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Mathematical solution for free vibration of sigmoid functionally graded beams with varying cross-section

Hassen Ait Atmane[1,2], Abdelouahed Tounsi[1]*, Noureddine Ziane[1], and Ismail Mechab[1]

[1]*Laboratoire des Matériaux et Hydrologie, Université de Sidi Bel Abbes, BP 89 Cité Ben M'hidi 22000 Sidi Bel Abbes, Algérie.*
[2]*Département de génie civil, Faculté des Sciences de l'Ingénieur, Univesité Hassiba Benbouali de Chlef, Algérie*

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Abstract. This paper presents a theoretical investigation in free vibration of sigmoid functionally graded beams with variable cross-section by using Bernoulli-Euler beam theory. The mechanical properties are assumed to vary continuously through the thickness of the beam, and obey a two power law of the volume fraction of the constituents. Governing equation is reduced to an ordinary differential equation in spatial coordinate for a family of cross-section geometries with exponentially varying width. Analytical solutions of the vibration of the S-FGM beam are obtained for three different types of boundary conditions associated with simply supported, clamped and free ends. Results show that, all other parameters remaining the same, the natural frequencies of S-FGM beams are always proportional to those of homogeneous isotropic beams. Therefore, one can predict the behaviour of S-FGM beams knowing that of similar homogeneous beams.

Keywords: functionally graded materials; beams; variable cross-section; free vibration

1. Introduction

The concept of functionally graded materials (FGMs) was first introduced in 1984 as ultrahigh temperature-resistant materials for aircrafts, space vehicles, nuclear and other engineering applications. Since then, FGMs have attracted much interest as heat-resistant materials. Functionally graded materials are heterogeneous composite materials, in which the material properties vary continuously from one interface to the other. This is achieved by gradually varying the volume fraction of the constituent materials. The continuity of the material properties reduces the influence of the presence of interfaces and avoids high interfacial stresses. The outcome of this is that this class of materials can survive environments with high-temperature gradients, while maintaining the desired structural integrity. However, in the case of adding an FGM of a single power-law function to the multi-layered composite, stress concentrations appear on one of the interfaces where the material is continuous but changes rapidly (Lee and Erdogan 1995, Bao and Wang 1995). Therefore, Chung and Chi (2001) defined the volume fraction using two power-law functions to ensure smooth distribution of stresses among all the interfaces and this functionally graded material is thus called sigmoid functionally graded material (S-FGM). Most researchers use the power-law function or exponential function to describe the volume fractions. However, only a few studies used sigmoid function to describe the volume fractions. Therefore, FGM

* Corresponding author, Professor, E-mail: tou_abdel@yahoo.com

beams with sigmoid function will be considered in this paper in detail. Many studies have been conducted on the static behaviour of FGM structures (Zhong and Yu 2007, Benatta *et al*. 2008, Yang *et al*. 2008, Jabbari *et al*. 2008, Sallai *et al*. 2009, Benatta *et al*. 2009). However, knowledge of free vibration characteristics of beams forms an important aspect in assessing the structural integrity. In addition, the research effort devoted to free vibration of FG beams has been very limited. Ying *et al*. (2008) obtained the exact solutions for bending and free vibration of FG beams resting on a Winkler–Pasternak elastic foundation based on the two-dimensional elasticity theory by assuming that the beam is orthotropic at any point and the material properties vary exponentially along the thickness direction. Li (2008) proposed a new unified approach to investigate the static and the free vibration behavior of Euler-Bernoulli and Timoshenko beams. Sina *et al*. (2009) used a new beam theory different from the traditional first-order shear deformation beam theory to analyze the free vibration of FG beams. Pradhan and Sarkar (2009) studied the bending, buckling and vibration of tapered FGM beams using Eringen non-local elasticity theory. Both Euler-Bernoulli and Timoshenko beam theories are considered in their study and the associated differential equations are solved employing Rayleigh-Ritz method. Pradhan and Phadikar (2009) used general differential quadrature (GDQ) and non-local elasticity theory to study bending, buckling and vibration behaviors of nonhomogeneous naotubes. Thermal post-buckling behaviour of uniform slender FGM beams is investigated by Sanjay Anandrao *et al*. (2010) using the classical Rayleigh-Ritz (RR) formulation and the versatile Finite Element Analysis (FEA) formulation. Şimşek and Kocaturk (2009) have investigated the free and forced vibration characteristics of an FG Euler-Bernoulli beam under a moving harmonic load. In a recent study, Şimşek (2010) has studied the dynamic deflections and the stresses of an FG simply-supported beam subjected to a moving mass by using Euler–Bernoulli, Timoshenko and the parabolic shear deformation beam theory. Şimşek (2010) studied the free vibration of FG beams having different boundary conditions using the classical, the first-order and different higher-order shear deformation beam theories. The non-linear dynamic analysis of a FG beam with pinned-pinned supports due to a moving harmonic load has been examined by Şimşek (2010) using Timoshenko beam theory. Yas *et al*. (2011) presented three dimensional solutions for free vibration analysis of functionally graded fiber reinforced cylindrical panel by using differential quadrature method (DQM).

In modern engineering design, there is increasing use of composite beams or of beams made of FGM. Hence, beams are used as structural component in many engineering applications and a large number of studies can be found in literature about transverse vibration of uniform isotropic beams (Gorman 1975).

Non-uniform beams may provide a better or more suitable distribution of mass and strength than uniform beams and therefore can meet special functional requirements in architecture, robotics, aeronautics and other innovative engineering applications and they have been the subject of numerous studies. However, it is more difficult to obtain general closed form solutions for the static and dynamic response of beams with arbitrary non-homogeneity and arbitrary varying cross-sections, since the governing equations of such beams possess variable coefficients. In the past, many methods have been proposed for investigating the dynamic response of non-uniform Euler-Bernoulli beams; for example, the transfer matrix method (Chu and Pilkey 1979) the finite element method (Bathe 1982), the boundary element method (Beskos 1987), the dynamic stiffness method (Just 1977), the dynamic method in conjunction with modal analysis (Ovunk 1974, Beskos 1979), the transformed dynamic stiffness method combined with the Laplace transform (Beskos and Narayanan 1983), the step-reduction method (Yeh 1979, Yeh *et al*. 1992), and the semi-analytic method (Lee 1990). Laura *et al*. (1996) used approximate numerical approaches to determine the natural frequencies of Bernoulli beams with constant width and bilinearly varying thickness. Datta and Sil (1996) numerically determined the natural frequencies of cantilever beams with constant width and linearly varying depth. Caruntu (2000) examined the nonlinear vibrations of beams with rectangular cross section and parabolic thickness variation. Recently, Elishako and Johnson (2005) investigated the vibration problem of a beam which has axially non-uniform material properties. Free vibration of stepped beams has also received a considerable attention and a comprehensive review is given by Jang and Bert (1989a, b). Some of these results can also be found in the monograph by Elishako (2005). Hence, the dynamic behaviour of these beams with varying cross-section has been a subject of active research. However, research work on FGM structures with varying cross-section is scarce. By using the Rayleigh-Ritz method, Guven *et al*. (2004) considered the transverse vibration of a polar orthotropic rotating solid disk whose thickness varies exponentially with any power of the radius. The disk was assumed to be under a constant radial stress. Toso and Baz (2004) presented numerical solutions for the wave propagation problem of a periodic shell with tapered wall thickness. Theoretical results predicted by the transfer matrix method and the wavelet transformation method were compared with and verified against the experimental data. Their study demonstrated that a combination of the use of FGMs and tapered geometry gives more flexibility in designing a structure with better performance. All of these studies, except (Guven *et al*. 2004), focused on the free vibration of tapered structures only. Previous studies clearly show that vibration characteristics of beams structures with continuously changing cross-section have significant features and are not yet fully addressed.

The objective of this paper is to study the free vibration of S-FGM beams with exponentially varying width. The classical Bernoulli-Euler beam theory is used in the present study. In this paper, we assume that the S-FGM beams are made from two constituent materials, whose material properties are graded in the thickness direction according to a power-law distribution of material composition. Comprehensive numerical results are obtained analytically for beams with clamped-free, hinged-hinged, and clamped-clamped boundary conditions. Results show that the natural frequencies of S-FGM beams can be obtained from the corresponding results for isotropic beams so that a direct analysis of S-FGM beams is not necessary. These results confirm those obtained by Abrate (2006) in the case of P-FGM structures with uniform section and in which material properties vary along the beam thickness only according to power law distributions.

2. Material properties of S-FGM beams

In the case of adding an FGM of a single power-law function to the multi-layered composite, stress concentrations appear on one of the interfaces where the material is continuous but changes rapidly (Lee and Erdogan 1995, Bao and Wang 1995). Therefore, Chung and Chi (2001) defined the volume fraction using two power-law functions to ensure smooth distribution of stresses among all the interfaces. The two power-law functions are defined by

$$\mathbf{g}_1(\mathbf{z}) = 1 - \frac{1}{2}\left(\frac{\mathbf{h}/2 - \mathbf{z}}{\mathbf{h}/2}\right)^{\mathbf{p}} \text{for } 0 \le \mathbf{z} \le \mathbf{h}/2 \tag{1a}$$

$$\mathbf{g}_2(\mathbf{z}) = \frac{1}{2}\left(\frac{\mathbf{h}/2 + \mathbf{z}}{\mathbf{h}/2}\right)^{\mathbf{p}} \text{for } -\mathbf{h}/2 \le \mathbf{z} \le 0 \tag{1b}$$

where g_i, $(i = 1,2)$ is the volume fraction and p is the power law index which takes values greater than or equal to zero.

By using the rule of mixture, the effective material properties p, such as Young's modulus E, the Poisson ratio ν, and mass density ρ can be expressed as

$$\mathbf{P}(\mathbf{z}) = \mathbf{g}_1(\mathbf{z})\mathbf{P}_2 + [1 - \mathbf{g}_1(\mathbf{z})]\mathbf{P}_1 \text{ for } 0 \le \mathbf{z} \le \mathbf{h}/2 \tag{2a}$$

$$\mathbf{P}(\mathbf{z}) = \mathbf{g}_2(\mathbf{z})\mathbf{P}_2 + [1 - \mathbf{g}_2(\mathbf{z})]\mathbf{P}_1 \text{ for } -h/2 \le \mathbf{z} \le 0 \tag{2b}$$

where $\mathbf{P}_1$ and $\mathbf{P}_2$ denote the materials properties of the bottom and top surfaces of the S-FGM beam, respectively ($z = \pm \boldsymbol{h}/2$).

Fig. 1 shows the volume fraction distribution of ceramic phase through the thickness for several values of the power law index. The value of p equal to zero represents a fully ceramic beam and infinite p, a fully metallic beam. The variation of the composition of ceramics and metal is linear for $p = 1$**.** The volume fraction rapidly changes near the top and bottom surfaces for $\mathbf{p} < 1$ but vary rapidly near the middle surface for $\mathbf{p} > 1$. Therefore, if the S-FGM plate is used as the undercoat in a laminated material, the material distribution with $\mathbf{p} > 1$ is the better choice.

3. Theoretical formulations

Consider an elastic S-FGM beam of length L and constant thickness h, with exponentially varying width. Based on the Euler-Bernoulli hypothesis, the displacements parallel to the x -and z -axes of an arbitrary point in the beam, denoted by $\bar{u}(x, z, t)$ and $\bar{w}(x, z, t)$, respectively, take the form of

$$\bar{\mathbf{u}}(\mathbf{x}, \mathbf{z}, \mathbf{t}) = \mathbf{u}(\mathbf{x}, \mathbf{t}) - z\frac{\partial \bar{\mathbf{w}}}{\partial \mathbf{x}} \tag{3a}$$

$$\bar{\mathbf{w}}(\mathbf{x}, \mathbf{z}, \mathbf{t}) = \mathbf{w}(\mathbf{x}, t) \tag{3b}$$

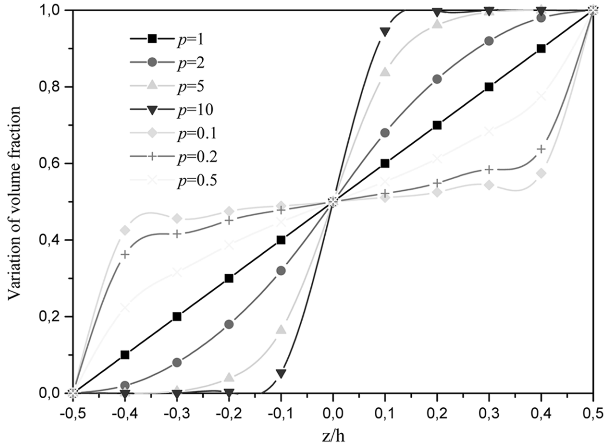

Fig. 1 Variation of the volume fraction through the thickness of a S-FGM beam with differing material parameters $\boldsymbol{p}$.

where $u(x, t)$ and $w(x, t)$ are the displacement components of a point in the mid-plane. The normal resultant force N, bending moment$M$, and transverse shear force Q are related to the normal strain $\boldsymbol{\varepsilon}_0 = \frac{\partial \mathbf{u}}{\partial \mathbf{x}}$ and flexural curvature $k_x = \frac{\partial^2 w}{\partial x^2}$ by

$$\begin{Bmatrix} \mathbf{N} \\ \mathbf{M} \end{Bmatrix} = \mathbf{b}(\mathbf{x})\begin{bmatrix} \mathbf{A}_{11} & \mathbf{B}_{11} \\ \mathbf{B}_{11} & \mathbf{D}_{11} \end{bmatrix}\begin{Bmatrix} \varepsilon_0 \\ -\mathbf{k}_x \end{Bmatrix}, \mathbf{Q} = \frac{\partial \mathbf{M}}{\partial \mathbf{x}} = \mathbf{B}_{11}\frac{\partial(\mathbf{b}\varepsilon_0)}{\partial \mathbf{x}} - \mathbf{D}_{11}\frac{\partial(\mathbf{b}\mathbf{k}_x)}{\partial \mathbf{x}} \tag{4}$$

and

$$(\mathbf{A}_{11}, \mathbf{B}_{11}, \mathbf{D}_{11}) = \int_{-\mathbf{h}/2}^{\mathbf{h}/2} \frac{\mathbf{E}(\mathbf{z})}{1 - \nu(\mathbf{z})^2}(1, z, z^2)\mathbf{d}z \tag{5}$$

whrer b is the width of the cross-section which is assumed to vary exponentially along the length of the beam.

The equations of motion for the beam, with the axial inertia term being neglected, can be derived as follows

$$\mathbf{A}_{11}\frac{\partial}{\partial \mathbf{x}}\left(\mathbf{b}\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right) - \mathbf{B}_{11}\frac{\partial}{\partial \mathbf{x}}\left(\mathbf{b}\frac{\partial^2 \mathbf{w}}{\partial \mathbf{x}^2}\right) = 0 \tag{6a}$$

$$\left(\mathbf{D}_{11} - \frac{\mathrm{B}_{11}^2}{\mathbf{A}_{11}}\right)\frac{\partial^2}{\partial \mathbf{x}^2}\left(\mathbf{b}\frac{\partial^2 \mathbf{w}}{\partial \mathbf{x}^2}\right) + \mathbf{I}_1\mathbf{b}\frac{\partial^2 \mathbf{w}}{\partial \mathbf{t}^2} = 0 \tag{6b}$$

where

$$\mathbf{I}_1 = \int_{-\mathbf{h}/2}^{\mathbf{h}/2} \rho(\mathbf{z})\mathbf{dz} \tag{7}$$

The Eq. (6b) can be rewritten as follows

$$\zeta\frac{\partial^4 \mathbf{w}}{\partial \mathbf{x}^4} + 2\zeta\frac{\mathbf{b}'(\mathbf{x})}{\mathbf{b}(\mathbf{x})}\frac{\partial^3 \mathbf{w}}{\partial \mathbf{w}^3} + \zeta\frac{\mathbf{b}''(\mathbf{x})}{\mathbf{b}(\mathbf{x})}\frac{\partial^2 \mathbf{w}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{w}}{\partial \mathbf{t}^2} = 0 \tag{8}$$

with $\zeta = \frac{D_{11}}{I_1} - \frac{B_{11}^2}{I_1 A_{11}}$

For harmonic vibrations, the displacement can be expressed as

$$\mathbf{w}(\mathbf{x}, \mathbf{t}) = \mathbf{W}(\mathbf{x})\mathbf{e}^{\mathbf{i}\omega\mathbf{t}} \tag{9}$$

where ω is the natural frequency of the FGM beam. Substitution of Eq. (9) into Eq. (8) leads to the following ordinary differential equation.

$$W^{(4)} + 2\frac{b'(x)}{b(x)}W''' + \frac{b''(x)}{b(x)}W'' - \mu^2 W = 0 \tag{10}$$

Here μ is a real constant and defined as $\mu^2 = \omega^2 / \zeta$.

Solution of Eq. (10) requires the geometry of the cross-section of the beam to be specified. The characteristic height of the cross-section or the thickness of the beam is kept constant and the characteristic width of the cross-section is assumed to vary exponentially along the length of the beam so that $b(x) = b_0 e^{\delta x}$. Here δ is the non-uniformity parameter and b_0 is the width of the cross-section of the beam at the left end of the beam where $x = 0$ that is $b_0 = b(0)$.

For the family of the cross-sections with exponentially varying characteristic width and constant characteristic height, Eq. (10) reduces to

$$W^{(4)} + 2\delta W''' + \delta^2 W'' - \mu^2 W = 0 \tag{11}$$

Solution of Eq. (11) can be obtained as

$$W(x) = e^{-\frac{\delta}{2}x}\left[B_1 \cos(\lambda_1 x) + B_2 \sin(\lambda_1 x) + B_3 \cosh(\lambda_2 x) + B_4 \sinh(\lambda_2 x)\right] \tag{12}$$

where

$$\lambda_1 = \frac{\sqrt{4\mu - \delta^2}}{2}, \lambda_2 = \frac{\sqrt{4\mu + \delta^2}}{2}$$

The present study considers the S-FGM beams with three different end supports, i.e., a beam with left end clamped and the other end free (clamped-free), a beam hinged at both ends (hinged-hinged), and a beam clamped at both ends (clamped-clamped):

- For a clamped-free beam

$$W = 0, \frac{dW}{dx} = 0 \quad \text{at} \quad x = 0 \tag{14a}$$

$$M = 0, Q = 0 \quad \text{at} \quad x = L \tag{14b}$$

- For a hinged-hinged beam

$$W = 0, M = 0 \quad \text{at} \quad x = 0 \tag{15a}$$

$$M = 0, M = 0 \quad \text{at} \quad x = L \tag{15b}$$

- For a clamped-clamped beam

$$W = 0, \frac{dW}{dx} = 0 \quad \text{at} \quad x = 0 \tag{16a}$$

$$\mathbf{M} = 0, \frac{\mathbf{dW}}{\mathbf{dx}} = 0 \ \text{at} \ \mathbf{x} = \mathbf{L} \tag{16b}$$

4. Mathematical solutions

Solution of Eq. (11) subjected to either one of the boundary conditions given by Eqs. (14) – (16) can be written as

$$\mathbf{W}(\mathbf{x}) = \mathbf{B}_2\mathbf{e}^{-\frac{\delta}{2}\mathbf{x}}[\mathbf{b}_1\cos(\lambda_1\mathbf{x}) + \sin(\lambda_1\mathbf{x}) - \mathbf{b}_1\cosh(\lambda_2\mathbf{x}) + \mathbf{b}_4\sinh(\lambda_2\mathbf{x})]. \tag{17}$$

Here the coefficients b_1 and b_4 depend on δ, $\boldsymbol{p}$ and ω. Application of the boundary conditions in each case yields an implicit equation for the determination of the natural frequency ω for a given non-uniformity parameter δ and the material index $\boldsymbol{p}$. The coefficients b_1, b_4 and the natural frequency equations are given below for each physical case considered in the present study:

- Case 1: S-FGM beam hinged at both ends (hinged-hinged)

$$b_1 = -\frac{\delta(2\lambda_1\sinh\lambda_2 - 2\lambda_2\sin\lambda_1)}{\delta\lambda_2(2\cosh\lambda_2 - 2\cos\lambda_1) + \mu\sinh\lambda_2} \tag{18}$$

$$\mathbf{b}_4 = -\frac{\lambda_1}{\lambda_2} - \frac{2\mu}{\delta\lambda_2}\mathbf{b}_1 \tag{19}$$

$$4\delta^2\lambda_1\lambda_2\cosh\lambda_2\cos\lambda_1 + (8\mu^2 - \delta^4)\sinh\lambda_2\sin\lambda_1 - 4\delta^2\lambda_1\lambda_2 = 0 \tag{20}$$

- Case 2: S-FGM beam clamped at both ends (clamped-clamped)

$$\mathbf{b}_1 = -\frac{\lambda_1\sinh\lambda_2 - \lambda_2\sin\lambda_1}{\lambda_2(\cosh\lambda_2 - \cos\lambda_1)} \tag{21}$$

$$\mathbf{b}_4 = -\frac{\lambda_1}{\lambda_2} \tag{22}$$

$$4\lambda_1\lambda_2\cosh\lambda_2\cos\lambda_1 - \delta^2\sinh\lambda_2\sin\lambda_1 - 4\lambda_1\lambda_2 = 0 \tag{23}$$

- Case 3: The left end of the S-FGM beam is clamped while the right end is free (clamped-free)

$$\mathbf{b}_1 = \frac{2\lambda_1(2\delta\lambda_2 - \delta^2 - 2\mu)e^{2\lambda_2} - 4\lambda_2[2\delta\lambda_1\cos\lambda_1 + (2\mu - \delta^2)\sin\lambda_1]e^{\lambda_2} + \lambda_1(2\lambda_2 + \delta)^2}{\lambda_2[2(2\delta\lambda_2 - \delta^2 - 2\mu)e^{2\lambda_2} + 4[(\delta^2 - 2\mu)\cos\lambda_1 + 2\delta\lambda_1\sin\lambda_1]e^{\lambda_2} - (2\lambda_2 + \delta)^2]} \tag{24}$$

$$\mathbf{b}_4 = -\frac{\lambda_1}{\lambda_2} \tag{25}$$

$$2\lambda_1(16(\lambda_2-\delta)\mu^2\mathbf{e}^{2\lambda_2}-(2\lambda_2+\delta)^2[2(\delta^2-2\mu)\lambda_2-\delta^3])\cos\lambda_1+\delta\,[8(3\delta-4\lambda_2)\mu^2e^{2\lambda_2}+(2\lambda_2+\delta)^2 \times(8\lambda_1^2\lambda_2-\delta^3+2\delta\mu)]\sin\lambda_1+4\lambda_1\lambda_2e^{\lambda_2}[(2\lambda_2+\delta)^2(-2\delta\lambda_2+\delta^2+2\mu)+8\mu^2] = 0 \tag{26}$$

5. Physical meaning of the quantities A_{11}, B_{11}, D_{11} and I_1

If both the Young's modulus and the Poisson's ratio are considered for calculating the coefficients (A_{11}, B_{11}, D_{11}), the integration will turn out to be very complicate. Delale and Erdogan (1983) indicated that the effect of Poisson's ratio on the deformation is much less than that of Young's modulus. Thus, Poisson's ratio of the beams is assumed to be constant.

Substituting the gradation of the Young's modulus and mass density of S-FGM beams in (2) into the definition of coefficients in Eqs. (5) and (7) respectively, we obtain the coefficients of S-FGM beams

$$\mathbf{A}_{11} = \frac{\mathbf{h}}{1-\nu^2}\left(\frac{\mathbf{E}_1+\mathbf{E}_2}{2}\right) \tag{27a}$$

$$\mathbf{B}_{11} = \frac{\mathbf{h}^2(\mathbf{E}_1-\mathbf{E}_2)}{8(1-\nu^2)}\frac{(\mathbf{p}^2+3\mathbf{p})}{(\mathbf{p}+1)(\mathbf{p}+2)} \tag{27b}$$

$$\mathbf{D}_{11} = \frac{\mathbf{h}^3}{12(1-\nu^2)}\left[\frac{\mathbf{E}_1+\mathbf{E}_2}{2}\right] \tag{27c}$$

$$\mathbf{I}_1 = \mathbf{h}\left(\frac{\rho_1+\rho_2}{2}\right) \tag{27d}$$

For S-FGM beams with constant Poisson's ratio, the parameters A_{11}, B_{11} and D_{11} are defined in Eq. (5) as

$$(\mathbf{A}_{11},\mathbf{B}_{11},\mathbf{D}_{11}) = \frac{1}{1-\nu^2}\int_{-\mathbf{h}/2}^{\mathbf{h}/2}(\mathbf{E}(\mathbf{z}),\mathbf{z}\mathbf{E}(\mathbf{z}),\mathbf{z}^2\mathbf{E}(\mathbf{z}))\mathbf{dz} \tag{28}$$

Therefore, it is clear that $(1-\nu^2)\mathbf{A}_{11}$ equals the area under the $\mathbf{E}(\mathbf{z})$ curve from $\mathbf{z}=-\mathbf{h}/2$ to $\mathbf{z}=\mathbf{h}/2$, as indicated in Ref (Delale and Erdogan 1983). Similarly, the parameters $\mathbf{B}_{11}$ and $\mathbf{D}_{11}$ are related to the first and second moments of the area under the $\mathbf{E}(\mathbf{z})$ curve from $\mathbf{z}=-\mathbf{h}/2$ to $\mathbf{z}=\mathbf{h}/2$ with respect to the $\mathbf{z}=0$ axis. In the same way, the parameter $\mathbf{I}_1$ from Eq. (7) is related the area under the $\rho(\mathbf{z})$ curve from $\mathbf{z}=-\mathbf{h}/2$ to $\mathbf{z}=\mathbf{h}/2$. They are simplified as

- $(1-\nu^2)\mathbf{A}_{11}$ = the area under the $\mathbf{E}(\mathbf{z})$ curve from $\mathbf{z}=-\mathbf{h}/2$ to $\mathbf{z}=\mathbf{h}/2$ (29a)

- $\mathbf{I}_{11}$ = the area under the $\rho(\mathbf{z})$ curve from $\mathbf{z} = -\mathbf{h}/2$ to $\mathbf{z} = \mathbf{h}/2$ (29b)

- $(1-\nu^2)\mathbf{B}_{11} = (1-\nu^2)\mathbf{A}_{11} \times \bar{\mathbf{z}}$ (29c)

- $(1-\nu^2)\mathbf{D}_{11} = \bar{I}(1-\nu^2)\mathbf{A}_{11} \times \bar{\mathbf{z}}^2$ (29d)

where $\bar{\mathbf{z}}$ is the distance from the centroid of the area $(1-\nu^2)\mathbf{A}_{11}$ to the axis $\mathbf{z} = 0$, and $\bar{I}$ is the second moment of the area $(1-\nu^2)\mathbf{A}_{11}$ with respect to the axis passing through the centroid. It can be seen from Eq. (29c) that the location of the centroid $\bar{z}$ can be expressed by the parameters $\mathbf{A}_{11}$ and $\mathbf{B}_{11}$ as

$$\bar{\mathbf{z}} = \frac{\mathbf{B}_{11}}{\mathbf{A}_{11}} \tag{30}$$

From Eqs. (27), the quantity $\mathbf{B}_{11}$ is positive if the Young's moduli satisfy $\mathbf{E}_1 > \mathbf{E}_2$; in this case the location of the centroid $\bar{\mathbf{z}}$ is also positive. Based on work presented by Sankar (2001) where $\mathbf{Z}_{NA} = -\mathbf{B}_{11}/\mathbf{A}_{11}$ is the location of the neutral surface of the FGM beams, it can be observed that the axes of the physical neutral surface and the centroid (Eq. (30)) of the area under the $\mathbf{E}(\mathbf{z})$ curve coincide. The neutral surfaces versus the material parameter **p** with different ratios of Young's moduli are plotted in Fig. 2 for S-FGM beams. The results indicate that the neutral axes move far away from the $\mathbf{z} = 0$ axis as the parameter **p** increases for $\mathbf{E}_1/\mathbf{E}_2 > 1$ (with $\mathbf{E}_1$ fixed). With the same parameter **p** and Young's moduli $\mathbf{E}_1$ and $\mathbf{E}_2$, the locations of the neutral surfaces of the S-FGM beams are closer to the middle surfaces.

6. Numerical results and discussion

We suppose that the S-FGM beam is made from a mixture of ceramic and metal and the composition varies from the top to the bottom surface; i.e. the top surface ($z = h/2$) of the beam is ceramic-rich (alumina), whereas the bottom surface ($z = -h/2$) is metal-rich (aluminum). Typical values for alumina

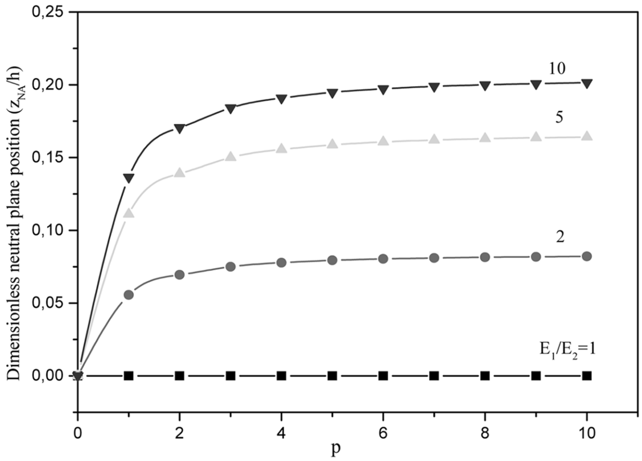

Fig. 2 Locations of the neutral surfaces versus the material parameter ***p*** for $\boldsymbol{E}_1 = 70$ GPa and varying $\boldsymbol{E}_2$.

and aluminum are listed in Table 1 (Sallai *et al.* 2009, Huang and Shen 2004). It is assumed in the following example that the S-FGM beam thickness $h = 0.1\ m$ the slenderness ratio $L / h = 10$;

Table 2 lists the natural frequencies ($\mu_n = \omega_n/\sqrt{\zeta}$) of an exponential narrowing isotropic homogeneous beam ($\delta = -1$ and $\mathbf{E}_1 / \mathbf{E}_2 = 1$). The present results agree very well with those given by Cranch and Adler (1956) and by Tong *et al.* (1995).

The three first normalized natural frequencies ($\bar{\omega}_n = \omega_n/\sqrt{\zeta_0}$) of S-FGM beams, where ζ_0 denote the value of ζ of an isotropic homogeneous beam ($\mathbf{E}_1 / \mathbf{E}_2 = 1$).

Table 3 lists the first three dimensionless natural frequencies ($\bar{\omega}_n = \omega_n/\sqrt{\zeta_0}$) of clamped-free (C-F), hinged-hinged (H-H), and clamped-clamped (C-C) S-FGM beam for a given non-uniformity parameter δ, where ζ_0 denote the value of ζ of a fully metallic beam. It can be observed that the natural frequencies decrease with an increase in the volume fraction index **p** because this means a reduction in the volumetric percentage of alumina whose Young's modulus is much higher than aluminum.

For the three cases of the boundary conditions (H-H, C-C, C-F), the natural frequencies of S-FGM beams increase with the mode numbers. It is found that the isotropic homogeneous beams have higher frequencies than the graded beams. The natural frequencies for the hinged-hinged and clamped-clamped boundary conditions are independent from the sign of δ since the implicit equations for the natural frequency involve δ^2 only. All the natural frequencies of the non-uniform beam are greater than those of the uniform beam for the clamped-clamped boundary conditions and the natural frequencies increase with the non-uniformity parameter δ. The fundamental natural frequency of the non-uniform beam for the hinged-hinged boundary conditions is observed to be decreasing with the non-uniformity parameter δ while the higher frequencies are increasing. All the natural frequencies of an exponentially narrowing beam are greater than those of the uniform beam for the clamped-free boundary conditions and increase with the increasing magnitude of the non-uniformity parameter δ as is shown in Table 1. However, the natural frequencies of an exponentially widening beam are smaller than those of the uniform beam for the clamped-free boundary conditions.

In Figs (3)-(5) the normalized natural frequencies $\bar{\omega}_n$ of the non-uniform S-FGM beams are plotted versus the normalized frequencies of the non-uniform aluminium beams for the H-H, C-C and C-F boundary conditions respectively. These results show remarkable proportionality between the natural frequencies of non-uniform S-FGM beams and those of isotropic non-uniform beams. The present observation will dramatically reduce the need for extensive numerical analysis of non-uniform S-FGM beams since

Table 1 Material properties (Sallai *et al.* 2009, Huang and Shen 2004)

Materials	Property		
	$E\ (GPa)$	$\rho\ (kg/m^3)$	ν
Aluminum	70	2707	0.3
Alumina	380	3800	0.3

Table 2 Natural frequencies (μ_n) for an exponentially narrowing beam ($\delta = -1$) under the C-F conditions.

Mode number	Present	Cranch and Adler (1956)	Tong *et al.* (1995)
1	4.723	4.735	4.7347
2	24.2017	24.2025	24.2005
3	63.8645	63.85	63.8608
4	123.098	–	123.091

Table 3 First three dimensionless natural frequencies of non-uniform FGM beam with exponential width variation

δ	ρ	C-F			H-H			C-C		
		$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$
0	0	6.9143	43.3312	121.3286	19.4087	77.6350	174.6787	43.9974	121.2806	237.7585
	0.5	5.2682	33.0154	92.4442	14.7881	59.1526	133.0933	33.5231	92.4076	181.1559
	1	5.0326	31.5388	88.3097	14.1268	56.5070	127.1408	32.0238	88.2748	173.0538
	5	4.5137	28.2868	79.2038	12.6701	50.6804	114.0310	28.7217	79.1725	155.2098
	10	4.4385	27.8153	77.8838	12.4589	49.8358	112.1305	28.2430	77.8530	152.6229
		3.5160	22.0345	61.6972	09.8696	39.4784	88.8264	22.3733	61.6728	120.9034
1	0	5,6210	39,4074	117,7370	19.2186	77.8158	174.9620	44,2696	121.6481	238.1609
	0.5	4,2828	30,0257	89,7076	14.6433	59.2903	133.3092	33,7300	92,6880	181,4624
	1	4.0912	28.6829	85.6955	13.9884	56.6386	127.3470	32.2219	88,5422	173,3467
	5	3,6694	25,7253	76,8592	12.5460	50.7985	114.2159	28,8994	79,4124	155.4724
	10	3,6082	25,2965	75,5782	12.3369	49.9518	112.3123	28,4177	78,0889	152,8812
		2.8583	20.0392	59.8708	09.7729	39.5704	88.9705	22.5117	61.8597	121.1080
2	0	5,7205	35,7418	114,8223	18,6568	78,3702	175,8168	45,1074	122,7553	239,3698
	0.5	4,3586	27,2330	87,4868	14,2152	59,7128	133,9605	34,3688	93,5312	182,3835
	1	4.1637	26,0149	83,5740	13,5795	57,0422	127,9692	32,8317	89,3481	174.2266
	5	3,7343	23,3324	74,9565	12,1793	51,1604	114,7740	29,4463	80,1352	156,2616
	10	3,6721	22,9435	73,7072	11,9763	50,3077	112,8611	28,9555	78,7996	153,6572
		2.9089	18.1752	58.3887	9.4873	39.8523	89.4052	22.9377	62.4227	121.7227
-1	0	9.2878	47.5930	125.5906						
	0.5	7.0767	36.2626	95.6915						
	1	6,7602	34,6408	91,4118						
	5	6.0631	31.0689	81.9861						
	10	5,9621	30,5511	80,6196						
		4.7230	24.2017	63.8645						
-2	0	12,3080	52,2769	129,9688						
	0.5	9,3778	39,8315	99,4524						
	1	8.9584	38.0500	95.0044						
	5	8,0347	34,1266	85,2083						
	10	7,9008	33,5578	83,7881						
		6.2588	26.5835	66.3745						

their natural frequencies can be deduced from those of the isotropic plate.

Fig. 6 shows that the normalized natural frequencies of the non-uniform beam are almost proportional to the normalized natural frequencies of the uniform beam ($\delta = 0$) with the same boundary condition especially for high natural frequencies. Hence, the high natural frequencies of non-uniform beam (with exponentially varying width) can be obtained from the corresponding results for uniform beam so that a direct analysis of non-uniform beam is not necessary.

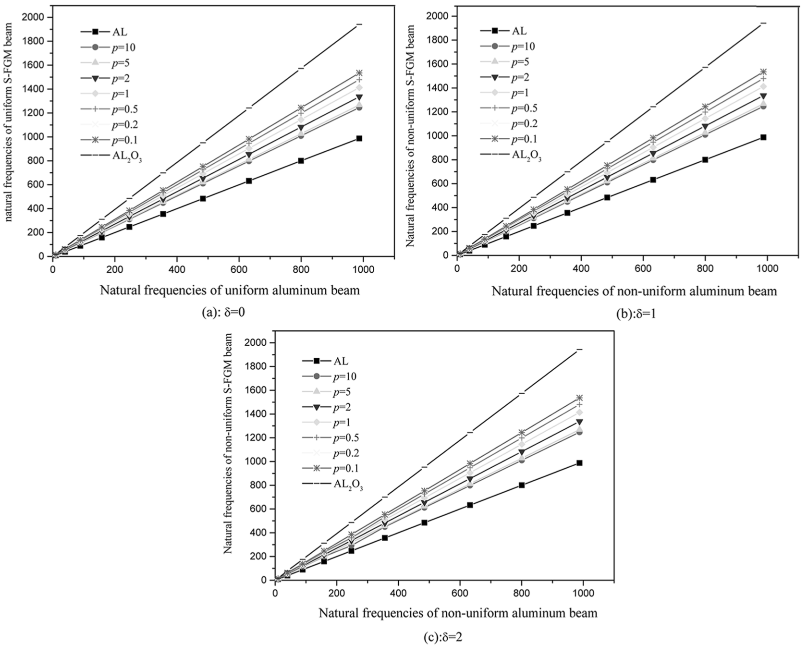

Fig. 3 Natural frequencies of hinged-hinged S-FGM beams compared to natural frequencies of aluminum beams with similar boundary conditions: (a) $\delta = 0$; (b)$\delta = 1$; (c)$\delta = 2$.

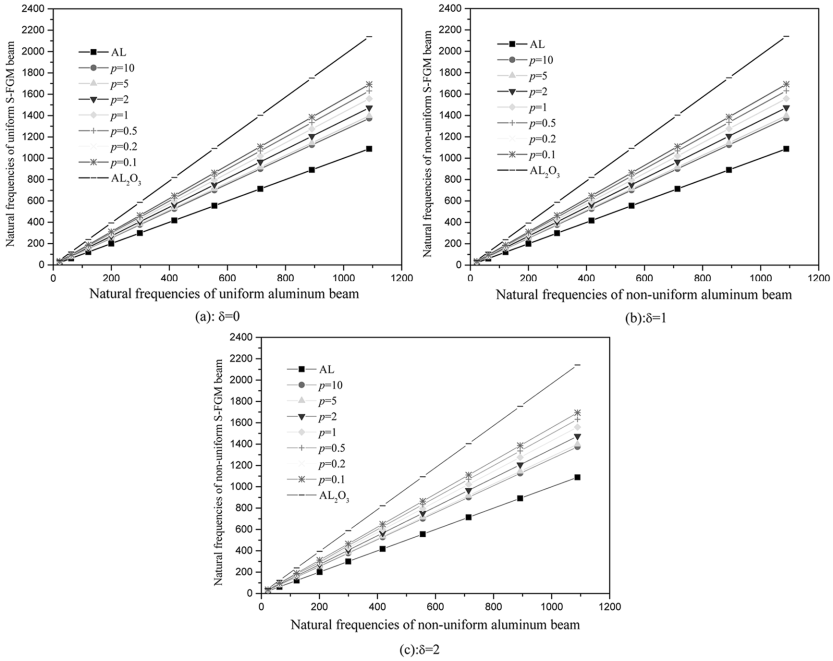

Fig. 4 Natural frequencies of clamped-clamped S-FGM beams compared to natural frequencies of aluminum beams with similar boundary conditions: (a) $\delta = 0$; (b)$\delta = 1$; (c)$\delta = 2$.

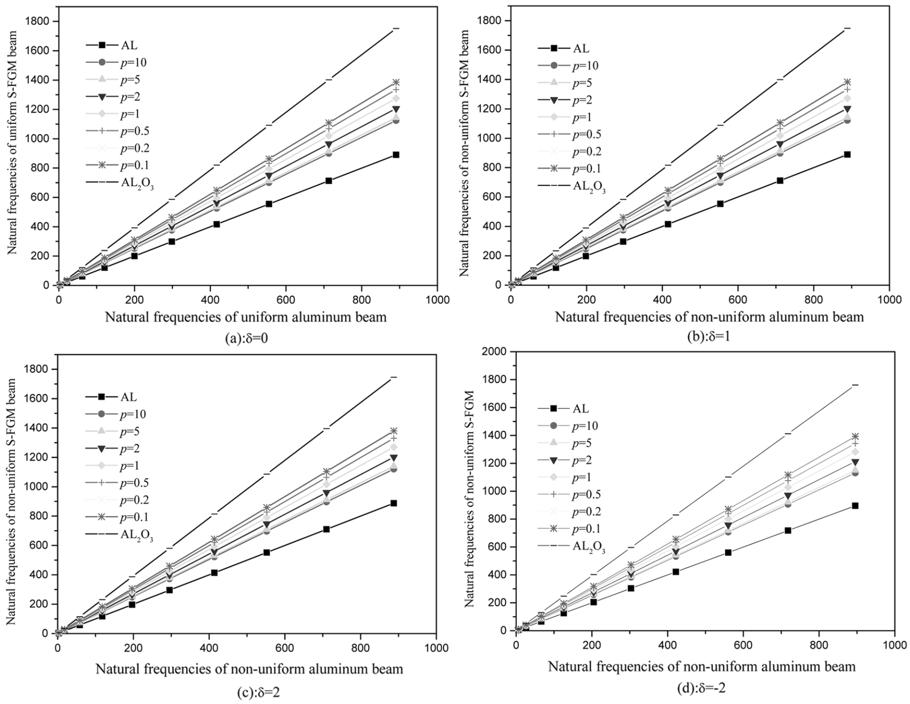

Fig. 5 Natural frequencies of clamped-free S-FGM beams compared to natural frequencies of aluminum beams with similar boundary conditions: (a) $\delta = 0$; (b)$\delta = 1$; (c)$\delta = 2$, (d) $\delta = -2$.

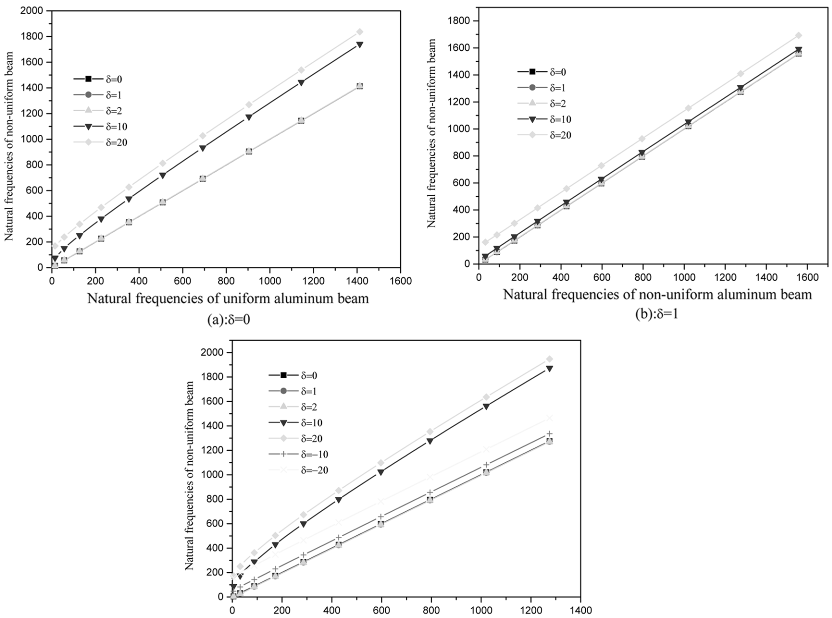

Fig. 6 Natural frequencies of non-uniform beam compared to natural frequencies of uniform beams with similar boundary condition ($\boldsymbol{p} = 1$): (a) hinged-hinged; (b) clamped-clamped; (c) clamped-free.

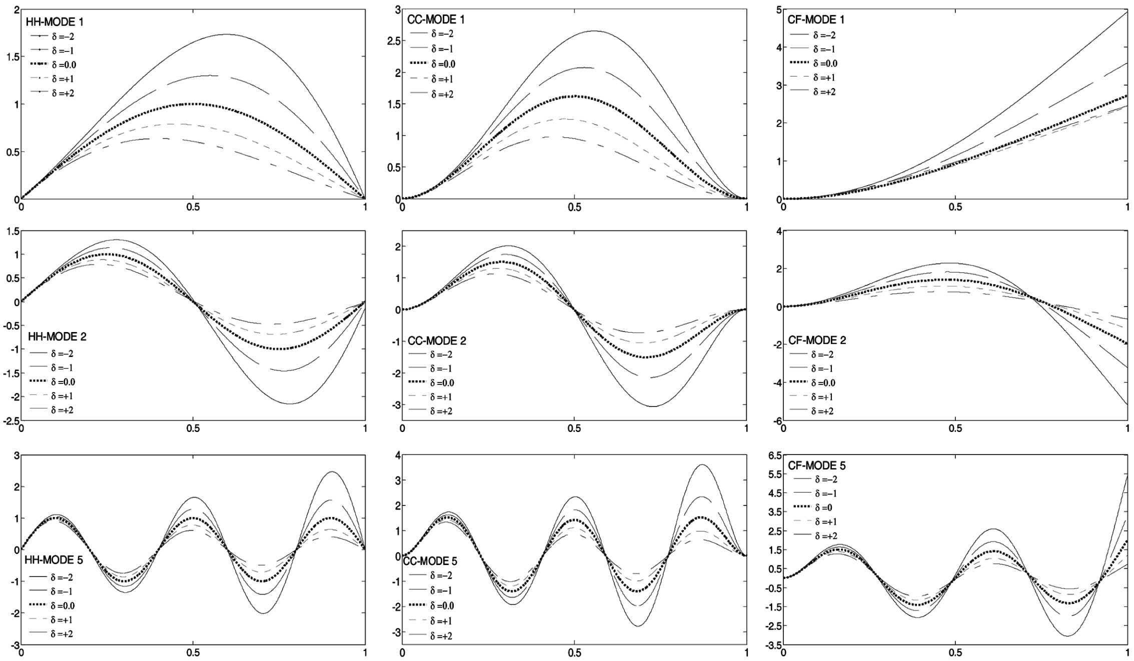

Fig. 7 First, second and fifth mode shapes of non-uniform beam ($p = 1$) for the considered cases.

Fig. 7 shows the 1st, 2nd, and 5th mode shapes of beams with different boundary conditions (H-H, C-C, C-F) and with various values of the non-uniformity parameter (δ). Note that the mode shape associated with $\delta = 0$ corresponds to the mode shape for the uniform beam. It is found that varying width and boundary conditions have a significant influence on the mode shapes. It may be seen from Eq. (17) that the amplitude of the transverse vibrations is proportional to $e^{-\frac{\delta}{2}x}$. Therefore amplitude of the mode shapes for a given non-uniformity parameter δ increases with $\mathbf{x}$ for narrowing beams ($\delta < 0$) and decreases with x for widening beams ($\delta > 0$).

7. Conclusions

Free vibration behavior of sigmoid functionally graded beams with exponentially varying width is investigated by using Bernoulli–Euler beam theory. Young's modulus of the assumed beam varies in the thickness direction according to two power law. It is found that the natural frequencies of non-uniform S-FGM beams were proportional to those of the corresponding non-uniform homogenous beam. Then, S-FGM beams behave like homogeneous beams which mean that no special techniques or software needs to be developed for their analysis.

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