Steel and Composite Structures, Vol. 11, No. 2 (2011) 115-130
DOI: http://dx.doi.org/10.12989/scs.2011.11.2.115

Stochastic finite element analysis of composite plates considering spatial randomness of material properties and their correlations

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(Received January 27, 2010, Accepted February 14, 2011)

Abstract Considering the randomness of material parameters in the laminated composite plate, a scheme of stochastic finite element method to analyze the displacement response variability is suggested. In the formulation we adopted the concept of the weighted integral where the random variable is defined as integration of stochastic field function multiplied by a deterministic function over a finite element. In general the elastic modulus of composite materials has distinct value along an individual axis. Accordingly, we need to assume 5 material parameters as random. The correlations between these random parameters are modeled by means of correlation functions, and the degree of correlation is defined in terms of correlation coefficients. For the verification of the proposed scheme, we employ an independent analysis of Monte Carlo simulation with which statistical results can be obtained. Comparison is made between the proposed scheme and Monte Carlo simulation.

Keywords: Composite laminates, stochastic finite element analysis, spatial randomness, correlation, coefficient of variation.

1. Introduction

It is of importance to consider the stochasticity of composite structures because the number of parameters which can have randomness is, in general, greater than that of isotropic materials. Accordingly, the effect of correlation between these random parameters on the response variability is also of high concern. This is caused by complex manufacturing processes which render composite materials random in nature. Even with this intrinsic randomness, the composite material has nevertheless been used extensively in a variety of structures due to high specific strength and stiffness to weight ratios, high design flexibility, durability to fatigue and corrosion, and so on (Singh *et al.*, 2002). Therefore, the reliable estimation of degree of uncertain response of the structures built with composite materials is of great importance.

While the stochastic analysis on the effect of random excitation to the structure has been extensively established (Nigam, 1994), the investigation on the effect of material randomness on the response is to a degree limited (Singh *et al.*, 2002) due to the complexity in taking into account the various random parameters. Even though the Monte Carlo analysis provides a general way of computing random responses owing to the simplicity of the scheme, it needs to generate numerically the complex random fields complying with the pre-assigned probabilistic characteristics of the analysis domain and more importantly, an additional statistical operation is necessary on a set of deterministic analyses. In

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response to these drawbacks, some non-statistical methodologies have been developed (Antonio and Hoffbauer 2007, Cavdar *et al*. 2008, Lal *et al*. 2007, Noh 2004, Noh and Park 2006, Ngah and Young 2007, Papadopoulous *et al.* 2006). The weighted integral method, among others, which has been applied to analyze truss, beam, plane-problems, plate bending and so on, is known to give a more accurate response variability than the other schemes available in the literature (Deodatis 1991, Graham and Deodatis 1998, Noh and Park 2006, Schuëller 2001, Stefanou 2009). In the weighted integral method, the random variable is defined as an integration of a stochastic field function multiplied by a deterministic function over a finite element domain.

In this study, we suggest a formulation using a weighted integral scheme to the analysis of composite laminate plates considering material parameters as random. The random parameters we take are elastic moduli E_1, E_2 and in-plane and out-of-plane shear moduli G_{12}, G_{13} and G_{23}. The correlation between aforementioned 5 random parameters are modeled with corresponding auto- and cross-correlation functions

The proposed scheme is verified with some numerical illustrations, and is also compared with the Monte Carlo analysis based on the local averaging scheme, which is told to give more accurate results than the mid-point rule.

2. Constitutive relation of laminate composite

2.1 Constitutive relation matrix

When we deal with the randomness in material parameters we need to investigate into constitutive relations since effects of the material parameters on the response is condensed in these relations. The constitutive relation for composite material is widely known to be as follows

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}^{(k)} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix}^{(k)}, \begin{Bmatrix} \sigma_4 \\ \sigma_5 \end{Bmatrix}^{(k)} = \begin{bmatrix} Q_{44} & 0 \\ 0 & Q_{55} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_4 \\ \varepsilon_5 \end{Bmatrix}^{(k)}, \tag{1a}$$

or using brief matrix notations we get

$$\boldsymbol{\sigma}_b^{(k)} = \mathbf{Q}_b^{(k)} \varepsilon_b^{(k)}, \boldsymbol{\sigma}_s^{(k)} = \mathbf{Q}_s^{(k)} \varepsilon_s^{(k)} \tag{1b}$$

The first part has to do with in-plane behavior and the second part is for shear deformation. The expression in Eq. (1) is for (k)-th lamina. The respective entries in Eq. (1) is given as Eq. (2).

$$Q_{11} = \frac{E_1}{1 - v_{12}v_{21}}, Q_{12} = \frac{v_{12}E_2}{1 - v_{12}v_{21}}, Q_{22} = \frac{E_2}{1 - v_{12}v_{21}},$$
$$Q_{44} = G_{23}, Q_{55} = G_{13}, Q_{66} = G_{12} \tag{2}$$

The elastic moduli E_1, E_2 correspond to two principal axes of the material. The value of the parameters depends on the volumetric portion of two materials that constitute the composite. Due to different material characteristics along independent axes, two Poisson's ratios v_{12}, v_{21}, where v_{ij} defines a ratio of lateral strain along j-direction to axial strain along i-direction, appear. The modulus for in-plane shear is G_{12}, and those of out-of-plane are G_{23}, G_{13}.

According to the rule of mixture, the material properties of composite can be determined in terms of volume fractions for two materials, fiber and matrix, which constitute the composite laminate. If we denote elastic moduli of materials a and b as E_a, E_b, the total volume as $V = V_a + V_b$, and the volume ratio of these materials as $r_a = V_a / V$, $r_b = V_b / V$, then the elastic moduli along the fiber direction E_1 and perpendicular to fiber, E_2, are obtained as follows

$$E_1 = E_a r_a + E_b r_b \tag{3a}$$

$$E_2 = \frac{E_a E_b}{E_a r_b + E_b r_a} \tag{3b}$$

The in-plane shear modulus G_{12} and the Poisson's ratio v_{12} can both be given in terms of volume ratios as follows

$$G_{12} = \frac{1}{2}\frac{E_a E_b}{E_a r_b(1+v_b) + E_b r_a(1+v_a)} \tag{4}$$

$$v_{12} = v_a r_a + v_b r_b \tag{5}$$

2.2 Transformation of constitutive relation

One of the characteristics of the laminated composites is that the material axes do not coincide with the global axes of a structure, which forms an angle θ known as a stacking angle, between these two sets of coordinate systems. Accordingly, the constitutive relation has to be transformed into appropriate directions. Performing the transformation, the elements of the constitutive relation can be found to be a function of original coefficients of constitutive relation and the angle between global and material axes,

$$Q_{ij}' = f(Q_{ij}, c, s) \tag{6}$$

where, c and s denote $\cos\theta$ and $\sin\theta$, respectively.

The detailed expressions for coefficients of the constitutive matrix in the transformed coordinate are given as follows

$$\begin{aligned}
Q'_{11} &= Q_{11}c^4 + 2(Q_{12}+2Q_{66})s^c c^2 + Q_{22}s^4 \\
Q'_{22} &= Q_{11}s^4 + 2(Q_{12}+2Q_{66})s^c c^2 + Q_{22}c^4 \\
Q'_{12} &= (Q_{11}+Q_{22}-4Q_{66})s^c c^2 + Q_{12}(s^4+c^4) \\
Q'_{16} &= (Q_{11}-Q_{12}-2Q_{66})sc^3 + (Q_{12}-Q_{22}+2Q_{66})s^3 c \\
Q'_{26} &= (Q_{11}-Q_{12}-2Q_{66})s^3 c + (Q_{12}-Q_{22}+2Q_{66})sc^3 \\
Q'_{66} &= (Q_{11}+Q_{22}-2Q_{12}-2Q_{66})s^2c^2 + Q_{66}(s^4+c^4)
\end{aligned} \tag{7a}$$

$$Q'_{44} = Q_{44}c^2 + Q_{55}s^2$$

$$Q'_{55} = Q_{44}s^2 + Q_{55}c^2$$

$$Q'_{45} = (-Q_{44} + Q_{55})sc \tag{7b}$$

2.3 Material parameters with stochasticity

In this study we assume that all the elastic moduli are uncertain in the spatial domain. Owing to the representative expression for random material properties, where the modulus is assumed to consist of mean parameter plus the deviator component, the two elastic and three shear moduli can be expressed as

$$E_1 = \bar{E}_1[1 + \alpha_1(\mathbf{x})], E_2 = \bar{E}_2[1 + \alpha_2(\mathbf{x})], G_{12} = \bar{G}_{12}[1 + \alpha_3(\mathbf{x})] \tag{8a}$$

$$G_{23} = \bar{G}_{23}[1 + \alpha_4(\mathbf{x})], G_{13} = \bar{G}_{13}[1 + \alpha_5(\mathbf{x})] \tag{8b}$$

where $\alpha_i(\mathbf{x})$, $i = 1,2,...,5$ denote stochastic field functions for respective random material parameters, which represent the fluctuating component of randomness over the structural domain. The mean of the respective random parameters are designated with over-bar symbol as $\bar{E}_1, \bar{E}_2, \bar{G}_{12}, \bar{G}_{23}, \bar{G}_{13}$, and accordingly we can note that the mean of respective stochastic field function α_i vanishes.

With a direct substitution of Eq. (8) into Eq. (2), and using the relation given in Eq. (7), we can rearrange the constitutive relations as gathered together having corresponding stochastic field function α_i, with an omission of position vector **x** for the simplicity of the notation as follows

$$\mathbf{Q}'_b = \mathbf{Q}'_{bo} + \sum_{i=1}^{3} \alpha_i(\mathbf{x})\mathbf{Q}'_{\alpha_i}, \quad \mathbf{Q}'_s = \mathbf{Q}'_{so} + \sum_{i=4}^{5} \alpha_i(\mathbf{x})\mathbf{Q}'_{\alpha_i} \tag{9}$$

where, $\mathbf{Q}'_{bo} = Q'_{ij}(\alpha_l{=}0.0)$ where $i,j = 1,2,6$, $l = 1,2,3$ and $\mathbf{Q}'_{so} = Q'_{ij}(\alpha_l{=}0.0)$ where $i,j,l = 4,5$ denote the mean contribution; the remainders are as follows

$$\mathbf{Q}'_{\alpha_1} = Q_{11}\begin{bmatrix} c^4 & s^2c^2 & sc^3 \\ & s^4 & s^3c \\ symm. & & s^2c^2 \end{bmatrix} \tag{10a}$$

$$\mathbf{Q}'_{\alpha_2} = Q_{12}\begin{bmatrix} 2s^2c^2 & s^4 + c^4 & -sc^3 + s^3c \\ & 2s^2c^2 & -s^3c + sc^3 \\ symm. & & -2s^2c^2 \end{bmatrix} + Q_{22}\begin{bmatrix} s^4 & s^2c^2 & -s^3c \\ & c^4 & -sc^3 \\ symm. & & s^2c^2 \end{bmatrix} \tag{10b}$$

$$\mathbf{Q}'_{\alpha_3} = Q_{66}\begin{bmatrix} 4s^2c^2 & -4s^2c^2 & -2sc^3+2s^3c \\ & 4s^2c^2 & -2s^3c+2sc^3 \\ symm. & & -2s^2c^2+s^4+c^4 \end{bmatrix} \tag{10c}$$

$$\mathbf{Q}'_{\alpha_4} = Q_{44}\begin{bmatrix} c^2 & -sc \\ -sc & s^2 \end{bmatrix}, \quad \mathbf{Q}'_{\alpha_5} = Q_{55}\begin{bmatrix} s^2 & sc \\ sc & c^2 \end{bmatrix} \tag{10d}$$

The stress resultants for extensional, bending and extensional-bending coupling parts, A_{ij}, B_{ij} and D_{ij} respectively, can be obtained by means of integration over thickness as follows

$$\begin{aligned} A_{ij} &= \int_{-t/2}^{t/2} Q'_{ij}\, dz = \sum_{k=1}^{N} Q'^{(k)}_{ij}(z_{k+1}-z_k) \\ B_{ij} &= \int_{-t/2}^{t/2} Q'_{ij} z\, dz = \frac{1}{2}\sum_{k=1}^{N} Q'^{(k)}_{ij}(z^2_{k+1}-z^2_k) \\ D_{ij} &= \int_{-t/2}^{t/2} Q'_{\alpha_5} z^2\, dz = \frac{1}{3}\sum_{k=1}^{N} Q'^{(k)}_{ij}(z^3_{k+1}-z^3_k) \end{aligned} \tag{11}$$

where, i, j = 1,2,6 and 't' denotes the thickness of the laminate.

In a similar way, the coefficients of stress resultant for shear are obtained as follows

$$A_{Sij} = \int_{-t/2}^{t/2} Q'_{ij} dz = \kappa \sum_{k=1}^{N} Q'^{(k)}_{ij}(z_{k+1}-z_k),\; i,j = 4.5 \tag{12}$$

In Eq. (12), κ denotes shear correction factor, and a value of 5 / 6 is employed.

Consequently, total constitutive matrices for an element are given in terms of matrices **A**, **B**, **D** and $\mathbf{A}_s$ as follows

$$\mathbf{C}_{ABD} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} = \mathbf{C}_{ABD_o} + \sum_{i=1}^{3} \alpha_i C_{\alpha_i}, \; \mathbf{C}_S = [\mathbf{A}_S] = \mathbf{C}_{So} + \sum_{i=4}^{5} \alpha_i C_{\alpha_i} \tag{13}$$

3. Stochastic stiffness matrix of composite plate

As noted in Eq. (13), the constitutive matrix includes the stochastic field functions which render the stiffness matrix to have randomness. The contribution of uncertainty due to stochastic field affects not only the extension and bending part but also the shear part. In fact, the stiffness is divided into two parts: the first includes the extensional, bending and extension-bending coupling behavior, and the second is only for in-plane shear behavior.

$$\mathbf{k} = \int_V \mathbf{B}_{ABD}^T \mathbf{C}_{ABD} \mathbf{B}_{ABD} dV + \int_V \mathbf{B}_S^T \mathbf{C}_S \mathbf{B}_S dV \tag{14}$$

If we directly substitute constitutive matrices in Eq. (13) into Eq. (14), the stiffness matrix is divided depending on the kinds of stochastic fields as follows

$$\mathbf{k} = \int_V \mathbf{B}_{ABD}^T \Big(\mathbf{C}_{ABD} + \sum_{i=1}^{3} \alpha_i C_{\alpha_i} \Big) \mathbf{B}_{ABD} dV + \int_V \mathbf{B}_S^T \Big(\mathbf{C}_S + \sum_{i=4}^{5} \alpha_i C_{\alpha_i} \Big) \mathbf{B}_S dV \tag{15}$$

In the context of weighted integral scheme, where the integration of the stochastic field that is multiplied by a deterministic function over the finite element domain is considered as a random variable, the stiffness matrix in Eq. (15) becomes a function of random variable $X_{ij}^{(l)}$. The random variable is defined as follows for the k-th stochastic field α_k and deterministic function g_{ij}.

$$X_{ij}^{(l)} = \int_V \alpha_l(\mathbf{x}) g_{ij}(\mathbf{x}) dV \tag{16}$$

If we see the detailed form of Eq. (15) as given in Eq. (17),

$$\begin{aligned} \mathbf{k} &= \int_V \mathbf{B}_{ABD}^T \mathbf{C}_{ABDo} \mathbf{B}_{ABD} dV + \int_V \mathbf{B}_S^T \mathbf{C}_{So} \mathbf{B}_S \, dV + \int_V \mathbf{B}_{ABD}^T \Big(\sum_{i=1}^{3} \alpha_i \mathbf{C}_{\alpha_i} \Big) \mathbf{B}_{ABD} \, dV + \int_V \mathbf{B}_S^T \Big(\sum_{i=4}^{5} \alpha_i \mathbf{C}_{\alpha_i} \Big) \mathbf{B}_S \, dV, \\ &= \mathbf{k}_o + \Delta \mathbf{k} \end{aligned} \tag{17}$$

it is quite apparent that the stiffness is decomposed into two parts, i.e. the mean and the deviator, and only the deviator stiffness $\Delta\mathbf{k}$ is a function of random variable $X_{ij}^{(l)}$. We can also note that the deterministic function g_{ij}, which is used in defining the random variable, is linked to strain-displacement matrix, i.e., $\mathbf{B}_{ABD}$ and $\mathbf{B}_S$. In defining the random variables, we assume that the strain-displacement matrix can be decomposed into a sum $B_i p_i$, $i = 1, 2, \ldots, n_p$ with n_p the number of independent polynomials for $\mathbf{B}_{ABD}$ and $\mathbf{B}_S$. Accordingly, the deviator stiffness in Eq. (17) can be written as

$$\Delta \mathbf{k} = \sum_{l=1}^{5} \sum_{i,j}^{n_p} \mathbf{B}_i^T \mathbf{C}_{\alpha_k} \mathbf{B}_j \mathbf{X}_{ij}^{(l)} \tag{18}$$

where the deterministic function $g_{ij}(\mathbf{x})$ is given as a multiplication of consecutive polynomials p_i, e.g., $g_{12}(\mathbf{x}) = p_1 p_2$, etc.

With out analysis of Eqs. (17) and (18) we discern that both the element stiffness and the global stiffness matrix $\mathbf{K} = \sum_{i=1}^{N_e} \mathbf{k}_i$, where N_e denotes the number of finite elements in the domain, are functions of random variable $X_{ij}^{(l)}$ so that the displacement vector is to be a function of the random variable. This is because the displacement is related to the inverse of the global stiffness matrix. Due to the fact we are considering 5 independent material parameters as random, referring to Eq. (18), the number of random variables for an individual finite element is estimated to be $n_r = 5 r_{\alpha_k}$, where $r_{\alpha_k} = n_p(n_p + 1)$. In the following, the random variable will be denoted simply as X_r, where the index r ranges from 1 to n_r.

4. Response statistics of the displacement vector

4.1 Taylor's expansion on the displacement vector

Employing first order Taylor's expansion on the displacement vector with respect to the mean random variable $\bar{X}_r$, we obtain the following

$$\mathbf{U} = \mathbf{U}\Big|_{\varepsilon} + \sum_e \sum_r (X_r^e - \bar{X}_r^e) \frac{\partial \mathbf{U}}{\partial X_r^e}\Big|_{\varepsilon} \tag{19}$$

where, the subscript 'ε' denotes an evaluation at the mean, 'e' and 'r' are the indices for the finite elements and the random variable. In order to represent the expansion in a more direct way, we can replace the partial differentiation on the displacement vector with that on the global stiffness by performing partial differentiation on the equation $\mathbf{KU} = \mathbf{F}$, which results in

$$\mathbf{U} = \mathbf{U}\Big|_{\varepsilon} - \mathbf{K}_o^{-1} \sum_e \sum_r (X_r^e - \bar{X}_r^e) \frac{\partial \mathbf{U}}{\partial X_r^e} \mathbf{U}\Big|_{\varepsilon} \tag{20}$$

where $\mathbf{K}_o$ stands for the deterministic global stiffness matrix, which is exactly the same as the mean stiffness in the current formulation. It is noted here that the proposed scheme is applicable to the stochastic problems with small coefficient of variation because the first order linear Taylor's expansion is used in the formulation.

4.2 Mean and covariance of the displacement

The mean displacement can be obtained employing the mean operation on Eq. (20). With mean operation, it is apparent that the term in the parentheses vanishes because the operation changes X_r^e into $\bar{X}_r^e$. As a result, the mean displacement is the following

$$\bar{\mathbf{U}} = \varepsilon[\mathbf{U}] = \mathbf{U}|_{\varepsilon} \tag{21}$$

In estimating the covariance of the displacement, we need to obtain the variation in the displacement with respect to the mean, i.e., $\Delta\mathbf{U} = \mathbf{U} = \mathbf{U}$, which is given as follows

$$\Delta\mathbf{U} = -\mathbf{K}_o^{-1} \sum_e \sum_r (X_r^e - \bar{X}_r^e) \frac{\partial \mathbf{K}}{\partial X_r^e} \mathbf{U}|_{\varepsilon} \tag{22}$$

Following the definition of covariance of the displacement, the displacement covariance becomes

$$\begin{aligned} Cov[\mathbf{U}] &= \varepsilon[\Delta\mathbf{U}\Delta\mathbf{U}^T] \\ &= K_o^{-1} \varepsilon\left[\sum_e^{N_e} \sum_f^{N_e} \sum_r^{n_r} \sum_s^{n_r} (X_r^e - \bar{X}_r^e)(X_s^f - \bar{X}_s^f) \frac{\partial \mathbf{K}}{\partial X_r^e} \bar{\mathbf{U}} \bar{\mathbf{U}}^T \left(\frac{\partial \mathbf{K}}{\partial X_r^e}\right)^T\right] K_o^{-T} \end{aligned} \tag{23}$$

If we gather separately the contribution from the respective stochastic field functions, Eq. (23) can be rearranged as the following expression

$$Cov[\mathbf{U}] = \mathbf{K}_o^{-1}\varepsilon\left(\sum_e^{N_e}\sum_f^{N_e}\left\{\left(\sum_{k=1}^{5}\Delta\mathbf{K}_{\alpha_k}^{e}\right)\bar{\mathbf{U}}\bar{\mathbf{U}}^T\left(\sum_{k=1}^{5}\Delta\mathbf{K}_{\alpha_k}^{f}\right)^T\right\}\right)\mathbf{K}_o^{-T}. \tag{24}$$

Since the term $\Delta\mathbf{F}_{\alpha_i}^{e}\bar{\mathbf{U}}$ is in the form of an element-related load vector, we assign $\Delta\mathbf{F}_{\alpha_i}^{e} = \Delta\mathbf{k}_{\alpha_i}^{e}\bar{\mathbf{U}}$, thus the covariance operation becomes

$$Cov[\mathbf{U}] = \mathbf{K}_o^{-1}\varepsilon\sum_e^{N_e}\sum_f^{N_e}\left\{\sum_{k=1}^{5}\varepsilon[\Delta\mathbf{F}_{\alpha_k}^{e}\Delta\mathbf{F}_{\alpha_k}^{fT}] + 2\sum_{k=1}^{4}\sum_{l=k+1}^{5}\varepsilon[\Delta\mathbf{F}_{\alpha_k}^{e}\Delta\mathbf{F}_{\alpha_l}^{fT}]\right\}\mathbf{K}_o^{-T} \tag{25}$$

In total, we have 5 square terms plus 10 additional pair terms for the variance and covariance between extensional, bending, extension-bending coupling, and shear. The numerical evaluation of respective terms in Eq. (25) can be performed based on the following double integration

$$\varepsilon[\Delta\mathbf{F}_{\alpha_i}\Delta\mathbf{F}_{\alpha_j}^{T}] = \int_{V_i}\int_{V_j}\varepsilon[\alpha_i(\mathbf{r}_i)\alpha_j(\mathbf{r}_j)]\mathbf{B}^T\mathbf{C}_{\alpha_i}\mathbf{B}\bar{\mathbf{U}}\bar{\mathbf{U}}^T\mathbf{B}\mathbf{C}_{\alpha_i}\mathbf{B}^T dV_j\, dV_i \tag{26}$$

where $\mathbf{B}$ can be $\mathbf{B}_{ABD}$ or $\mathbf{B}_S$ depending on the effects that are under consideration.

4.3 Functions for correlation between random parameters

The random characteristics of respective random parameters can be directly represented if we use the data practically acquired in the field or the data obtained by means of the numerical generation. However, from the viewpoint of non-statistical stochastic analysis schemes, the random characteristics are taken into account in an indirect way. One of generally accepted methods is to use correlation functions which represent the random characteristics by an expectation on the data at two distinct points having relative distance d.

In this study, we employ exponentially decaying auto- and cross-correlation functions as in Eq. (27). With theses functions, the degree of correlation is assumed to decrease as the relative distance between two distinct points increases. The function used is as follows

$$R_{\alpha\alpha}(\xi) = \sigma_{\alpha\alpha}^2 e^{-|\xi|/d}e^{-|\eta|/d} \tag{27}$$

where, the coefficient of variation of the stochastic field α is denoted as $\sigma_{\alpha\alpha}$ and the correlation distance as d. The component of relative distance vector ξ is denoted as ξ, η as shown in Fig. 1.

The cross-correlation between individual random parameters is modeled using a similar correlation function by introducing the correlation coefficient $\rho_{\alpha,\beta}$ as follows

$$R_{\alpha\beta}(\xi) = \rho_{\alpha\beta}R_{\alpha\alpha}(\xi) \ \text{ or } \ \rho_{\alpha\beta}R_{\beta\beta}(\xi) \tag{28}$$

As expected, the correlation coefficient takes a value in the unit interval, i.e., $\rho_{\alpha\beta} \in [-1, 1]$.

For the numerical integration on Eq. (26) adopting Eq. (27) or Eq. (28), we employ a Lobatto integration scheme, which is known to give more accurate results for exponential functions such as in Eqs. (27), (28).

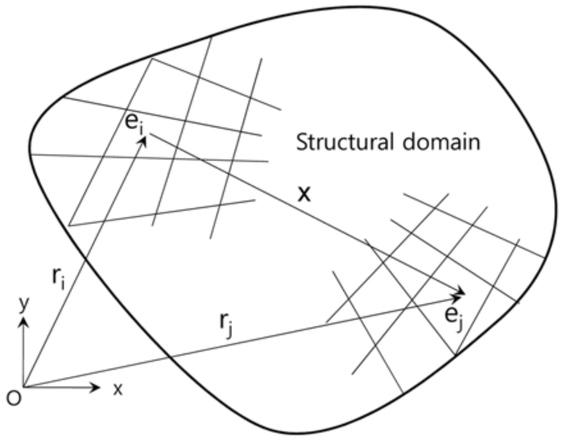


Fig. 1 Relative distance vector between two distinct points in the domain of analysis

4.4 State of correlation between random parameters

Since we have 5 random parameters, in total 10 (${}_5C_2$) correlation coefficients ρ_{ij} can be defined in between the random parameters. The number of independent correlation coefficients, however, is restricted to only 4. For instance, if we have correlation coefficient ρ_{12} between parameters 1 and 2 and ρ_{23} between 2 and 3, then the correlation ρ_{13} between 1 and 3 depends on the previous correlation coefficients. As a result, the other coefficients all depend on these 4 independent coefficients. Accordingly, there exists in total 16 cases of correlation state if we consider only the positive and negative perfect correlations for respective correlation coefficient, i.e., $\rho_{ij} = \pm 1$. Namely, for independent correlation coefficients ρ_{12}, ρ_{23}, ρ_{34}, ρ_{45}, we have 16 states as listed in Table 1.

Among the 16 correlation states, we will choose some representative ones, and the results are shown for chosen correlation states only. Obviously, if we do not consider the randomness of the out-of-plane shear moduli, then the correlation coefficients $\rho_{34} = \rho_{G_{12}G_{13}}$ and $\rho_{45} = \rho_{G_{13}G_{23}}$ become zero, and in this case the number of independent correlation coefficients and the 'Case numbers' in Table 1 reduce to 2 and 4, respectively.

Table 1 Number of cases of correlation state for 4 independent correlation coefficients for perfectly positive and negative correlations.

Case number	ρ_{12}	ρ_{23}	ρ_{34}	ρ_{45}	Case number	ρ_{12}	ρ_{23}	ρ_{34}	ρ_{45}
1	+1	+1	+1	+1	9	−1	+1	+1	+1
2	+1	+1	+1	−1	10	−1	+1	+1	−1
3	+1	+1	−1	+1	11	−1	+1	−1	+1
4	+1	+1	−1	−1	12	−1	+1	−1	−1
5	+1	−1	+1	+1	13	−1	−1	+1	+1
6	+1	−1	+1	−1	14	−1	−1	+1	−1
7	+1	−1	−1	+1	15	−1	−1	−1	+1
8	+1	−1	−1	−1	16	−1	−1	−1	−1

5. Numerical validation

5.1 Modeling of composite plate

5.1.1 Ply orientation and stacking schemes

In case of structures made of composite materials, the layering schemes affect the structural behavior resulting in characteristic responses. In the numerical verification for the proposed stochastic FE(finite element) analysis formulation, we investigate the random response of composite plates of cross-ply laminate and angle-ply laminate as well. As depicted in Fig. 2, the cross-ply laminate denotes stacking scheme having stacking angles of 0 and 90 degree for consecutive laminae, whereas the angle-ply laminate denotes stacking scheme having orientation of α and β for sequential lamina, where $\alpha, \beta \in [0, 90]$. From the viewpoint of stacking sequence with respect to the mid-surface, the composite plates can be divided into two kinds: symmetrical ones such as ($\alpha / \beta / \beta / \alpha$) and asymmetric ones such as ($\alpha / \beta / \alpha / \beta$) (Fig. 3).

5.1.2 Geometry and materials for example plate

For an example plate, a 20 × 20 (m^2) plate is taken which is exerted by a uniformly distributed load having density of 1.0 per unit area. The materials are Graphite- Epoxy (T300/934) and Glass-Epoxy. For a finite element analysis, we model the plate with 36 (6 × 6) elements exclusively. The thickness of

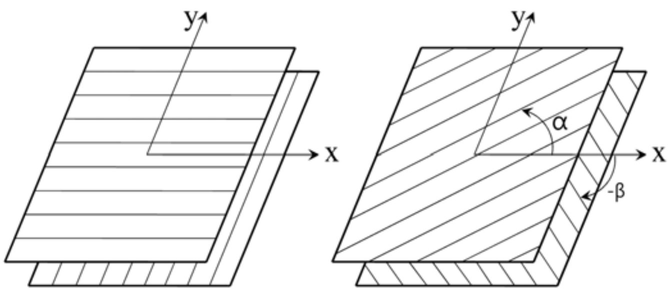


Fig. 2 Ply orientation: Schemes of cross-ply and angle-ply

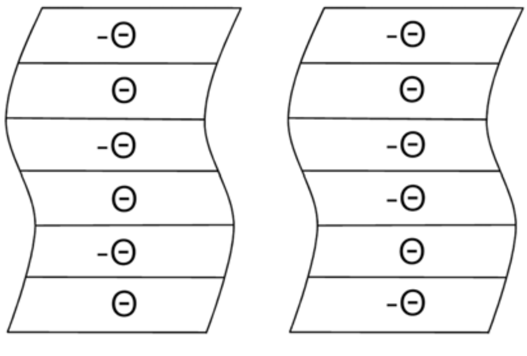


Fig. 3 Stacking schemes: symmetric and asymmetric

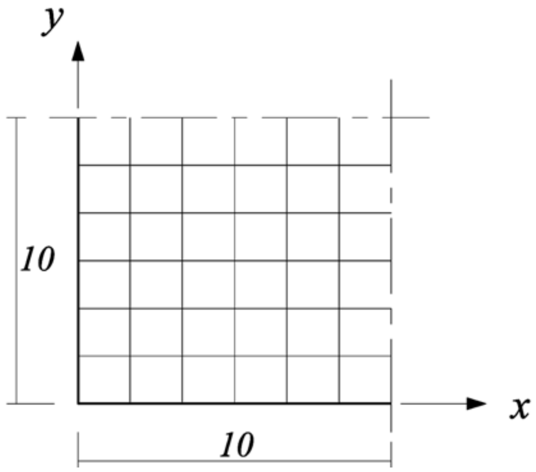


Fig. 4 Geometry of example plate

the plate is 0.1 if not mentioned otherwise. We used $(0/90)_4$ and $(45/-45)_4$ for cross-ply and angle-ply laminations, respectively.

The parameters assumed to have randomness is two elastic moduli, E_1, E_2, and three shear moduli, G_{12}, G_{13}, G_{23}. The deterministic values of the material constant for Graphite-Epoxy (Gr-Ep) are: $\overline{E}_1 =$ 131 GPa, $\overline{E}_1\ /\ \overline{E}_2 = 12.67$. The in-plane shear modulus $\overline{G}_{12} = 6.895$ GPa, and the out-of-plane shear moduli are assumed to be $\overline{G}_{23}$, $\overline{G}_{13} = 6.2055$ GPa. The Poisson's ratio ν_{12}, which is assumed as deterministic, is 0.22. The values of the material constant for Glass-Epoxy (Gl-Ep) are: $\overline{E}_1 =$ 38.612 GPa, $\overline{E}_1\ /\ \overline{E}_2 = 4.67$. The in-plane shear modulus $\overline{G}_{12}$ and out-of-plane shear moduli $\overline{G}_{23}$, $\overline{G}_{13}$ are 4.317 GPa, 3.448 GPa and 4.317 GPa, respectively. The Poisson's ratio ν_{12} is 0.26. For all individual random parameters, 0.1 of coefficient of variation is assumed. In the analysis, in order to investigate into the effect of constraints, not only a fixed but also a simply supported boundary condition is employed. The coefficient of variation (COV) of the displacement is obtained at the center of the plate.

For a numerical validation of the proposed weighted integral stochastic finite element scheme, we perform the crude Monte Carlo analysis adopting the so-called local averaging scheme, which is recognized to give a more accurate solution than the mid-point rule (Stefanou 2009). For Monte Carlo analysis, we generated 6,480 samples that comply with the spectral condition required by the correlation functions of Eqs. (27) and (28).

5.2 Numerical example analyses

5.2.1 Response variability depending on correlation states

Fig. 5 shows the variation of the COV of center displacement in terms of the 'correlation state' when the correlation distance d is fixed at 10.0. Even though the fluctuation of the two graphs look different from each other, we can clearly see that the variation patterns are similar, i.e. the tendency of increase and decrease between the two graphs. In particular, the pairs (2,4), (6,8), (10,12) and (14,16) for Graphite-Epoxy laminates, which correspond to the case where the sign of the correlation coefficient ρ_{34} in between G_{12} and G_{13} is reversed, show almost identical COVs. In fact, the situation is the same for the Glass-Epoxy laminates. This indicates that the in-plane and out-of-plane shear moduli are

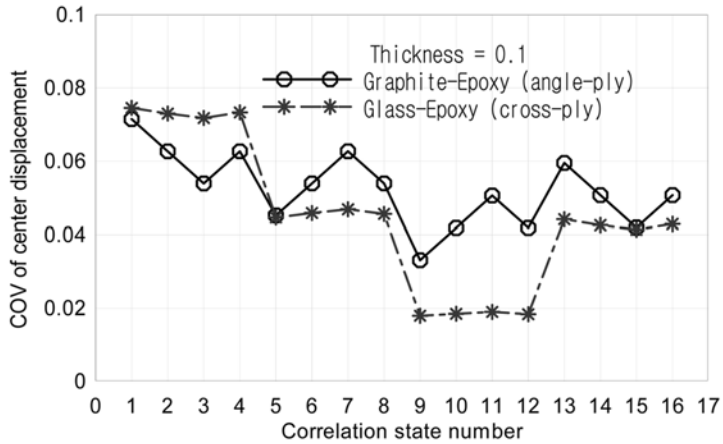


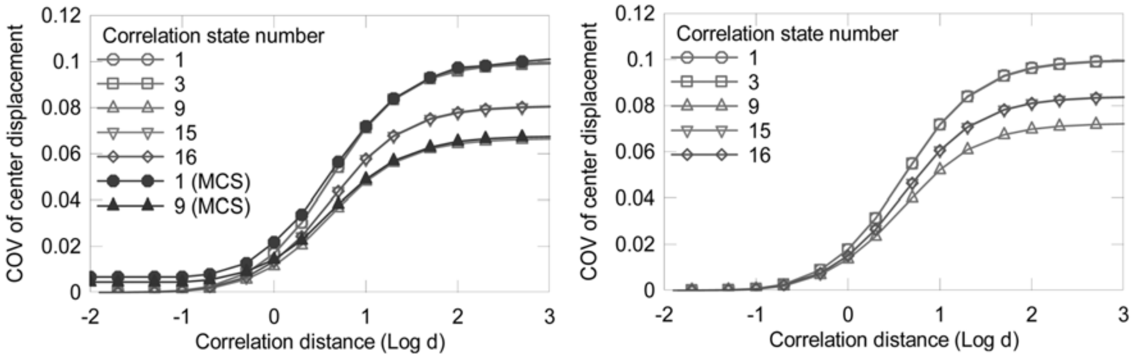
Fig. 5 COV of displacement depending on correlation cases: d=10.0, fixed support

almost independent from each other in the viewpoint of stochasticity.

The maximum and minimum response variability are obtained at the case number 1 and 9, respectively, where the sign of ρ_{12}, the correlation coefficient between E_1 and E_2, is reversed and the others are in perfect positive correlation. In the case when the correlations ρ_{23}, ρ_{34}, ρ_{45}, are mixed as perfect positive and negative or all negative, the effect of sign change in the correlation ρ_{12} is mitigated resulting in the small difference between the corresponding correlation cases (e.g. case pair (2, 10), (3,11), and so on). The same is observed for the Glass-Epoxy composites where the sign of correlation coefficient ρ_{12} remains the same for respective clustering. On the other hand, the effect of shear moduli is observed to be pronounced in the case of Graphite Epoxy composites. In passing, it should be noted that 'state 1' is an ideal state because all the material parameters are assumed to be in the same form of spatial variation over the analysis domain, such that it is far away from an actual circumstance of distribution of random material parameters.

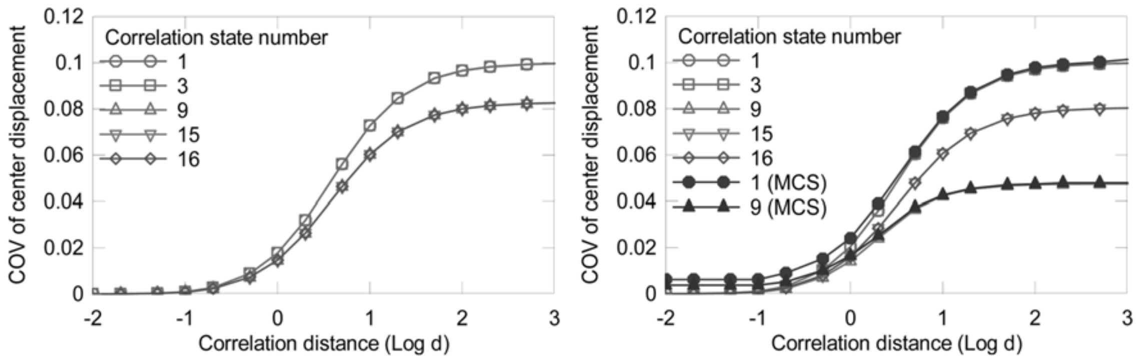
Figs. 6-8 show the response variability of composite plate for individual correlation states as the correlation distance d varies. As seen in the figures, even though the trends of COV are similar to each other for almost all the cases, the COV itself shows remarkable difference for respective correlation states. In particular, in the case of Glass-Epoxy composite, there appears decreasing COV for the correlation distance d after 10.0 for correlation state 9. As shown, the effect of correlation between shear moduli, i.e., the correlation coefficients ρ_{34}(in between correlation states 1 and 3), ρ_{45}(in between states 15 and 16), is as small as to be ignored. This is because the thickness of the plate is relatively small as 0.1. The adequacy of the proposed results is approved by MCS analyses, which are given with solid symbols in the figures. Even though we observed good agreements between the proposed scheme and the MCS for all the cases, only a few is given in Figs. 6-8 for the sake of brevity of presentation.

We need to note here that 'state 1', in fact, is an ideal state where all the random parameters take exactly the same fluctuation fashion over the analysis domain. This means that the other states are, at least, more realistic representation of the correlation state in the composite plates between the 5 random parameters under consideration. Furthermore, since we have the largest response variability for correlation state 1, we can deduce that the composite plates will show lower response variability than the plate of isotropic materials. In the case of isotropic plate the variation of the response variability shows a similar trend to that of correlation state 1 (Graham and Deodatis 1998, Noh and Park 2006).



(a) Angle-ply lamination (b) Cross-ply lamination

Fig. 6 Symmetrically stacked plates with fixed support (Gr-Ep)



(a) Angle-ply lamination (b) Cross-ply lamination

Fig. 7 Asymmetrically stacked plates with simple support (Gr-Ep)

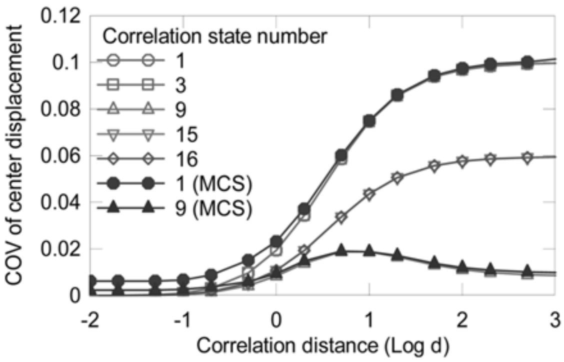


Fig. 8 Asymmetrically stacked cross-ply lamination with simple support (Gl-Ep)

This fact indicates that the composite plates show more positive behavior than the isotropic plates from the perspective of response variability.

5.2.2 Effect of thickness and randomness in the out-of-plane shear moduli

In order to investigate the effect of randomness of out-of-plane shear moduli G_{23}, G_{13} on the response variability for relatively thin plates, the thickness of the plate is considered as a variable, and two out-of-plane shear moduli are assumed to be random or deterministic.

The results for fixed support composite plate are shown in Table 2. It is clear that the COV is affected only when the out-of-plane shear moduli are assumed to be deterministic. If the modulus is random, the response COV is practically constant irrespective of the change in the plate thickness. It is also apparent that the change in COV is larger for the thick plate than that of the thin plate. In short, the deterministic assumption on the out-of-plane shear moduli underestimates the response COV, especially for the relatively thick plates. Consequently, we need to consider the out-of-plane shear moduli as random in particular when the plate is relatively thick.

Figs. 9 and 10 further show the effects of thickness especially for the case of moderately thick plates. To this end, we assume the thickness of the plate as 1.0. Contrary to Figs. 6-8, discrepancies in COV between correlation states 1 and 3, and between 15 and 16, respectively, are large and thus not possible to ignore. According to the results in Table 2, it is apparent that the effect of shear moduli is so small as to be ignorable for thin plates, even though it increases as the thickness grows. For moderate thick plates, we note that the randomness in shear moduli plays an important role in determining the uncertain behavior of laminate composites.

Table 2 Effect of the plate thickness and out-of-plane shear modulus to the COV of displacement for moderately thin laminates with thickness 0.2 and 0.02.

Ply scheme	Thickness	Correlation distance d				% increase in (a)/(b)			
		10.0		1000.0		d = 10.0		d = 1000.0	
		random	Deter.	random	Deter.	random	Deter.	random	Deter.
Angle	0.02[(b)]	7.2062	7.2056	9.9642	9.9634	0.0	0.72	0.0	0.74
	0.2[(a)]	7.2067	7.1544	9.9642	9.8900				
Cross	0.02[(b)]	7.1463	7.1459	9.9630	9.9625	0.0	0.54	0.0	0.54
	0.2[(a)]	7.1464	7.1076	9.9630	9.9092				

Note: (1) COV × 100 is given; (2) 'Deter.' signifies only the out-of plane shear modulus is deterministic (the others are random); (3) Asymmetrically stacked and for correlation state 1.

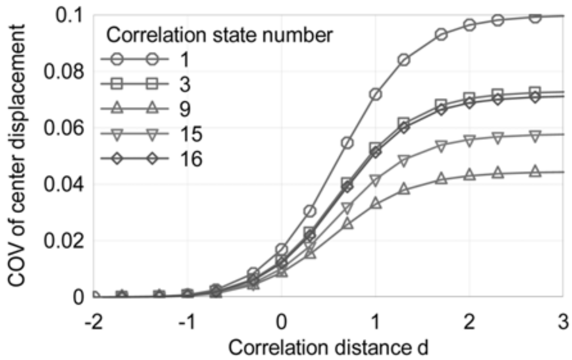


Fig. 9 Asymmetrically stacked angle-ply lamination with fixed support (Gr-Ep)

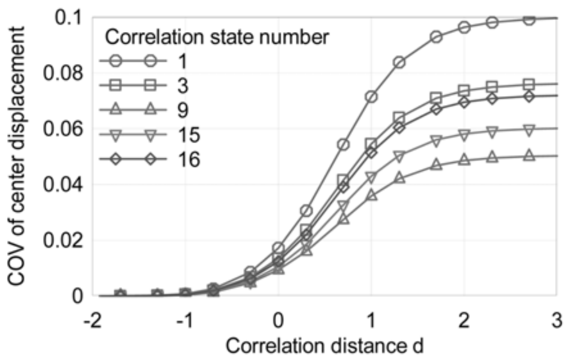


Fig. 10 Asymmetrically stacked cross-ply lamination with fixed support (Gr-Ep)

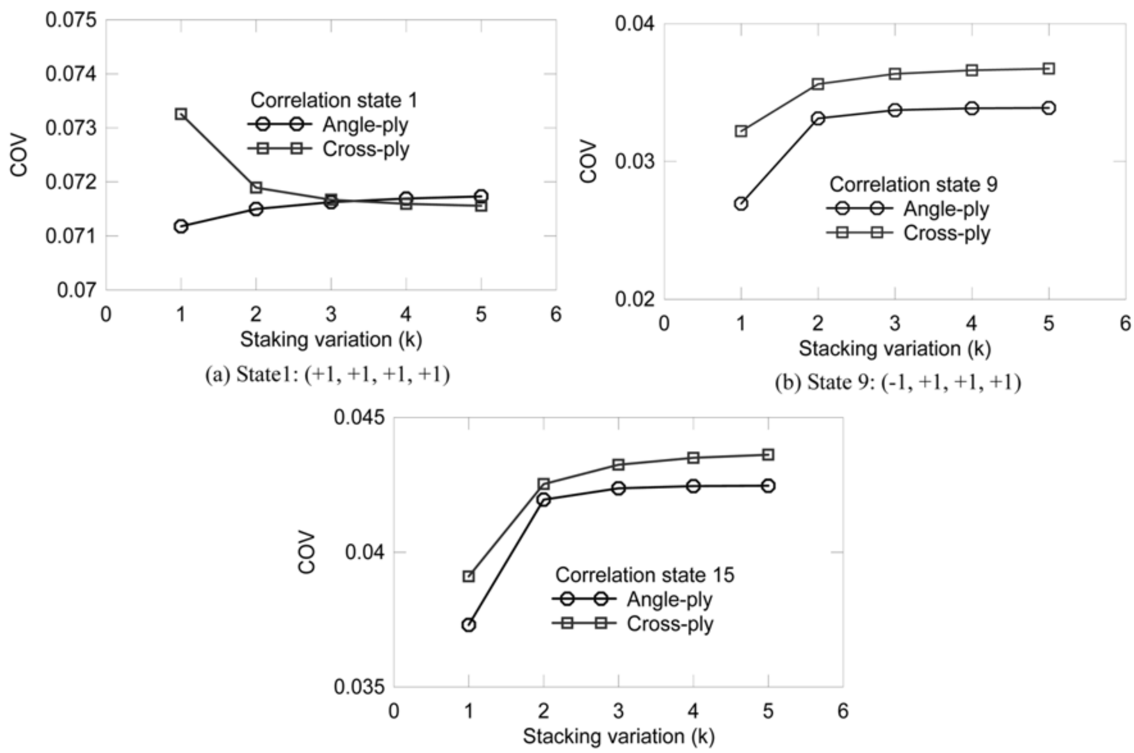
5.2.3 Investigation on the effect of the number of stacking layers

In order to investigate the effect of the number of stacking layers in the composite plates, we analyze the plate with varying number of stacking layers. In the case of cross-ply laminate, the stacking scheme of $(0/90)_{2k}$ is applied, and $(45/-45)_{2k}$ is employed for the angle-ply laminate, where the sub-index k varies from 1 to 5.

As seen in Fig. 11, though there is not any general rule in the trend of variation, there appears convergence in COV approaching specific values as the number of layers increases for the respective correlation state. In almost all cases, however, the increase of the number of layers tends to increase the COV of center displacement.

6. Conclusions

In this paper our aim was to propose a stochastic finite element scheme for an evaluation of the response variability of composite laminate plates. As a result, a formulation which takes 5 random material properties into account is suggested. The random variable is defined by the weighted integral (or the stochastic integral). Moreover the randomness of the parameters is considered in the numerical evaluation of the response variability by means of auto- or cross-correlation functions, which is integrated by adopting the Lobatto numerical integration scheme.



(a) State1: (+1, +1, +1, +1)

(b) State 9: (-1, +1, +1, +1)

(c) State 15: (-1, -1, -1, +1)

Fig. 11 Effect of number of stacking layers for Graphite Epoxy laminates (d=10.0)

From the numerical demonstrations on the Graphite Epoxy and Glass Epoxy composite laminate plates, we found that the response variability of the composite plates is, in general, less than that of the isotropic plates, which shows the advantages of the composite material over the conventional materials.

In addition, from the analysis on the plates having different thicknesses, we observed that the randomness of out-of-plane shear moduli need to be taken into account if the plate thickness is relatively large since the influence of these random parameters on the response variability is not so small as to be ignored. All the results from the proposed scheme are in good agreement with those of the MCS.

Acknowledgement

This work was supported by the Human Resources Development of the Korea Institute of Energy Technology Evaluation and Planning(KETEP) grant funded by the Korea government Ministry of Knowledge Economy (No. 20104010100520)

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