (assumed) real critical load (obtained by LBA) will always lead to a much smaller error in the ultimate load stability verification procedure (developed in Marques *et al*. 2012, for example, see Section 1). In addition, this difference will always be safe-sided considering that the developed formula in this paper was also calibrated to be conservative.

7. Conclusions

In this paper, a simple formula for calculation of the major axis critical axial force was developed.

An analytical derivation for elastic flexural buckling of non-prismatic columns was firstly presented. A parametric study of more than 250 LBA simulations was then carried out regarding simply supported linearly web-tapered column with constant axial force. After that, a displacement function for the in-plane critical mode was adjusted and the Rayleigh-Ritz method was then considered for development of a simple formula for calculation of the critical load of web-tapered columns leading to an excellent agreement with numerical LBA analysis. This formula is then adequate for application in recent proposals for the nonlinear stability verification of tapered columns.

Future research aims at solving the case of other support conditions and web height variation, not only at the critical load level, but also taking into account material and geometrical nonlinearities, i.e., ultimate load.

Acknowledgments

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Notations

Lowercases

a, A	Auxiliary terms for application of proposed formula for $N_{cr,\mathrm{Tap}}$
b	Cross section width
$b_{\max}$	Maximum cross section width
$b_{\min}$	Minimum cross section width
f	Function for the displacement
h	Cross section height
$h_{\max}$	Maximum cross section height
$h_{\min}$	Minimum cross section height
$n(x)$	Distributed axial force
$n_{ed}(x)$	Design distributed axial force
t_f	Flange thickness
t_w	Web thickness
x-x	Axis along the member
y-y	Cross section axis parallel to the flanges
z-z	Cross section axis perpendicular to the flanges

Uppercases

A	Cross section area
E	Modulus of elasticity
GMNIA	Geometrical and Material Non-linear Analysis with Imperfections
I	2^{nd} moment of area
I_y, I_z	Second moment of area, y-y axis and z-z axis
$I_{y,eq}$	Equivalent 2^{nd} moment of area, y-y axis
$I_{y,\max}$	Maximum 2^{nd} moment of area, yy axis
$I_{y,\min}$	Minimum 2^{nd} moment of area, yy axis
L	Member length
$L_{cr,z}$, $L_{cr,y}$	Member buckling length regarding flexural buckling, minor and major axis
LBA	Linear Buckling Analysis
L_{eq}	Equivalent member length
M	Bending moment
N	Normal force
N_{conc}	Concentrated axial force
$N_{cr,tap}^{LBA}$	Elastic critical force of a tapered column obtained by a LBA analysis
$N_{cr,z}$	Elastic critical force for out-of-plane buckling
$N_{cr,z,tap}$	Elastic critical force of the tapered column about the weak axis
N_{Ed}	Design normal force
Q	Shear force
U, U_b	Strain energy, due to bending
V, V_b	Potential energy, due to bending

Lowercase Greek letters

α_{cr}	Load multiplier which leads to the elastic critical resistance
γ_i	Taper ratio: γ_I – according to inertia; γ_h – according to height; γ_b – according to witdh
δ_{cr}	General displacement of the critical mode
δ_y	Displacement about y-y axis
δ_z	Displacement about z-z axis
$\bar{\lambda}$	Non-dimensional slenderness
$\bar{\lambda}_y$	Non-dimensional slenderness for flexural buckling, y-y axis
$\bar{\lambda}_z$	Non-dimensional slenderness for flexural buckling, z-z axis
$\xi.\ \eta$	Rectangular coordinates, longitudinal and transversal