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Nonlinear vibration of Euler-Bernoulli beams resting on linear elastic foundation

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Abstract. In this study simply supported nonlinear Euler-Bernoulli beams resting on linear elastic foundation and subjected to the axial loads is investigated. A new kind of analytical technique for a non-linear problem called He's Energy Balance Method (EBM) is used to obtain the analytical solution for non-linear vibration behavior of the problem. Analytical expressions for geometrically non-linear vibration of Euler-Bernoulli beams resting on linear elastic foundation and subjected to the axial loads are provided. The effect of vibration amplitude on the non-linear frequency and buckling load is discussed. The variation of different parameter to the nonlinear frequency is considered completely in this study. The nonlinear vibration equation is analyzed numerically using Runge-Kutta 4^{th} technique. Comparison of Energy Balance Method (EBM) with Runge-Kutta 4^{th} leads to highly accurate solutions.

Keywords: elastic foundation; nonlinear vibration; analytical method; Runge-Kutta 4^{th}

1. Introduction

In the last decades many researchers have been worked on nonlinear free vibrations of beams resting on elastic foundations such as soil etc. Different approaches have been proposed to represent the foundations of these structures in which they supported along their main axis supported such as Winkler, Pasternak or Vlasov, Flonenko - Borodich foundations. One of the most important approaches is Winkler model. It was first proposed by Winkler in 1867.Winkler model considers the normal displacement of the structure. In this method a linear algebraic relationship is introduces between the normal displacement of the structure and the contact pressure (Gorbunov-Posadov 1973). A set of mutually parallel independent spring elements are used in the Winkler model to represent the soil medium (Al-Hosani *et al.* 1999). Therefore, analyzing the nonlinear behavior of the system could be more easily compared to other methods (Soldatos and Selvadurai 1985).

Many researchers have been worked on the Winkler elastic foundation modeling in the past few decades. Zhou (1993) developed the work on the vibration frequencies of beams on a variable Winkler elastic foundation and tried to provide a general solution. A compression between the

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infinite beams on half-space and finite and infinite beams on a Winkler support was done by Auersch (2008). Eisenberger and Clastornik (1987) considered the vibrations and buckling of a beam on a variable Winkler elastic foundation. A complete study was done by Gupta *et al*. (2006) to extend the study for buckling and vibrational behavior of polar orthotropic circular plates with linearly varying thickness. Ruge and Birk (2007) compared the different infinite beam models on Winkler foundation and try to analysis the approximate dynamic behavior of the models at high frequencies. Lee (1998) investigated the response of a Timoshenko beam with a moving concentrated mass.

Generally, finding an exact solution for nonlinear problems are very difficult, therefore some approximate analytical methods have been proposed by many researchers to solve nonlinear vibration problems such as: Parameter Expansion Method (Xu 2007), Differential Transform Method (Arikoglu and Ozkol 2006), Variational Iteration Method (Liu 2009), Homotopy Perturbation Method (Shou 2009), Max-Min Approach (Bayat *et al*. 2012, He 2008, Pirbodaghi *et al*. 2009) and other analytical techniques (He 2007, Rao 2007, Tse *et al*. 1987, Azrar *et al*. 1999, He 2002, Ren and Gui 2011, Bayat *et al*. 2012, 2013, Bayat and Pakar 2013a, b, Pakar *et al*. 2012, Pakar and Bayat 2012, 2013a, b).

In this paper, we try to provide an approximate analytical solution for geometrically non-linear vibration of Euler-Bernoulli beams resting on a Winkler elastic foundation and subjected to the axial loads in time domain. The elastic variation through the beam is considered as constant .Galerkin method is used for discretization to obtain an ordinary nonlinear differential equation from the governing non- linear partial differential equation. It was then assumed that only fundamental mode was excited. In order to find the nonlinear frequency of the vibration the Energy Balance Method (EBM) is applied to obtain analytical solution and the results compared with Runge-Kutta. The EBM results are accurate and only one iteration leads to high accuracy of solutions for whole domain and can be a powerful approach for solving high nonlinear engineering problems.

2. Description of the problem

Fig. 1 represents a simply-supported buckled Euler-Bernoulli beam fixed at one end resting on Winkler foundation. The basic assumptions of the beam theory are considered such as (Rao 2007).

(1) The beam is isotropic and elastic.
(2) The beam deformation is dominated by bending and the distribution and rotation are negligible.
(3) The beam is along as slender with a constant section along the axis.

The equation of motion for an axially loaded Euler-Bernoulli beam by considering the mid-plane stretching effect is

$$EI\frac{\partial^4 W'}{\partial X'^4}+M\frac{\partial^2 W'}{\partial t'^2}+\overline{P}\frac{\partial^2 W'}{\partial X'^2}+K'(X)W'-\frac{EA}{2L}\frac{\partial^2 W'}{\partial X'^2}\int_0^L\left(\frac{\partial W'}{\partial X'}\right)^2 dX'=U(X',t') \tag{1}$$

where K' is a foundation modulus and U is a distributed load in the transverse direction.

Assume the non-conservative forces were equal to zero. Therefore Eq. (1) can be written as follows

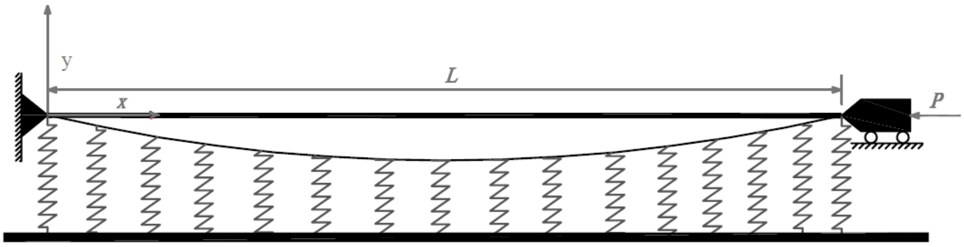


Fig. 1 Schematic representation of an axially loaded Euler-Bernoulli beam resting on Winkler foundation

$$EI\frac{\partial^4 W'}{\partial X'^4}+M\frac{\partial^2 W'}{\partial t'^2}+\overline{P}\frac{\partial^2 W'}{\partial X'^2}+K'(X)W'-\frac{EA}{2L}\frac{\partial^2 W'}{\partial X'^2}\int_0^L\left(\frac{\partial W'}{\partial X'}\right)^2 dX'=0 \tag{2}$$

Here we introduce the following non-dimensional variables

$$X=\frac{X'}{L},\ \ W=\frac{W'}{R},\ \ t=t'\sqrt{\frac{EI}{ML^4}},\ \ P=\frac{\overline{P}L^2}{EI},\ \ K=\frac{K'L^4}{EI} \tag{3}$$

where $R=\sqrt{I/A}$ is the radius of gyration of the cross-section. We assume the elastic coefficient of Winkler foundation is constant $K'(X)=K_0$. Then Eq. (2) can be written as follows

$$\frac{\partial^4 W}{\partial X^4}+\frac{\partial^2 W}{\partial t^2}+P\frac{\partial^2 W}{\partial X^2}+K_0 W-\frac{1}{2}\frac{\partial^2 W'}{\partial X^2}\int_0^L\left(\frac{\partial W}{\partial X}\right)^2 dX=0 \tag{4}$$

If we assume $W(X,\ t)=w(t)\,\phi(X)$ in which $w(t)$ is an unknown time dependent function and $\phi(X)$ is a trial function which must satisfy the kinematic boundary conditions and using the Galerkin method, then we will have the following governing nonlinear vibration equation of motion for an axially loaded Euler-Bernoulli beam

$$\frac{d^2 w(t)}{dt^2}+(\varepsilon_1+P\varepsilon_2+K_0)w(t)+\varepsilon_3 w^3(t)=0 \tag{5}$$

The initial conditions for center of the beam are

$$t=0,\ \ w=a,\ \ dw/dt=0 \tag{6}$$

The value of the ε_1, ε_2 and ε_3 can be obtained as follow

$$\varepsilon_1=\left(\int_0^1\left(\frac{\partial^4\phi(X)}{\partial X^4}\right)\phi(X)dX\right)\Bigg/\int_0^1\phi^2(X)dX \tag{7a}$$

$$\varepsilon_2=\left(\int_0^1\left(\frac{\partial^2\phi(X)}{\partial X^2}\right)\phi(X)dX\right)\Bigg/\int_0^1\phi^2(X)dX \tag{7b}$$

$$\varepsilon_3 = \left(\left(-\frac{1}{2} \right) \int_0^1 \left(\frac{\partial^2 \phi(X)}{\partial X^2} \int_0^1 \left(\frac{\partial^2 \phi(X)}{\partial X^2} \right)^2 dX \right) \phi(X) dX \right) \Big/ \int_0^1 \phi^2(X) dX \tag{7c}$$

3. Basic concept of energy balance method

In the present paper, we consider a general nonlinear oscillator in the Form (He 2002)

$$\ddot{w} + f\big((w(t)\big) = 0 \tag{8}$$

In which w and t are generalized dimensionless displacement and time variables, respectively. Its variational principle can be easily obtained

$$J(w) = \int_0^t \left(-\frac{1}{2}\dot{w}^2 + F(w) \right) dt \tag{9}$$

where $T = 2\pi/\omega$ is period of the nonlinear oscillator, $F(w) = \int f(w)dw$.
Its Hamiltonian, therefore, can be written in the form

$$H = \frac{1}{2}\dot{w}^2 + F(w) + F(a) \tag{10}$$

or

$$R(t) = -\frac{1}{2}\dot{w}^2 + F(w) - F(a) = 0 \tag{11}$$

Oscillatory systems contain two important physical parameters, i.e.,
The frequency ω and the amplitude of oscillation a. So let us consider such initial conditions

$$w(0) = a, \quad \dot{w}(0) = 0 \tag{12}$$

We use the following trial function to determine the angular frequency ω

$$w(t) = a\cos\omega t \tag{13}$$

Substituting (13) into w term of (11), yield

$$R(t) = \frac{1}{2}\omega^2 a^2 \sin^2(\omega t) + F\big(a\cos(\omega t)\big) - F(a) = 0 \tag{14}$$

If, by chance, the exact solution had been chosen as the trial function, then it would be possible to make R zero for all values of t by appropriate choice of ω. Since Eq. (13) is only an approximation to the exact solution, R cannot be made zero everywhere. Collocation at $\omega t = \pi/4$ gives

$$\omega = \sqrt{\frac{2\big(F(a)\big) - F\big(a\cos(\omega t)\big)}{a^2\sin^2(\omega t)}} \tag{15}$$

Its period can be written in the form

$$T = \frac{2\pi}{\sqrt{\dfrac{2\big(F(a)\big) - F\big(a\cos(\omega t)\big)}{a^2 \sin^2(\omega t)}}} \tag{16}$$

4. Application of the energy balance method

In Eq. (5), its variational formulation can be easily obtained as follow

$$J\big(w(t)\big) = \int_0^t \left(-\frac{1}{2}\left(\frac{dw}{dt}\right)^2 - + \frac{1}{2}(\varepsilon_1 + p\varepsilon_2 + K_0) w^2(t) + \frac{1}{4}\varepsilon_3 w^4(t) \right) dt. \tag{17}$$

Its Hamiltonian, therefore, can be written in the form

$$H = \frac{1}{2}\left(\frac{dw}{dt}\right)^2 + \frac{1}{2}(\varepsilon_1 + p\varepsilon_2 + K_0) w^2(t) + \frac{1}{4}\varepsilon_3 w^4(t) = \frac{1}{2}(\varepsilon_1 + p\varepsilon_2 + K_0) a^2 + \frac{1}{4}\varepsilon_3 a^4 \tag{18}$$

or

$$H = \frac{1}{2}\left(\frac{dw}{dt}\right)^2 + \frac{1}{2}(\varepsilon_1 + p\varepsilon_2 + K_0) w^2(t) + \frac{1}{4}\varepsilon_3 w^4(t) - \frac{1}{2}(\varepsilon_1 + p\varepsilon_2 + K_0) a^2 - \frac{1}{4}\varepsilon_3 a^4 = 0 \tag{19}$$

In Eqs. (19) and (18) the kinetic energy (E) and potential energy (T) can be respectively expressed as $(dw/dt)^2/2$, $(\varepsilon_1 + p\varepsilon_2 + K_0)\, w^2(t)/2 + \varepsilon_3 w^4(t)/4$ throughout the oscillation, it holds that $H = E + T$ constant.

We use the trial function to determine the angular frequency ω

$$w(t) = a\cos(\omega t) \tag{20}$$

If we substitute Eq. (20) into Eq. (19), it results the following residual equation

$$\begin{aligned} R(t) = &\left(\frac{1}{2} a^2 \omega^2 \sin^2(\omega t) + \frac{1}{2}(\varepsilon_1 + p\varepsilon_3 + K_0) a^2 \cos^2(\omega t) + \frac{1}{4}\varepsilon_3 a^4 \cos^4(\omega t) \right) \\ &- \left(\frac{1}{2}(\varepsilon_1 + p\varepsilon_2 + K_0) a^2 + \frac{1}{4}\varepsilon_3 a^4 \right) \end{aligned} \tag{21}$$

If we collocate at $\omega t = \pi/4$ we obtain

$$\frac{1}{4} a^2 \omega^2 - \frac{1}{4}(\varepsilon_1 + p\varepsilon_2 + K_0) a^2 - \frac{3}{16}\varepsilon_3 a^4 = 0 \tag{22}$$

This leads to the following result

$$\omega = \sqrt{(\varepsilon_1 + p\varepsilon_2 + K_0) + \frac{3}{4}\varepsilon_3 a^2} \tag{23}$$

Non-linear to linear frequency ratio is

$$\frac{\omega_{NL}}{\omega_L} = \frac{\sqrt{(\varepsilon_1 + p\varepsilon_2 + K_0) + \frac{3}{4}\varepsilon_3 a^2}}{\sqrt{\varepsilon_1 + p\varepsilon_2 + K_0}} \tag{24}$$

According to Eqs. (20) and (23), we can obtain the following approximate solution

$$w(t) = a\cos\left(\sqrt{(\varepsilon_1 + p\varepsilon_2 + K_0) + \frac{3}{4}\varepsilon_3 a^2 t}\right) \tag{25}$$

5. Results and discussion

The Energy Balance Method (EBM) is used to obtain an analytical solution for simply supported beam at constant elastic modulus.

To obtain numerical solution we must specify the parameter β. This parameter depends on value of ε_1, ε_2, ε_3 and p, then we have

$$\beta = \frac{\varepsilon_3}{(\varepsilon_1 + p\varepsilon_2 + K_0)} \tag{26}$$

So Eq. (21) become

$$\frac{\omega_{NL}}{\omega_L} = \sqrt{1 + \frac{3}{4}\beta a^2} \tag{27}$$

For simply supported beam the trial function $\phi(X) = \sin(\pi X)$ is assumed.

Table 1 represents the comparison of present study with the results obtained by (Azrar *et al*. 1999) for different values of amplitude and β. Fig. 2 considered the comparisons of energy balance method and Runge-Kutta algorithm (Appendix A) for two cases to show the effects of the elastic soil stiffness on the displacement response. Fig. 3 is one of the important results in this section;

Table 1 Comparison of nonlinear to linear frequency ratio (ω_{NL} / ω_L) for simply-supported beams

α	β	Present study (EBM)	Pade approximate P{4,2} (Azrar *et al*. 1999)	Pade approximate P{6,4} (Azrar *et al*. 1999)
0.2	3	1.04403	1.04388	1.04388
0.6	3	1.34536	1.33973	1.33970
1	3	1.80277	1.78468	1.78442
1.5	3	2.46221	2.42618	2.42541
2	3	3.16227	3.10845	3.10712
2.5	3	3.88104	3.80991	3.80802
3	3	4.60977	4.52172	4.51927

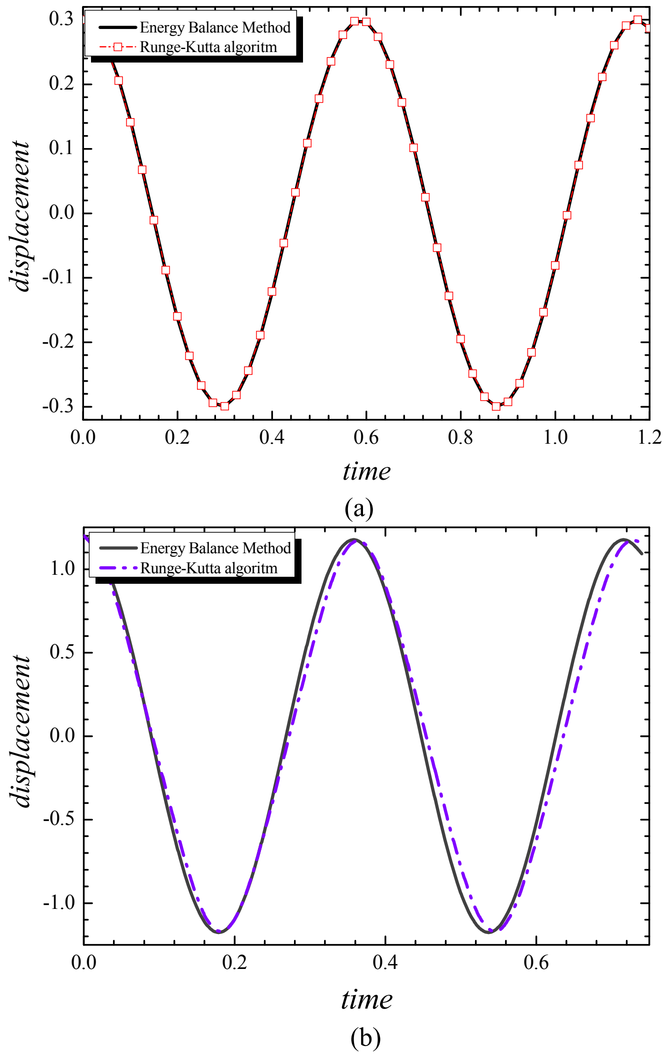


Fig. 2 Comparison of analytical solution of time history response with the numerical solution for simply supported beam (a) $a = 0.3$, $p = 10$, $K_0 = 100$; (b) $a = 1.2$, $p = 10$, $K_0 = 50$

because it is shown the effects of K_0 on ration the nonlinear to linear frequency of the beam vibration. It can be observed from the figure when the stiffness of the elastic soil grows up; the linear and nonlinear frequency comes to be closer and the ratio is also closer to 1. Fig. 4 is shown the axial load of the system on the nonlinear to linear frequency of the system. The axial load has a great effect on increasing the nonlinear frequency of the system.

Fig. 5 is a sensitivity analysis of nonlinear frequency of Energy Balance solution for various parameters for two cases. Case (a) is for a, K_0, ω and case (b) is for a, p, ω.

The phase plan of the problem is considered in figure 6 for two important cases. Case (a) shows the effects of the K_0 on the phase plan of the problem for different cases and the case (b) is the effect of p on the phase plan diagram.

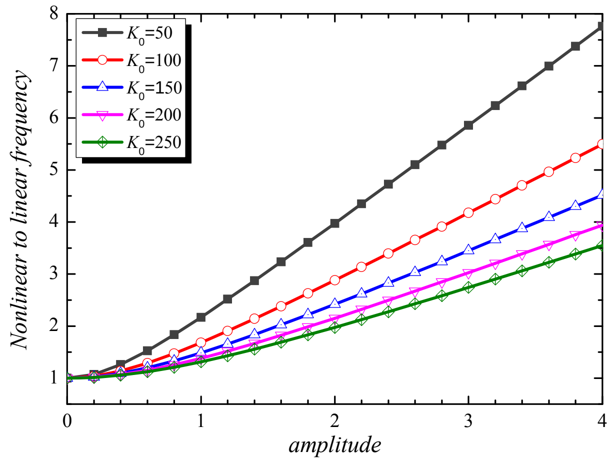


Fig. 3 Influence of K_0 on nonlinear to linear frequency base on amplitude for $p = 10$

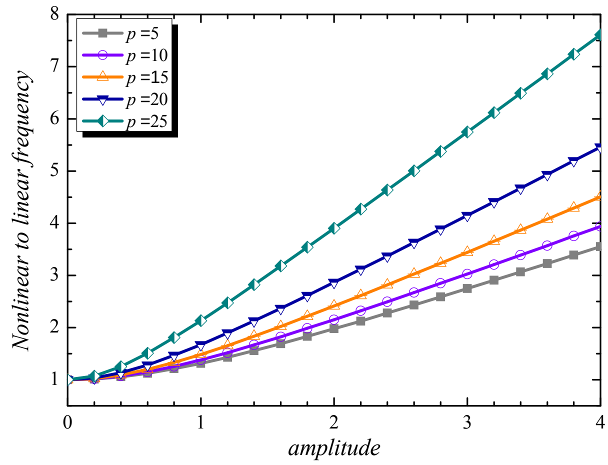
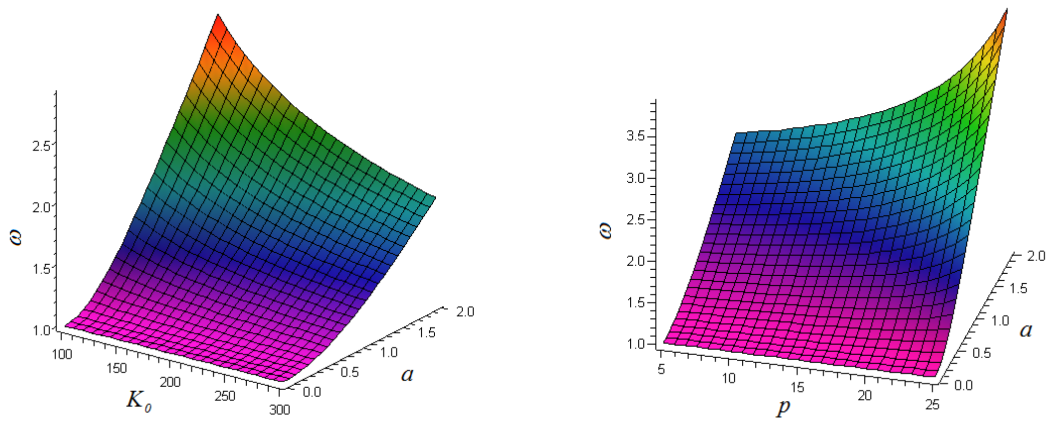


Fig. 4 Influence of axial load on nonlinear to linear frequency base on amplitude for $K_0 = 200$



(a) (b)

Fig. 5 Sensitivity analysis of frequency of energy balance solution for various parameters

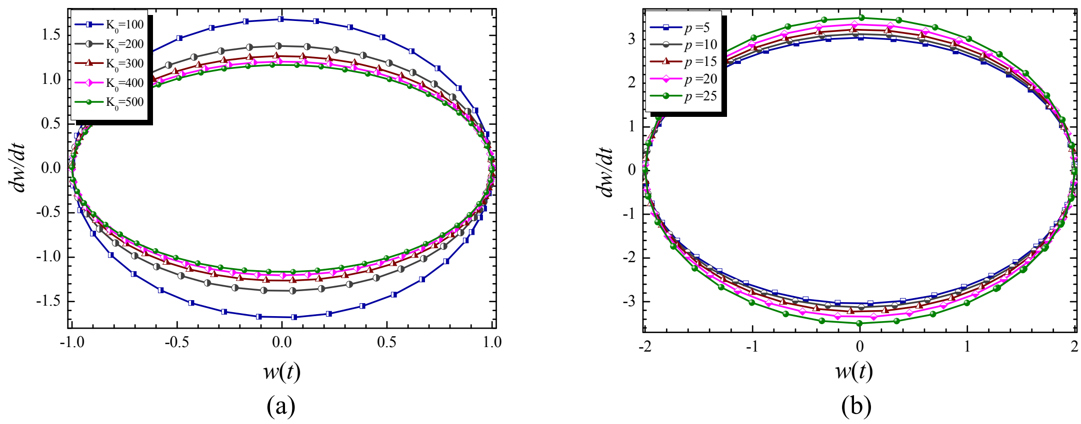


Fig. 6 Effect of K_0 and p parameters on phase-plan diagram for the cases: (a) a = 1, p = 10; (b) a = 2, K_0 = 500

6. Conclusions

Nonlinear dynamic response of an Euler-Bernoulli beam Euler-Bernoulli beams resting on a Winkler elastic foundation and subjected to the axial loads has been solved analytically by using a new novel method called Energy Balance Method (EBM) in time domain. Winkler approach is used widely to the beams and pipelines resting on an elastic soil. In the present work we assume the elastic coefficient of the springs is constant. As shown in this paper, the results of EBM have an excellent agreement with the numerical solutions. Its excellent accuracy in the whole range of oscillation amplitude values is one of the most significant features of this method. The successful application of the Energy Balance Method (EBM) for the large-amplitude beam vibration problem is considered in this study. The EBM could be easily extended to any nonlinear vibration problems which provide an easy and direct procedure for determining approximations to the periodic solutions.

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Nomenclature

A	cross-sectional area
L	beam length
W'	normal displacement
E	Young's modulus
X	axial coordinate
$\overline{P}$	axial load
M	mass per unit length
$\phi(X)$	trial function
t	time
K'	elastic coefficient of Winkler foundation
EA	axial rigidity of the beam cross section
EI	bending rigidity of the beam cross section
$w(t)$	time-dependent deflection parameter
A	dimensionless maximum amplitude of oscillation
β	parameter of boundary condition of beam
ω_{NL}	nonlinear frequency
ω_L	linear frequency

Appendix A: Basic idea of Runge-Kutta's algorithm

For such a boundary value problem given by boundary condition, some numerical methods have been developed. Here we apply the fourth-order RK algorithm to solve governing equations subject to the given boundary conditions. RK iterative formulae for the second-order differential equations are

$$\begin{aligned} \dot{w}_{(i+1)} &= \dot{w}_i + \frac{\Delta t}{6}(h_1 + 2h_2 + 2h_3 + h_4), \\ w_{(i+1)} &= w_i + \Delta t\left[\dot{w}_i + \frac{\Delta t}{6}(h_1 + h_2 + h_3)\right], \end{aligned} \tag{A.1}$$

where Δt is the increment of the time and h_1, h_2, h_3 and h_4 are determined from the following formulas

$$\begin{aligned} h_1 &= f\left(t_i\ ,\ w_i\ ,\ \dot{w}_i\right)\ , \\ h_2 &= f\left(t_i + \frac{\Delta t}{2},\ \ w_i + \frac{\Delta t}{2},\ \ \dot{w}_i + \frac{\Delta t}{2}h_1\right)\ , \\ h_3 &= f\left(t_i + \frac{\Delta t}{2},\ \ w_i + \frac{\Delta t}{2}\dot{w},\ \ \frac{1}{4}\Delta t^2 h_1\ ,\ \ \dot{w}_i + \frac{\Delta t}{2}h_2\right)\ , \end{aligned} \tag{A.2}$$

The numerical solution starts from the boundary at the initial time, where the first value of the displacement function and its first-order derivative is determined from the initial conditions. Then, with a small time increment [Δt], the displacement function and its first-order derivative at the new position can be obtained using (A.2). This process continues to the end of time.