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Buckling and stability of elastic-plastic sandwich conical shells

Jerzy Zielnica

Poznan University of Technology, Institute of Applied Mechanics, Piotrowo 3, 60-965 Poznan, Poland

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Abstract. Shell structures are very interesting from the design point of view and these are well recognized in the scientific literature. In this paper the analysis of the buckling loads and stability paths of a sandwich conical shell with unsymmetrical faces under combined load based on the assumptions of moderately large deflections (geometrically nonlinear theory) is considered and elastic-plastic properties of the material of the faces are taken into considerations. External load is assumed to be two-parametrical one and it is assumed that the shell deforms into the plastic range before buckling. Constitutive relations in the analysis are those of the Nadai-Hencky deformation theory of plasticity and Prandtl-Reuss plastic flow theory with the H-M-H (Huber-Mises-Hencky) yield condition. The governing stability equations are obtained by strain energy approach and Ritz method is used to solve the equations with the help of analytical-numerical methods using computer.

Keywords: shells; elastic-plastic stability; deformation theory of plasticity; large displacements; strain energy.

1. Introduction

Sandwich shells have a high structural efficiency and they are often used in the design of airplanes. They are characterized by a comparatively high load to weight ratio and they have thermo and vibroisolation properties. From a practical point of view one can find a very interesting case of sandwich open conical shell under combined loads. Shell structures are very interesting from the design point of view and these are well recognized in the scientific literature (Zielnica 2002). A very significant problem in linear and nonlinear analysis of shell structures is stability and associated phenomena. One can find here multilayered thin-walled structures, and primarily shell structures. In order to obtain a proper design of new structures, or to check the carrying capacity of the existing structures a comprehensive understanding of the phenomena of post-critical equilibrium paths is needed because of restrictive requirements during their design and manufacture. In Refs (Vinson 2001), (Zielnica 2001) the linear and nonlinear buckling analyses are described and discussed.

Thin-walled multilayered shells are also used in the manufacturing of modern vehicles, planes, cisterns, tanks, and in civil engineering as well. These are subjected to widely varying combinations of hydrostatic pressure and axial load, thus, a stability problem for such structures is of great importance.

* Corresponding author, Ph.D., E-mail: jerzy.zielnica@put.poznan.pl

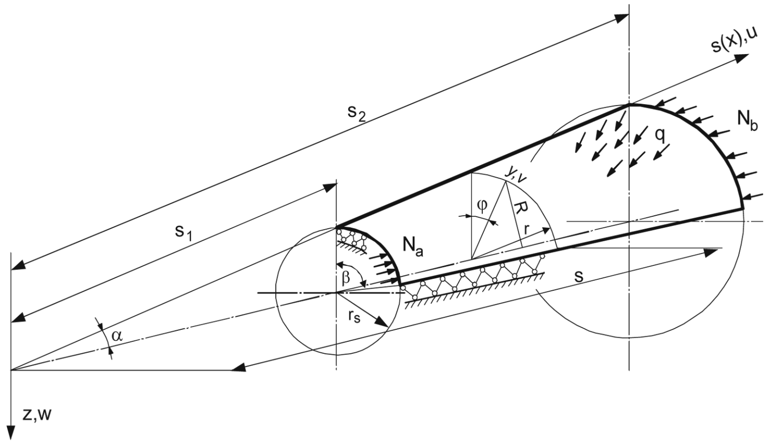

Fig. 1 Considered open conical shell

The purpose of this study is investigation of large displacement stability loss of a sandwich conical panel (see Fig. 1) loaded by longitudinal forces and uniformly distributed external pressure. It is assumed that the shell under consideration is made of a compressible material with linear and exponential strain hardening. Thus, it is also assumed that the effective stress in prebuckling state of stress in the shell can exceed the yield limit of the shell material.

The constitutive relations used in the elastic-plastic analysis follow the incremental J_2 Nadai - Hencky deformation theory of plasticity with the Huber-Mises yield condition. The K-L (Kirchhoff-Love) hypotheses are accepted and the active deformation processes according to Shanley concept are considered. Also, the Prandtl-Reuss plastic flow theory was used in the theoretical formulation of the problem. The system of stability equations expressed by the displacements does not have an exact solution. Any approximate solution, e.g., by Galerkin method is complicated because the appropriate calculations are time consuming. The necessity to satisfy the kinematic and static boundary conditions leads to the assumption of approximate functions in a very complicated form. Thus, the virtual work principle was used to derive the total strain energy in the shell, and the analysis is based on the strain energy minimization, where the total strain in the shell can be expressed in terms of displacement vector components. Ritz method is accepted to derive the stability equations for the considered shell. The final solution is a transcendental function of the deflection function parameter, which makes it possible to trace the equilibrium paths for the shell. An iterative computer algorithm was elaborated to facilitate the numerical analysis for the shells in elastic, elastic-plastic, and in totally plastic prebuckling state of stress. The algorithm reflects a specific feature of the elastic-plastic shell stability problem, where the stability equation is a transcendental function, where the stiffness coefficients of this equation depend on the load acting the shell. The elaboration is supplied with numerical examples which show the influence of geometrical parameters and material properties on the equilibrium paths of the shell. In the presented paper the stability paths of sandwich conical shells under combined load are analyzed. To find a solution of the problem, the assumptions of geometrically nonlinear theory and elastic-plastic properties of the faces are taken into account. In Refs Damatty A. A. *et al.* (1998), Zielnica J. (1981, 2001) both, linear and nonlinear buckling analyses of elastic-plastic conical and cylindrical shells are presented. In Ref. Vinson J. R. (2001), the up-to-date methods for sandwich structures analysis are described and discussed, and large reference list is given there. Kim S. E. and Kim, S. C. (2002), Pinna and Ronalds (2003), Shen (1998), and Siad (1999) discuss stability problems of cylindrical shells under various external loadings, also include imperfections.

The analyzed object is a sandwich conical shell presented in Fig. 1. The shell element is shown in Fig. 2. The shell consists of three layers: two thin face-layers, which are of different thickness h_1, h_2, and one core layer with thickness *2h*. The face layers can be made of different materials, which are compressible and isotropic.

The core layer is assumed to be elastic, incompressible in the normal *z* direction and it resists transverse shear only. The middle surface of the core layer is taken as the reference surface of the shell. The main assumptions for the accepted model include that the shell is thin-layered and shallow one, and the post-buckling stress state can be elastic or elastic-plastic.

2. Basic Relations

We consider a sandwich conical shell of circular cross section; the element of the shell is shown in Fig. 2.

The following basic assumptions hold for the accepted model: (*i*) the shell is thin-layered, the core is elastic, incompressible in the *z* direction, and it resists transverse shear only, the faces are of different thickness and they are made of different materials; (*ii*) the shell is shallow, the radii of curvatures of the layers are assumed to be equal; (*iii*) strains in the shell are described by nonlinear geometrical relations of the theory of moderately large deflections; (*iv*) the strains in postbuckling stress state in shell faces are elastic or elastic-plastic; (*v*) the displacements in normal direction do not depend on the *z* coordinate, and prebuckling stress state is the membrane one; (*vi*) constitutive relations in the analysis are those of the J_2 Nadai - Hencky deformation theory of plasticity with the H-M-H (Huber-Mises-Hencky) yield condition and bilinear stress-strain relation holds. Also, the J_2 incremental plastic flow theory will be used in the analysis.

The J_2 deformation theory of plasticity postulates a strict correspondence between the stress and strain. Thus, the components of the total plastic strain are taken to be proportional to the deviatoric stresses.

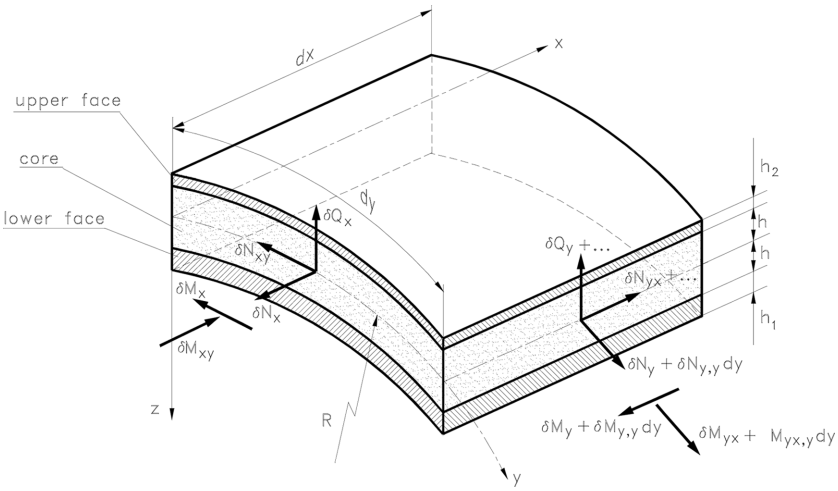

Fig. 2 The element of a sandwich conical shell

Assuming ε small strains, the plastic stress-strain relation proposed by Hencky may be written as

$$\varepsilon_{ij}^{p} = \lambda s_{ij} \tag{1}$$

where s_{ij} is deviatoric stress tensor, and parameter λ is positive during loading and zero during unloading. For a non-hardening material λ may be treated as an unspecified factor of proportionality. When the material work-hardens, λ depends on the equivalent stress $\bar{\sigma}$.

$$\bar{\sigma} = \sqrt{\frac{3}{2}}\sqrt{s_{ij}s_{ij}} \tag{2}$$

which may be regarded as a function of an equivalent total plastic strain $\bar{\varepsilon}^{p}$, defined as

$$\bar{\varepsilon}^{p} = \sqrt{\frac{2}{3}\varepsilon_{ij}^{p}\varepsilon_{ij}^{p}} \tag{3}$$

The relationship between $\bar{\sigma}$ and $\bar{\varepsilon}^{p}$ is given by the uniaxial stress - plastic strain curve. Substituting from (1) and using (2) for $\bar{\sigma}$, which implies the Huber-Mises-Hencky yield criterion, we have

$$\sqrt{\frac{3}{2}}\bar{\varepsilon}^{p} = \lambda\sqrt{s_{ij}s_{ij}} = \lambda\sqrt{\frac{2}{3}}\bar{\sigma} \tag{4}$$

giving $\lambda = 3\bar{\varepsilon}^{p}/2\bar{\sigma}$. The stress-strain relation (1) may therefore be expressed in the form

$$\varepsilon_{ij}^{p} = \frac{3\bar{\varepsilon}^{p}}{2\bar{\sigma}}s_{ij}\frac{3}{2}\left(\frac{1}{E_s} - \frac{1}{E}\right)s_{ij} \tag{5}$$

Where E_s is the secant modulus of the uniaxial stress-strain curve at $\sigma = \bar{\sigma}$. For an incompressible material, $\varepsilon_{ij}^{e} = 3s_{ij}/2E$ by Hooke's law, and (5) then furnishes $\varepsilon_{ij} = 3s_{ij}/2E_s$. The incremental form of (5) is as follows

$$d\varepsilon_{ij}^{p} = \frac{3}{2\bar{\sigma}}\left[\left(d\bar{\varepsilon}^{p} - \frac{\bar{\varepsilon}^{p}d\bar{\sigma}}{\bar{\sigma}}\right)s_{ij} + \bar{\varepsilon}^{p}ds_{ij}\right] \tag{6}$$

The Nadai-Hencky deformation theory of plasticity can be extended to large strains by using a suitable definition of the strain tensor ε_{ij}.

When the J_2 incremental plastic flow theory is used then the stresses and stress increments are related with strain increments or strain rates by a plasticity law which is flow rule and H-M-H yield condition generalized on the case of strain hardening. The Prandtl-Reuss equations can be expressed in the following incremental form

$$d\varepsilon_{ij} = \frac{1}{2G}\left(d\sigma_{ij} - \delta_{ij}\frac{3\nu}{1+\nu}d\sigma_m\right) + 3d\lambda(\sigma_{ij} - \delta_{ij}\sigma_{ij}),\ \sigma_m = \frac{1}{3}\sigma_{kk},\ d\lambda = \frac{1}{2}\frac{d\bar{\varepsilon}_i^{p}}{\sigma_i} \tag{7}$$

Parameter λ can be determined basing on the increment of plastic strain work

$$\delta W^{p} = \sigma_s\delta\varepsilon_s^{p} = \sigma_s\left(\frac{1}{E_t} - \frac{1}{E}\right)\delta\sigma_s \tag{8}$$

Thus, we have

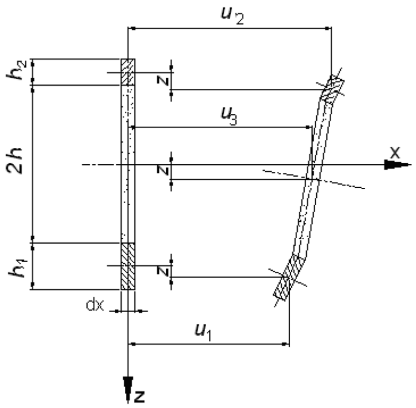

Fig. 3. Displacements of the sandwich shell element.

$$\lambda = \frac{3}{2}\left(\frac{1}{E_t} - \frac{1}{E}\right)\frac{\delta\sigma_i}{\sigma_i} \tag{9}$$

Here E_t is tangent modulus of stress hardening and σ_i is effective stress. The assumptions presented above allow us to apply the so called 'broken line' hypothesis, presented graphically in Fig. 3, see Zielnica (2001), to describe the geometrical relations between the displacement vector components u, v, and w of arbitrary point of the shell and the displacements of the points on the middle surfaces of the specific faces: u_a, v_a, w ($\alpha = 1, 2$).

outer face [$-c - b_1 \le z \le -c$]:

$$u = u_\alpha + u_\beta - \left(z + c + \frac{b_1}{2}\right)\frac{\partial w}{\partial x}, \qquad v = v_\alpha + v_\beta - \left(z + c + \frac{b_1}{2}\right)\frac{1}{r}\frac{\partial w}{\partial \phi}$$

inner face [$c \le z \le c + b_2$]:

$$u = u_\alpha - u_\beta - \left(z - c - \frac{b_2}{2}\right)\frac{\partial w}{\partial x}, \qquad v = v_\alpha - v_\beta - \left(z - c - \frac{b_2}{2}\right)\frac{1}{r}\frac{\partial w}{\partial \phi} \tag{10}$$

core [$-c \le z \le c$]:

$$u = u_\alpha + \frac{1}{4}(b_2 - b_1)\frac{\partial w}{\partial x} - \frac{z}{c}\left[u_\beta - \frac{1}{4}(b_1 + b_2)\frac{\partial w}{\partial x}\right],$$

$$v = v_\alpha + \frac{1}{4r}(b_2 - b_1)\frac{\partial w}{\partial \phi} - \frac{z}{c}\left[v_\beta - \frac{1}{4r}(b_1 + b_2)\frac{\partial w}{\partial \phi}\right]$$

Here: u_α, u_β, v_α, v_β are relations between displacements of the points on the middle surfaces of the specific faces

$$u_\alpha = \frac{u_1 + u_2}{2}, \quad u_\beta = \frac{u_1 - u_2}{2}, \quad v_\alpha = \frac{v_1 + v_2}{2}, \quad v_\beta = \frac{v_1 - v_2}{2} \tag{11}$$

Strains and changes in curvature are expressed in terms of displacements by non-linear geometrical relations for a conical shell

$$\varepsilon_{xj} = \frac{\partial u_j}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \qquad \varepsilon_{\phi j} = \frac{1}{r}\frac{\partial v_j}{\partial \phi} - \frac{w}{R} + \frac{u_j}{r}sin(\alpha) + \frac{1}{2r^2}\left(\frac{\partial w_j}{\partial \phi}\right)^2 \tag{12}$$

$$\gamma_{x\phi j} = \frac{1}{r}\frac{\partial u_j}{\partial \phi} + \frac{\partial v_j}{\partial x} - \frac{v_j}{r}sin(\alpha) + \frac{1}{r}\left(\frac{\partial^2 w}{\partial x \partial \phi}\right), \quad \gamma_{xzj} = \frac{\partial u_j}{\partial z} + \frac{\partial w}{\partial x}, \quad \gamma_{\phi zj} = \frac{\partial v_j}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \phi}$$

$$\kappa_{xj} = \frac{\partial^2 w}{\partial x^2}, \quad \kappa_{\phi j} = \frac{sin(\alpha)}{r}\frac{\partial w}{\partial x} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \phi^2} + \frac{cos(\alpha)}{r^2}\frac{\partial v_j}{\partial \phi}$$

$$\kappa_{x\phi j} = \frac{1}{r}\frac{\partial^2 w}{\partial x \partial \phi} - \frac{sin(\alpha)}{r^2}\frac{\partial w}{\partial \phi} + \frac{cos(\alpha)}{4r}\frac{\partial v_j}{\partial x} - \frac{5\,sin(\alpha)cos(\alpha)}{4r^2}v_j + \frac{cos(\alpha)}{4r^2}\frac{\partial u_j}{\partial \phi}$$

Here: $j = 1$ for upper face, $j = 2$ for lower face.

3. Loading and stresses

The considered shell is loaded by surface pressure q directed perpendicular to the shell main surface, and longitudinal forces N_a, N_b applied at the edges of the shell. In agreement witch main assumptions, the effective stress in the shell faces can exceed the yield stresses for the shell face material. In this case the stress-strain relations of appropriate plasticity theory have to be used. In the case of the Nadai-Hencky deformation theory the basic equations of this plasticity theory, developed by expanding relation (5), and separating the elastic and plastic terms in the constitutive relations, we get

$$\varepsilon_x = \varepsilon_x^{el} + \varepsilon_x^{pl} = \frac{1}{E}[\sigma_x - \nu(\sigma_\phi + \sigma_z)] + \frac{\phi(\sigma_i)}{3G}[\sigma_x - \nu(\sigma_\phi + \sigma_z)] \tag{13}$$

$$\varepsilon_\phi = \varepsilon_\phi^{el} + \varepsilon_\phi^{pl} = \frac{1}{E}[\sigma_\phi - \nu(\sigma_x + \sigma_z)] + \frac{\phi(\sigma_i)}{3G}[\sigma_\phi - \nu(\sigma_x + \sigma_z)]$$

$$\varepsilon_z = \varepsilon_z^{el} + \varepsilon_z^{pl} = \frac{1}{E}[\sigma_z - \nu(\sigma_\phi + \sigma_z)] + \frac{\phi(\sigma_i)}{3G}[\sigma_z - \nu(\sigma_\phi + \sigma_x)]$$

$$\gamma_{x\phi} = \frac{1}{G}\tau_{x\phi} + \frac{\phi(\sigma_i)}{G}\tau_{x\phi}, \quad \gamma_{z\phi} = \frac{1}{G}\tau_{z\phi} + \frac{\phi(\sigma_i)}{G}\tau_{z\phi}, \quad \gamma_{zs} = \frac{1}{G}\tau_{zx} + \frac{\phi(\sigma_i)}{G}\tau_{zx}, \quad \phi(\sigma_i) = \frac{3G}{E_s} - 1$$

The resultant middle surface forces and moments, developed by stability loss, in the shell faces are defined as follows

$$\delta N_{\alpha\beta} = \delta N_{\alpha\beta}^{+} + \delta N_{\alpha\beta}^{-} = \int_{-(h+h_2)}^{-h} \delta\sigma_{\alpha\beta} dz + \int_{h}^{h+h_1} \delta\sigma_{\alpha\beta} z dz \tag{14}$$

$$\delta M_{\alpha\beta} = \delta M_{\alpha\beta}^{+} + \delta M_{\alpha\beta}^{-} = \int_{-(h+h_2)}^{-h} \delta\sigma_{\alpha\beta} dz + \int_{h}^{h+h_1} \delta\sigma_{\alpha\beta} z dz$$

The equations describing internal forces and moments developed by buckling are derived by integration of Eq. (13) in incremental form. The simplified form of internal forces and moments after integration was made is as follows

$$\delta N_x = (b_{11}^{+} + b_{11}^{-})\delta\varepsilon_{11} + (b_{12}^{+} + b_{12}^{-})\delta\varepsilon_{22} - (b_{13}^{+} + b_{13}^{-})\delta\gamma_{12}\,, \tag{15}$$

$$\delta N_y = (b_{21}^{+} + b_{21}^{-})\delta\varepsilon_{11} + (b_{22}^{+} + b_{22}^{-})\delta\varepsilon_{22} - (b_{23}^{+} + b_{23}^{-})\delta\gamma_{12}\,,$$

$$\delta N_{xy} = -(b_{31}^{+} + b_{31}^{-})\delta\varepsilon_{11} - (b_{32}^{+} + b_{32}^{-})\delta\varepsilon_{22} + (b_{33}^{+} + b_{33}^{-})\delta\gamma_{12}\,,$$

$$\delta M_x = -(d_{11}^{+} + d_{11}^{-})\delta\kappa_1 - (d_{12}^{+} + d_{12}^{-})\delta\kappa_2 + (d_{13}^{+} + d_{13}^{-})\delta\kappa_{12}\,,$$

$$\delta M_y = -(d_{21}^{+} + d_{21}^{-})\delta\kappa_1 - (d_{22}^{+} + d_{22}^{-})\delta\kappa_2 + (d_{23}^{+} + d_{23}^{-})\delta\kappa_{12}\,,$$

$$\delta M_{xy} = (d_{31}^{+} + d_{31}^{-})\delta\kappa_1 + (d_{32}^{+} + d_{32}^{-})\delta\kappa_2 - (d_{33}^{+} + d_{33}^{-})\delta\kappa_{12}$$

Here: "+" is for upper face, "–" is for lower face.

The coefficients in Eq. (15) are the local stiffnesses that follow the Nadai-Hencky deformation theory of plasticity. They have the form as follows

$$b_{11}^{\pm} = \frac{12}{h_{1(2)}^2}d_{11}^{\pm} = B_s\beta_{11} + \bar{K}\bar{B}_{11}\,,\quad b_{12}^{\pm} = b_{21}^{\pm} = \frac{12}{h_{1(2)}^2}d_{12}^{\pm} = \frac{12}{h_{1(2)}^2}d_{21}^{\pm} = B_s\beta_{12} + \bar{K}\bar{\beta}_{12} \tag{16}$$

$$b_{13}^{\pm} = b_{31}^{\pm} = \frac{6}{h_{1(2)}^2}d_{13}^{\pm} = \frac{6}{h_{1(2)}^2}d_{31}^{\pm} = \bar{K}\bar{\beta}_{13}\,,\quad b_{23}^{\pm} = b_{32}^{\pm} = \frac{6}{h_{1(2)}^2}d_{23}^{\pm} = \frac{6}{h_{1(2)}^2}d_{32}^{\pm} = \bar{K}\bar{\beta}_{23}$$

$$b_{22}^{\pm} = \frac{12}{h_{1(2)}^2}d_{11}^{\pm} = B_s\beta_{22} + \bar{K}\bar{\beta}_{22}\,,\quad b_{33}^{\pm} = \frac{12}{h_{1(2)}^2}d_{33}^{\pm} = B_s\beta_{33} + \bar{K}\bar{\beta}_{33}$$

where

$$B_s = \frac{E_s}{a^o},\quad \bar{K} = \frac{E_s - E_t}{a^o},\quad \alpha_o = \frac{E_s}{E}(1-2v) \tag{17}$$

$$a^o = \left(1 - \frac{\alpha_o}{3}\right)\left\{1 + \frac{\alpha_o}{3}\left[2 - \alpha_o\frac{E_t}{E_s} - 3\left(1 - \frac{E_t}{E_s}\right)(\sigma_x^o\sigma_y^o - \tau_{xy}^{o2})\right]\right\}$$

$$\beta_{11} = \frac{4}{3}\left(1 - \frac{\alpha_o}{3}\right),\quad \bar{\beta}_{11} = \frac{4}{9}\alpha_o - \sigma_x^{o2} - \frac{4}{9}\alpha_o\sigma_y^{o2},\quad \beta_{12} = \beta_{21} = \frac{2}{3} - \frac{5}{9}\alpha_o + \frac{\alpha_o^2}{9} \tag{18}$$

$$\bar{\beta}_{12} = \bar{\beta}_{21} = -\left[\sigma_x^o\sigma_y^o - \frac{4}{9}\alpha_o\left(\bar{\sigma}_x\bar{\sigma}_y - \frac{9}{2}\tau_{xy}^{o2} + \frac{5}{4} - \frac{\alpha_o}{4}\right)\right]$$

$$\bar{\beta}_{13} = \bar{\beta}_{31} = \tau_{xy}^o\left(\sigma_x^o - \frac{2}{3}\alpha_o\bar{\sigma}_y\right),\quad \bar{\beta}_{23} = \bar{\beta}_{32} = \tau_{xy}^o\left(\sigma_y^o - \frac{2}{3}\alpha_o\bar{\sigma}_x\right)$$

$$\beta_{33} = \frac{1}{3} + \frac{2}{9}\alpha_o - \frac{\alpha_o^9}{9}, \quad \bar{\beta}_{33} = -\left(\tau_{xy}^{o\,2} + \frac{2}{9}\alpha_o + \frac{4}{9}\alpha_o \bar{\sigma}_x \bar{\sigma}_y\right)$$

Barred symbols in Eq. (18) are the relative prebuckling stresses related with the effective stress $\bar{\sigma}$

$$\sigma_x^o = \frac{\sigma_x}{\bar{\sigma}}, \quad \sigma_y^o = \frac{\sigma_y}{\bar{\sigma}}, \quad \tau_{xy}^o = \bar{\tau}_{xy} = \frac{\tau_{xy}}{\bar{\sigma}}, \quad \bar{\sigma}_x = \sigma_x^o - \frac{1}{2}\sigma_y^o, \quad \bar{\sigma}_y = \sigma_y^o - \frac{1}{2}\sigma_x^o \tag{19}$$

The factors of the local stiffness matrices are the functions of tangent modulus E_t, and secant modulus E_s. When plastic flow theory is used, the local stiffness matrix coefficients in Eq. (15) are as follows

$$b_{11}^{(i)} = \frac{12}{h_i^2} d_{11}^{(i)} = \psi_o^{(i)}\left\{2(1+\nu) + \psi_t^{(i)}\left[\frac{1+\nu}{2}(2\bar{\sigma}_\varphi - \bar{\sigma}_x)^2 + 9\bar{\tau}_{x\varphi}^2\right]\right\} \tag{20}$$

$$b_{12}^{(i)} = b_{21}^{(i)} = \frac{12}{h_i^2} d_{12}^{(i)} = \frac{12}{h_i^2} d_{21}^{(i)} = \psi_o^{(i)}\left\{2\nu(1+\nu) - \psi_t^{(i)}\left[\frac{1+\nu}{2}(2\bar{\sigma}_x - \bar{\sigma}_\varphi)(2\bar{\sigma}_\varphi - \bar{\sigma}_x)^2 + 9\nu\bar{\tau}_{x\varphi}^2\right]\right\}$$

$$b_{13}^{(i)} = b_{31}^{(i)} = \frac{6}{h_i^2} d_{13}^{(i)} = \frac{12}{h_i^2} d_{31}^{(i)} = 3\,\psi_o^{(i)}\psi_t^{(i)}\bar{\tau}_{x\varphi}[(2-\nu)\bar{\sigma}_x - (1-2\nu)\bar{\sigma}_\varphi]$$

$$b_{22}^{(i)} = \frac{12}{h_i^2} d_{22}^{(i)} = \psi_o^{(i)}\left\{2(1+\nu) + \psi_t^{(i)}\left[\frac{1+\nu}{2}(2\bar{\sigma}_x - \bar{\sigma}_\varphi)^2 + 9\bar{\tau}_{x\varphi}^2\right]\right\}$$

$$b_{23}^{(i)} = b_{32}^{(i)} = \frac{6}{h_i^2} d_{23}^{(i)} = \frac{12}{h_i^2} d_{32}^{(i)} = 3\,\psi_x \psi_t^{(i)}\bar{\tau}_{x\varphi}[(2-\nu)\bar{\sigma}_\varphi - (1-2\nu)\bar{\sigma}_x]$$

$$b_{33}^{(i)} = \frac{6}{h_i^2} d_{33}^{(i)} = \psi_o^{(i)}\left\{(1-\nu^2) + \frac{1}{4}\psi_t^i[(5-4\nu)(\bar{\sigma}_x^2 + \bar{\sigma}_\varphi^2) - 2(4-5\nu)\bar{\sigma}_x^s\bar{\sigma}_\varphi]\right\}$$

where

$$\psi_o^{(i)} = \frac{E_{(i)}h_{(i)}}{1+\nu}\{(5-4\nu)(\bar{\sigma}_x^2 + \bar{\sigma}_\varphi^2) - 2(4-5\nu)\bar{\sigma}_x\bar{\sigma}_\varphi + 18(1-\nu)\bar{\tau}_{x\varphi}^2\}^{-1}$$

$$\psi_t^{(i)} = \frac{E_{(i)}}{E_{t(i)}} - 1$$

$$\bar{\sigma}_x = \frac{\sigma_x}{\sigma_i} \quad \bar{\sigma}_\varphi = \frac{\sigma_\varphi}{\sigma_i} \quad \bar{\tau}_{x\varphi} = \frac{\tau_{x\varphi}}{\sigma_i} \quad \sigma_i = \sigma_{ef} = \sqrt{\sigma_x^2 - \sigma_x\sigma_\varphi + \sigma_\varphi^2 + 3\tau_{x\varphi}^2}$$

$i = 1$ is for upper face layer, and $i = 2$ is for lower face layer.

Internal forces of the membrane prebuckling stress state in the shell, resulting from the accepted external loadings (see Fig. 1), are as follows

$$N_s = H\sigma_s = \frac{1}{2}qs\,\mathrm{tg}\,\alpha\left[\left(\frac{s_1}{s}\right)^2 - 1\right] - N_a\frac{s_1}{s}, \quad \text{where } H = h_1 + h_2 + 2h \tag{21}$$

$$N_\varphi = h\sigma_\varphi = -qs\,\mathrm{tg}\,\alpha, \quad T_s = T_\varphi = 0$$

We introduce parameter k that represents the ration between lateral and longitudinal load

$$\kappa = \frac{N_a}{qs_1} \tag{22}$$

The stresses in prebuckling state of stress can be expressed by the external forces

$$\sigma_s = \frac{N_s}{H} = \frac{qs}{2H}tg\alpha\left[\left(\frac{s_1}{s}\right)^2 - 1\right] - N_a\left(\frac{s_1}{s}\right)\frac{1}{H} \tag{23}$$

$$\sigma_\varphi = \frac{N_\varphi}{H} = -\frac{qs}{H}tg\alpha$$

When additional quantity k_s is introduced and appropriate transformations are made, the following relations for the stresses are obtained

$$\sigma_s = \sigma_x = \frac{qs}{2H}k_s tg\alpha\,, \quad \sigma_\varphi = -\frac{qs}{H}tg\alpha \tag{24}$$

where

$$k_s = 1 - \left(\frac{s_1}{s}\right)^2\left(1 - \frac{1}{tg\alpha}\kappa\right)$$

4. Shell stability equations

The virtual work principle is used to derive the equilibrium equations for the considered shell.

$$\delta\Pi = \delta(U - A) = 0 \tag{25}$$

We conclude from Eq. (25) that if the shell is given the small virtual displacement, the equilibrium still persists if an increment of the total potential energy of the system $\delta\Pi$ is equal to zero. Relation Eq. (25) is the basis to derive the variational equation of equilibrium of a shell. For conical sandwich shell the total potential energy of internal forces is equal to the energy of the specified layers.

$$U = U^+ + U^- + U_c \tag{26}$$

The terms in Eq. (26) and the work of external forces A are as follows

$$U^\pm = \frac{1}{2}\int_0^l\int_0^{\beta R}(\delta N_x^\pm\delta\varepsilon_x^\pm + \delta N_y^\pm\delta\varepsilon_y^\pm + \delta N_{xy}^\pm\delta\varepsilon_{xy}^\pm + \delta M_x^\pm\delta\kappa_x + \delta M_y^\pm\delta\kappa_y + \delta M_{xy}^\pm\delta\kappa_{xy})dxdy\,, \tag{27}$$

$$U_c = \frac{1}{2}\int_V\left[\frac{E_c}{1-\nu_c^2}\left(\delta\varepsilon_{xc}^2 + \delta\varepsilon_{yc}^2 + 2\nu_c\delta\varepsilon_{xc}\delta\varepsilon_{yc} + \frac{1-\nu_c}{2}\gamma_{xyc}^2\right) + G_c(\delta\gamma_{yzc}^2 + \delta\gamma_{xzc}^2)\right]dv$$

$$A = \int_0^l\int_0^{\beta R}qwdxdy + \left\{\int_0^{\beta R}\left[N_x^o\left(u_\alpha + \frac{h_1 - h_2}{H}u_\beta + \frac{h(h_1 - h_2)}{2H}w_{,x}\right) + \right.\right. \tag{28}$$

$$+N_{xy}^{o}\left(v_{\alpha}+\frac{h_1-h_2}{H}v_{\beta}+\frac{h(h_1-h_2)}{2H}w_{,y}\right)\Big]dy\Bigg\}_0^l$$

Using the above relations we can present Eq. (25) in the form of

$$\delta\Pi = \int_0^l\int_0^{\beta R}\frac{\partial L}{\partial u_i}\delta u_i dxdy = 0 \tag{29}$$

where L is a function of the displacement vector components ui. Eq. (29) has to be satisfied for arbitrary values of δu_i; thus, we get a set of five equilibrium equations for the considered cylindrical shell with respect to deflection functions. Also, using Eq. (29), we may obtain a set of boundary conditions for the considered problem.

Using the above relations Eqs. (27-29) in Eq. (25), we obtain a full description of the shell potential energy Π. Next, we apply Ritz method to Eq. (29)

$$\delta\Pi = \sum_{i=1}^{5}\left(\frac{\partial\Pi}{\partial A_i}\right)\delta A_i = 0 \quad i=1, 2, ..., 5. \quad \text{Hence:} \quad \frac{\partial\Pi}{\partial A_i} = 0 \quad \text{for arbitrary values of } \delta A_i \tag{30}$$

We accept the following displacement functions which satisfy the boundary conditions for simply supported conical shell

$$u_{\alpha}(x,\varphi) = A_2 r^2 cos(k\psi)sin(p\varphi), \quad u_{\beta}(x,\phi) = A_3 r^2 cos(k\psi)sin(p\varphi) \tag{31}$$

$$v_{\alpha}(x,\varphi) = A_4 r^2 sin(k\psi)cos(p\varphi), \quad v_{\beta}(x,\phi) = A_5 r^2 sin(k\psi)cos(p\varphi)$$

$$w(x,\varphi) = A_1 r^2 sin(k\psi)sin(p\varphi)$$

Here: A_i are free parameters of the displacement functions to be determined, k and p are constants, $k=m\pi/l$, $p=n\pi/R\beta$. Integers m, and n are parameters corresponding to half-waves along shell generatrix and in circumferential direction, respectively, developed during stability loss.

The functions Eq. (31) satisfy kinematic boundary conditions for the displacements. Once we perform the prescribed operations in Eq. (30) the set of five algebraic equations is obtained for the considered shell. The general form of the equations is as follows

$$(a_{11}+a_{11p})A_1+a_{12}A_2+a_{13}A_3+a_{14}A_4+a_{15}A_5 \tag{32}$$

$$=b_{11}A_1^2+b_{12}A_1^3+b_{13}A_1A_2+b_{14}A_1A_3+b_{15}A_1A_4+b_{16}A_1A_5+b_{17}$$

$$a_{21}A_1+a_{22}A_2+a_{23}A_3+a_{24}A_4+a_{25}A_5 = b_{21}A_1^2$$

$$a_{31}A_1+a_{32}A_2+a_{33}A_3+a_{34}A_4+a_{35}A_5 = b_{31}A_1^2$$

$$a_{41}A_1+a_{42}A_2+a_{43}A_3+a_{44}A_4+a_{45}A_5 = b_{41}A_1^2$$

$$a_{51}A_1+a_{52}A_2+a_{53}A_3+a_{54}A_4+a_{55}A_5 = b_{51}A_1^2$$

Here: a_{ij}, b_{ij} are coefficients of the set of equations which depend on geometrical shell parameters, physical properties, buckling form, and on external loading. We eliminate from set of Eq. (32) parameters A_2-A_5, and after appropriate transformations and simplifications are made we obtain the final solution in the form of the following nonlinear algebraic equation

$$q = \frac{\tilde{e}_1 A_1 + \tilde{e}_2 A_1^2 + \tilde{e}_3 A_1^3}{A_1 \tilde{e}_4 \kappa + \tilde{e}_5}, \qquad \kappa = \frac{N_x^0}{q s_1} \tag{33}$$

Here: $\tilde{e}_i$ are coefficients of the stability equation which have a very complicated form and they depend on geometrical parameters, physical properties, buckling form, and external loading acting the shell. The above equation allows us to determine the nonlinear equilibrium paths for the considered shell.

The coefficients $\tilde{e}_i$ ($i = 1, ..., 5$) in the above equation depend also on elastic and plastic material constants of the material, and on parameters m, n which are the number of buckling mode half-waves. Subsequent step of the solution process is elaboration of special numerical iterative algorithm where the basis is stability Eq. (33). Also, the objective of the work is the analysis of the influence of geometrical and material parameters on critical loads.

In order to determine numerical results of the solution the elaborated algorithm was implemented in FORTRAN source code. The program of numerical calculations makes it possible to determine critical loads and the equilibrium paths for the considered conical sandwich shell being in elastic, plastic, or elastic-plastic state of stress. Thus, lateral pressure q, and longitudinal force N_s can be determined to be the functions of deflection w of the shell.

Solution algorithm and program of numerical calculation take into consideration a specific feature of

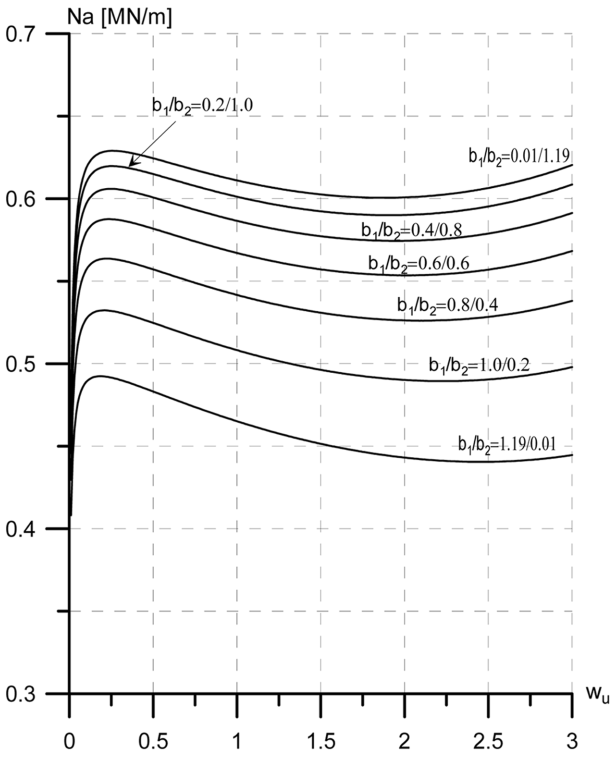

Fig. 5 Equilibrium paths for different thickness of the core

elastic-plastic stability of shells. Stability Eq. (33) is a transcendental function, where the coefficients of the local stiffness matrix depend on parameters of external loadings. So, we have to use some iterative techniques to build up equilibrium paths, and to determine upper and lower critical loads of the considered sandwich conical shell. Numerical calculations were performed by an algorithm where the branching is introduced for elastic or plastic paths which depend on the relation of equivalent stress to the yield limit at any point of the shell.

5. The results and conclusions

As it was pointed out, it is not possible to find a direct solution for Eq. (33) because this equation is transcendental one (the local stiffness matrix coefficients depend on the load). In order to get a numerical solution, we apply iterative methods, and a special computer algorithm was elaborated for the elastic-plastic analysis.

The obtained results present the relation between critical load (q, N_a), and the influence of some physical and/or geometrical parameters on the equilibrium paths. Fig. 5 shows an example of the results of a numerical solution for a sandwich conical shell with the faces made of structural steel, loaded by uniform external pressure and axial force. The following geometrical parameters and material constants were accepted for the considered shell in the numerical calculations: shell main radius $R = 990$ mm, shell length $l = 860$ mm, layer thickness: $h_1 = 1$ mm, $h_2 = 1, 1$ mm, $2h = 20$ mm, shear modulus of the core $G_3 = 25$ MPa, Young modulus for the faces material $E = 210000$ MPa, tangent modulus for bilinear stress-strain curve $E_t = 30000$ MPa (secant modulus E_s is variable and it depends on E_t), Poisson coefficient $\nu = 0.3$, yield limit $\sigma_Y = 240$ MPa. Fig. 5 shows a plot of curves which are the equilibrium paths for longitudinal force N_a versus maximum deflection of the shell. The curves represent how the shell face thickness ratio influences the equilibrium paths and the value of the upper and lower critical loads. One may observe here that the greater the face thickness ratio h_1/h_2 (here $h_1 = b_1$, $h_2 = b_2$), the lower the critical loads. Fig. 6 shows a diagram of the influence of load ratio parameter k on critical load and equilibrium paths. Once parameter κ increases the critical loads are greater and greater is the difference between upper N_a^+ and lower N_a^- critical loads.

The method presented in the paper is enclosed within a theoretical solution for elastic-plastic stability

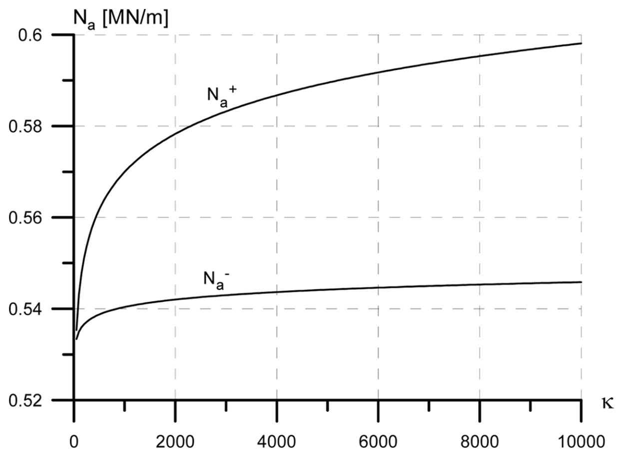

Fig. 6 The influence of load ratio parameter k on critical load and equilibrium paths

problem of a sandwich conical shell. The buckling of the analyzed shells occurs in the elastic-plastic state of stress. Appropriate selection of geometrical parameters of the shell and physical properties of shell material allows us to increase the critical load, thus, the buckling resistance of the shell becomes higher.

The presented method is a general one and it can be used in engineering analyses of sandwich shell structures.

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