

Mass transfer in cross-flow dialyzer with internal recycle

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(Received February 26, 2013, Revised May 29, 2013, Accepted June 24, 2013)

Abstract. The internal reflux effect on dialysis through the retentate phase of a countercurrently cross-flow rectangular module is investigated. Theoretical analysis of mass transfer in cross-flow devices with or without recycling is analogous to heat transfer in cross-flow heat exchangers. In contrast to a device without reflux, considerable mass transfer is achievable if cross-flow dialyzers are operated with reflux, which provides an increase in fluid velocity, resulting in a reduction in mass-transfer resistance. It is concluded that reflux can enhance mass transfer, especially for large flow rate and feed-concentration operated under high reflux ratio.

Keywords: dialysis; cross flow; countercurrently internal recycle; rectangular module

1. Introduction

Dialysis is a rate-governed membrane process in which a microsolutes is driven at a greater rate than macrosolutes across a semipermeable membrane using the difference in chemical potential between the feed solution and receiving dialysate (Mulder 1991). Early dialysis applications were recovered NaOH from cellulose steeping liquids and sulfuric acid from a copper leaching solution (Kessler and Klein 1992). The most famous dialysis application is hemodialysis for removing metabolic waste from the blood such as urea, creatine, phosphates, and uric acid (Mulder 1991).

Mass transfer in microporous membrane modules can be analogous to heat transfer in heat exchangers (Jacob 1957). A mathematical model describing the mass transfer in a parallel-plate dialyzer with cocurrently laminar flow in both channels was developed, based on the Navier-Stokes equations and Fick's second law of diffusion (Kolev and van der Linden 1992a, 1992b). It was reported that the application of multipass (Yeh and Chang 2005), or reflux (Yeh *et al.* 1987, Yeh 2008, Yeh and Yang 2012, Yeh and Liao 2013, Goto and Gaspillo 1992, Ho *et al.* 1998), has much influence on the heat and mass transfer, and that the operation of dialysis coupled with ultrafiltration will enhance the mass-transfer rate (Yeh *et al.* 1997, 2000). The effect of external (Yeh 2009) and internal (Yeh 2011) recycles on mass transfer in double-pass parallel-flow dialyzers were investigated. Recently, several devices of ultrafiltration (Costanzo *et al.* 2005, Bourge and Tallaj 2005, Costanzo *et al.* 2007, Kazory and Ross 2008) and peritoneal dialysis (Nakayama *et al.* 2010) for severe heart failure have been introduced. It is the purpose of present study to investigate the effect of internal recycle on the mass-transfer rate in

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countercurrently cross-flow rectangular dialyzers for improved performance.

2. Theory

Fig. 1 shows a schematic diagram of a countercurrently cross-flow rectangular dialyzer with internal recycle. This system consists of two flow channels for fluids *a* and *b*, respectively, which are separated by a microporous membrane sheet through which solute is dialyzed and transferred perpendicularly to its exposed surfaces. An impermeable plate with negligible thickness is placed vertical to the upper plate and the membrane sheet, at the centerline of channel *a* (phase *a*) to divide the feed solution into sub-channels *a*₁ (operation channel) and *a*₂ (reflux channel) of equal width, $W/2$, and a pump is installed for recycling with a reflux ratio of *R*.

2.1 Governing equations

Fig. 2 shows the flow sheet for the present module of interest. The assumptions made in this analysis are: steady state, no chemical reaction, uniform velocities and uniform concentrations [$C_a(x)$ and $C_b(x)$] over the cross section of flows, constant rates of flows, zero solvent flux through the membrane, and constant mass transfer coefficients. The overall mass balance for total mass transfer rate, *M*, related to volumetric flow rates, Q_a and Q_b , and inlet ($C_{a,i}$ and $C_{b,i}$) and outlet concentrations ($C_{a,e}$ and $C_{b,e}$), is

$$M = Q_a(C_{a,i} - C_{a,e}) = Q_b(C_{b,e} - C_{b,i}) \quad (1)$$

By taking the mass balances through a differential area, $(W/2) dx$, in sub-channel *a*₁, we have

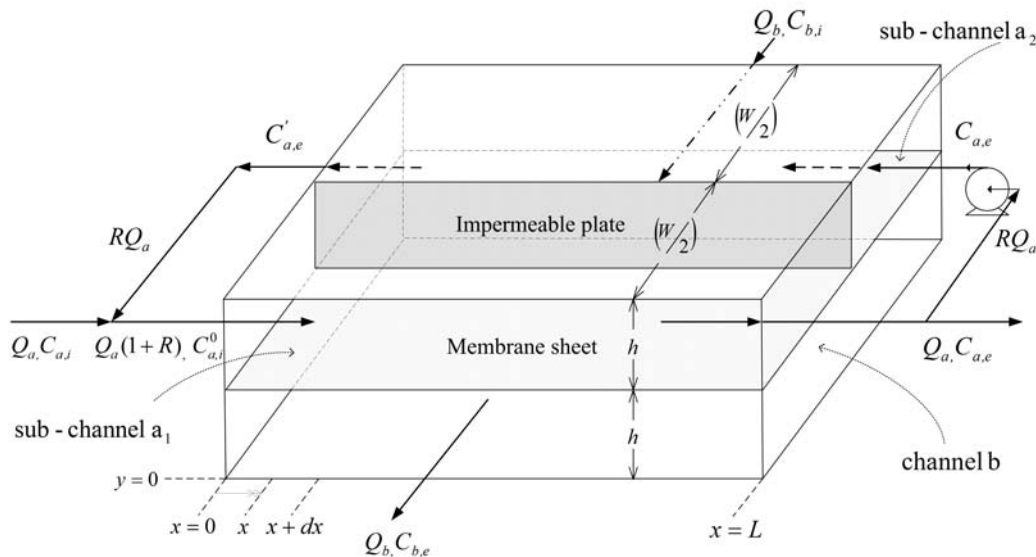


Fig. 1 Schematic diagram of a countercurrently cross-flow parallel-plate membrane dialyzer with internal reflux

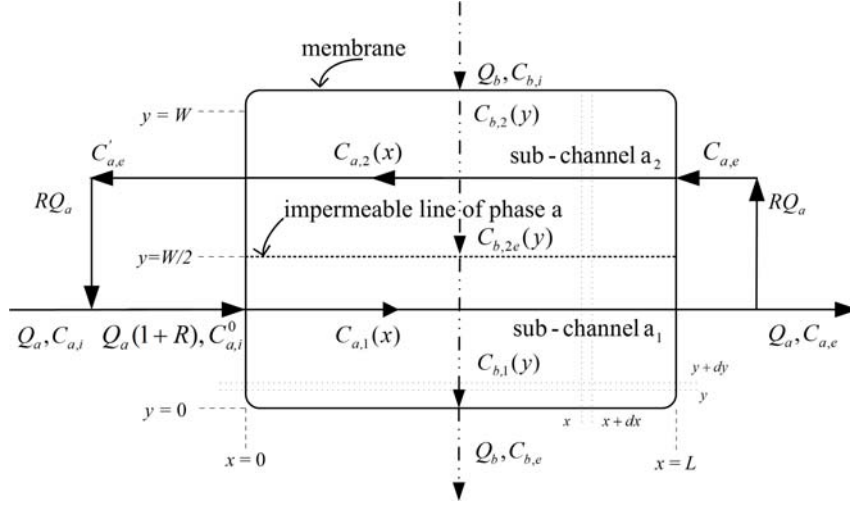


Fig. 2 Flow sheet of a countercurrently cross-flow parallel-plate membrane dialyzer with internal reflux

$$-(1+R)Q_a dC_{a,1} = K_1 \left(\frac{W}{2} \right) dx (C_{a,1} - C_{b,1}) \quad (2)$$

$$-Q_b dC_{b,1} = K_1 L (C_{a,1} - C_{b,1}) dy \quad (3)$$

Similarly, for sub-channel a_2

$$RQ_a dC_{a,2} = K_2 \left(\frac{W}{2} \right) dx (C_{a,2} - C_{b,2}) \quad (4)$$

$$-Q_b dC_{b,2} = K_2 L (C_{a,2} - C_{b,2}) dy \quad (5)$$

where K_1 and K_2 are the overall mass transfer coefficients in sub-channels a_1 and a_2 , respectively. The boundary conditions as shown in Fig. 2 for solving $C_{a,1}$, $C_{a,2}$, $C_{b,1}$, and $C_{b,2}$ from Eqs. (2)-(5) are

$$\text{at } x=0, \quad C_{a,1} = C_{a,i}^0 \quad (6)$$

$$\text{at } x=0, \quad C_{a,2} = C_{a,e}' \quad (7)$$

$$\text{at } x=L, \quad C_{a,1} = C_{a,e} \quad (8)$$

$$\text{at } x=L, \quad C_{a,2} = C_{a,e} \quad (9)$$

$$\text{at } y=0, \quad C_{b,1} = C_{b,e} \quad (10)$$

$$\text{at } y=W/2, \quad C_{b,1} = C_{b,2} = C_{b,2e} \quad (11)$$

$$\text{at } y=W, \quad C_{b,2} = C_{b,i} \quad (12)$$

Inspecting Eqs. (6)-(12) shows that the outlet concentrations ($C_{a,e}$, $C'_{a,e}$, $C_{b,2e}$ and $C_{b,e}$) as well as the mixed inlet concentration, $C_{a,i}^0$, are not specified a priori. Mathematically, more relations for the mass transfer rates, M_1 and M_2 , in sub-channels a_1 and a_2 , as well as for the mixing effect at the inlet, are needed to determine these values. With the mean concentration differences, $(\Delta C)_{1,m}$ and $(\Delta C)_{2,m}$, in regions 1 and 2, respectively, they are

$$M_1 = Q_a(1+R)(C_{a,i}^0 - C_{a,e}) = Q_b(C_{b,e} - C_{b,2e}) = K_1 L \left(\frac{W}{2} \right) (\Delta C)_{1,m} \quad (13)$$

$$M_2 = Q_a R (C_{a,e} - C'_{a,e}) = Q_b (C_{b,2e} - C_{b,i}) = K_2 L \left(\frac{W}{2} \right) (\Delta C)_{2,m} \quad (14)$$

$$R C'_{a,e} + C_{a,i} = (1+R) C_{a,i}^0 \quad (15)$$

where $C'_{a,e}$ denotes the outlet solute concentration in the reflux stream of phase a , and

$$(\Delta C)_{1,m} = \frac{(C_{a,e} - C_{b,e}) - (C_{a,i} - C_{b,i})}{\ln[(C_{a,e} - C_{b,e}) / (C_{a,i} - C_{b,i})]} \quad (16)$$

$$(\Delta C)_{2,m} = \frac{(C_{a,i} - C_{b,e}) - (C_{a,e} - C_{b,i})}{\ln[(C_{a,i} - C_{b,e}) / (C_{a,e} - C_{b,i})]} \quad (17)$$

2.2 Mass transfer rate

The outlet concentration $C_{a,e}$ in the retentate phase (phase a) can be obtained by solving Eqs. (1)-(5) and (13)-(15) coupled with the use of the boundary conditions, Eqs. (6)-(12). The analysis is a bit complex and the procedure is cumbersome and is presented in Appendix. The result is

$$C_{a,e} = \frac{C_{b,i} [(1+R)\xi_{a,e} + R\xi'_{a,e}] - C_{a,i} (1 + \xi_{a,e})}{R(\xi_{a,e} + \xi'_{a,e}) - 1} \quad (18)$$

Provided that

$$\xi'_{a,e} = \beta \xi_{a,e} - \gamma \quad (19)$$

$$\xi_{a,e} = \alpha \xi'_{a,e} - 1 \quad (20)$$

where

$$\alpha = \frac{R(Q_a/Q_b)}{1 - \exp(K_2 WL/2Q_b)} + \frac{1}{1 - \exp(K_2 WL/2Q_a R)} + \frac{R}{(K_2 WL/2Q_a)} - R \left(\frac{Q_a}{Q_b} \right) - 1 \quad (21)$$

$$\beta = \frac{(Q_a/Q_b)(1+R)}{1 - \exp(K_1 WL/2Q_b)} - \frac{1}{1 - \exp[-K_1 WL/2Q_a(1+R)]} + \frac{1+R}{(K_1 WL/2Q_a)} - \left(\frac{Q_a}{Q_b} \right) (1+R) \quad (22)$$

$$\gamma = \frac{1}{R(Q_a/Q_b)} \quad (23)$$

and that constants, $\xi_{a,e}$ and $\xi'_{a,e}$ in Eq. (18) can be solved from Eqs. (19) and (20), as

$$\xi_{a,e} = \frac{\alpha\gamma + 1}{\alpha\beta - 1} \quad (24)$$

$$\xi'_{a,e} = \frac{\beta + \gamma}{\alpha\beta - 1} \quad (25)$$

Once the outlet concentration in the retentate phase is obtained from Eq. (18), the total mass transfer rate will be determined using Eq. (1).

2.3 Mass transfer coefficients

The overall mass transfer coefficients K_1 and K_2 for dialysis may be defined as (Poter 1990)

$$\frac{1}{K_i} = \frac{1}{k_{a,i}} + \frac{1}{k_m} + \frac{1}{k_b}, \quad i = 1, 2 \quad (26)$$

in which the mass transfer coefficients in the respective phase adjust to the membrane side, $k_{a,1}$, $k_{a,2}$, and k_b , as well as in the membrane, k_m , are given as (Poter 1990)

$$k_{a,1} = 0.816 \left[\frac{6Q_a(1+R)D^2}{L(W/2)h^2} \right]^{\frac{1}{3}} \quad (27)$$

$$k_{a,2} = 0.816 \left[\frac{6Q_aRD^2}{L(W/2)h^2} \right]^{\frac{1}{3}} \quad (28)$$

$$k_b = 0.816 \left[\frac{6Q_bD^2}{LWh^2} \right]^{\frac{1}{3}} \quad (29)$$

$$k_m = \frac{D\varepsilon}{\tau t} \quad (30)$$

In the above equations, D is the mass diffusivity of the solute; ε , τ , and t are, respectively, the porosity of the membrane, the pore tortuosity of the membrane, and the membrane sheet thickness; and h denotes the height of the flow channels.

3. Improvement in performance

The improvement in performance I in the present device is based on that in the single-pass module without recycling where M_0 may be defined as

$$I = \frac{M - M_0}{M_0} \quad (31)$$

3.1 Module without reflux

Fig. 3 shows the flow sheet of the single-pass module without internal reflux, in which the retentate phase only has one flow channel (channel *a*). The expression for mass transfer rate in a cross-flow rectangular membrane dialyzer without reflux was derived as (Yeh and Chen 2000)

$$M_0 = \frac{Q_a(C_{a,i} - C_{b,i})}{1/[1 - \exp(-K_0WL/Q_a)] + (Q_a/Q_b)/[1 - \exp(-K_0WL/Q_b)] + Q_a/K_0WL} \quad (32)$$

where K_0 is the overall mass transfer coefficient of a single-pass device.

3.2 Numerical example

For the purpose of illustration, let us employ some numerical values for dialyzing urea from an aqueous solution using a membrane sheet ($L = W = 0.6$ m) made of microporous polypropylene ($\varepsilon = 0.7$, $t = 1.78 \times 10^{-4}$ m, $\tau = 2.6$) as the permeable barrier. Further, $h = 2 \times 10^{-2}$ m, $D = 1.378 \times 10^{-9}$ m²/s (Geankoplis 1983), and $C_{b,i} = 0$.

Using the above numerical values, the dialysis rate was calculated using the appropriate equations, while the overall mass transfer coefficient of a single-pass device without reflux should be modified from Eq. (26) as

$$\frac{1}{K_0} = \frac{1}{k_a} + \frac{1}{k_m} + \frac{1}{k_b} \quad (33)$$

where

$$k_a = 0.816 \left(\frac{6Q_a D^2}{LWh^2} \right)^{\frac{1}{3}} \quad (34)$$

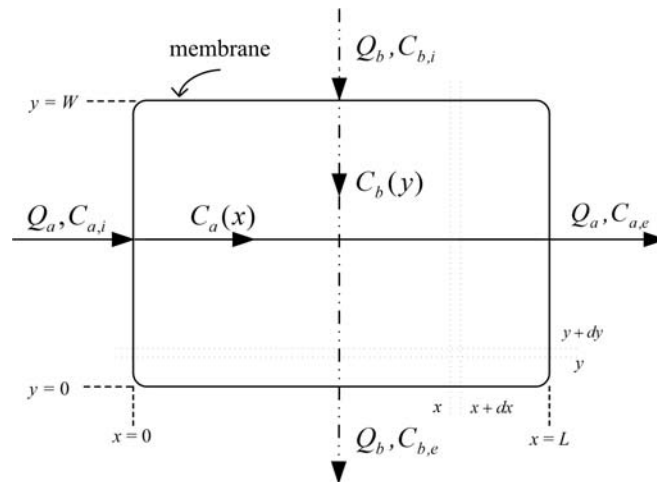


Fig. 3 Flow sheet of cross-flow parallel-plate membrane dialyzer without internal reflux

Table 1 Dialysis rates in countercurrently cross-flow device with internal reflux with $C_{a,i} = 1 \text{ kgmol/m}^3$ and $C_{b,i} = 0$

$Q_a \times 10^7$ (m ³ /s)	$Q_b \times 10^7$ (m ³ /s)	$M_0 \times 10^8$ (kgmol/s)	$R = 1$		$R = 3$		$R = 5$		$R = 7$		$R = 9$	
			$M \times 10^8$ (kgmol/s)	I (%)	$M \times 10^8$ (kgmol/s)	I (%)	$M \times 10^8$ (kgmol/s)	I (%)	$M \times 10^8$ (kgmol/s)	I (%)	$M \times 10^8$ (kgmol/s)	I (%)
1	1	2.1754	2.1662	-0.42	2.3390	7.52	2.4122	10.89	2.4575	12.97	2.4897	14.45
1	5	2.8809	2.8798	-0.04	3.1890	10.69	3.3270	15.49	3.4145	18.52	3.4776	20.71
1	10	3.1090	3.1137	0.15	3.4748	11.76	3.6390	17.05	3.7438	20.42	3.8197	22.86
5	1	2.8809	3.0373	5.43	3.2229	11.87	3.3019	14.61	3.3507	16.31	3.3853	17.51
5	5	4.2836	4.6504	8.56	5.1016	19.10	5.3050	23.85	5.4339	26.86	5.5268	29.02
5	10	4.8136	5.2853	9.80	5.8737	22.02	6.1450	27.66	6.3186	31.27	6.4447	33.88
10	1	3.1090	3.2852	5.67	3.4536	11.08	3.5250	13.38	3.5688	14.79	3.5998	15.79
10	5	4.8136	5.2605	9.28	5.7095	18.61	5.9101	22.78	6.0364	25.40	6.1271	27.29
10	10	5.4941	6.0869	10.79	6.6945	21.85	6.9720	26.90	7.1486	30.11	7.2762	32.44

Table 2 Dialysis rates in countercurrently cross-flow device with internal reflux with $C_{a,i} = 5 \text{ kgmol/m}^3$ and $C_{b,i} = 0$

$Q_a \times 10^7$ (m ³ /s)	$Q_b \times 10^7$ (m ³ /s)	$M_0 \times 10^8$ (kgmol/s)	$R = 1$		$R = 3$		$R = 5$		$R = 7$		$R = 9$	
			$M \times 10^8$ (kgmol/s)	I (%)	$M \times 10^8$ (kgmol/s)	I (%)	$M \times 10^8$ (kgmol/s)	I (%)	$M \times 10^8$ (kgmol/s)	I (%)	$M \times 10^8$ (kgmol/s)	I (%)
1	1	10.8770	10.8308	-0.42	11.6952	7.52	12.0610	10.89	12.2873	12.97	12.4483	14.45
1	5	14.4045	14.3989	-0.04	15.9448	10.69	16.6352	15.49	17.0726	18.52	17.3881	20.71
1	10	15.5451	15.5687	0.15	17.3739	11.76	18.1948	17.05	18.7188	20.42	19.0986	22.86
5	1	14.4045	15.1864	5.43	16.1143	11.87	16.5096	14.61	16.7536	16.31	16.9265	17.51
5	5	21.4178	23.2518	8.56	25.5079	19.10	26.5252	23.85	27.1697	26.86	27.6341	29.02
5	10	24.0681	26.4264	9.80	29.3685	22.02	30.7248	27.66	31.5932	31.27	32.2233	33.88
10	1	15.5451	16.4259	5.67	17.2679	11.08	17.6248	13.38	17.8442	14.79	17.9992	15.79
10	5	24.0681	26.3027	9.28	28.5473	18.61	29.5504	22.78	30.1821	25.40	30.6354	27.29
10	10	27.4704	30.4344	10.79	33.4724	21.85	34.8600	26.90	35.7431	30.11	36.3811	32.44

3.3 Results and discussion

The total mass transfer rates, M , for the devices with internal reflux were calculated from Eq. (1) with the use of Eq. (18) for outlet concentration and Eq. (32) for the device without reflux, M_0 . The improvements in performance based on the device without reflux were then predicted from Eq. (31). All results are listed in Tables 1 and 2. It is seen in these tables that all mass transfer rates increase when flow rate Q_a or feed concentration $C_{a,i}$ increases. Considerable improvement in performance, I , was obtained in the reflux devices, and the improvement increases with the reflux ratio R , as well as with the flow rate and feed concentration. In the case of $Q_a = Q_b = 1 \times 10^{-6} \text{ m}^3/\text{s}$, $C_{a,i} = 5 \text{ kgmol/m}^3$ and $R = 9$, the improvement of the total mass transfer rate in the device reaches

32.44%, as shown in Table 2.

Since the recycle operation is applied to the retentate phase (phase *a*) in the present interest, if the flow rate in phase *a* is not large enough, the effect of increase in mass-transfer coefficients produced under small reflux ratio operation cannot compensate for the effect of decreases in driving force of mass transfer due to mixing at the inlet, as shown by the minus sign of I ($M < M_0$) in the tables. In this case, the device without recycles is preferably employed, instead of the device with recycle. However, considerable improvement I is readily achievable under larger reflux-ratio operation. I increases as Q_a or Q_b or R increases.

4. Conclusions

Membrane dialysis performances through cross-flow rectangular modules with and without internal reflux were analyzed and compared. In the theoretical analysis of the present mass exchanger, equations for solute concentration distributions in the retentate and dialysate phases were derived based on the mass balances with the assumptions of uniform concentrations and fluid velocities over flow channel cross sections. The outlet concentrations were obtained by simultaneously solving the governing equations using appropriate boundary conditions. Once the outlet concentration for the reflux device was obtained from the derived equations, the overall mass transfer rates for the devices with and without internal reflux, M and M_0 , were calculated. A numerical example for dialyzing urea from an aqueous solution was given. The results show that considerable improvement in performance is obtainable if the cross-flow rectangular dialyzer is operated with internal reflux. The improvement increases when the reflux ratio, flow rate, and feed concentration increase. However, the increase in dialysis performance using reflux only exists for the case in which the increase in mass transfer coefficient can compensate for the reduction in solute concentration difference (mass transfer driving force), i.e., the operating conditions used in the present study. Therefore, the operating conditions should be suitably controlled for improved performance. Though considerable improvement in mass transfer can be achieved by the internal recycle operation, the hydraulic-dissipated energy by the internal recycle operation due to the friction loss of fluid flow should be taken into consideration. Nevertheless, it was reported that the hydraulic-dissipated powers in the recycled device are still too small to be concerned about (Yeh 2006).

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Appendix A. Mean concentration differences

Define the following average concentrations and dimensionless groups

$$C_{a,1m} = \frac{1}{L} \int_0^L C_{a,1} dx \quad (A1)$$

$$C_{a,2m} = \frac{1}{L} \int_0^L C_{a,2} dx \quad (A2)$$

$$C_{b,1m} = \frac{2}{W} \int_0^{W/2} C_{b,1} dy \quad (A3)$$

$$C_{b,2m} = \frac{2}{W} \int_{W/2}^W C_{b,2} dy \quad (A4)$$

$$\ell = \frac{Q_a}{Q_b} \quad (A5)$$

$$n_1 = \frac{K_1 WL}{Q_a} \quad (A6)$$

$$n_2 = \frac{K_2 WL}{Q_a} \quad (A7)$$

$$-\xi_{a,e} = \frac{C_{a,i}^0 - C_{a,e}}{C_{a,i}^0 - C_{b,i}} \quad (A8)$$

$$-\xi'_{a,e} = \frac{C_{a,e} - C'_{a,e}}{C_{a,i}^0 - C_{b,i}} \quad (A9)$$

Accordingly, with the use of Eqs. (13)-(15)

$$\begin{aligned} \frac{C_{b,e} - C_{b,2e}}{C_{a,i}^0 - C_{b,i}} &= \left[\left(\frac{Q_a}{Q_b} \right) (1 + R) \right] \frac{C_{a,i}^0 - C_{a,e}}{C_{a,i}^0 - C_{b,i}} \\ &= -\ell(1 + R)\xi_{a,e} \end{aligned} \quad (A10)$$

$$\begin{aligned} \frac{C_{b,2e} - C_{b,i}}{C_{a,i}^0 - C_{b,i}} &= \left(\frac{Q_a}{Q_b} \right) \left[\frac{R(C_{a,e} - C'_{a,e})}{C_{a,i}^0 - C_{b,i}} \right] \\ &= -\ell R \xi'_{a,e} \end{aligned} \quad (A11)$$

$$\frac{C'_{a,e} - C_{b,2e}}{C_{a,i}^0 - C_{b,i}} = \xi_{a,e} + \xi'_{a,e} + \ell(1 + R)\xi_{a,e} + 1 \quad (A12)$$

Integrating Eqs. (2) and (4) from $y = 0$ to $y = W/2$ and from $y = W/2$ to $y = W$, respectively, one obtains

$$-(1 + R)Q_a dC_{a,1} = K_1 \frac{W}{2} (C_{a,1} - C_{b,1m}) dx \quad (A13)$$

$$RQ_a dC_{a,2} = K_2 \frac{W}{2} (C_{a,2} - C_{b,2m}) dx \quad (A14)$$

Similarly, integrating Eqs. (3) and (5) from $x = 0$ to L , we have

$$-Q_b dC_{b,1} = K_1 L (C_{a,1m} - C_{b,1}) dy \quad (A15)$$

$$-Q_b dC_{b,2} = K_2 L (C_{a,2m} - C_{b,2}) dy \quad (A16)$$

Integration of Eqs. (A13) and (A14) from $x = 0$ to L with the use of Eqs. (6)-(9) and (A5)-(A7) and rearrangement yields, respectively

$$C_{b,1m} = \frac{C_{a,e} - C_{a,i}^0}{1 - \exp[-n_1/2(1+R)]} + C_{a,i}^0 \quad (A17)$$

$$C_{b,2m} = \frac{C_{a,e} - C_{a,e}'}{1 - \exp[n_2/2R]} + C_{a,e}' \quad (A18)$$

Integrating Eqs. (A15) and (A16) from $y = 0$ to $y = W/2$ and from $y = W/2$ to $y = W$, respectively, with the use of Eqs.(10)-(12) and (A5)-(A7), one obtains

$$C_{a,1m} = \frac{C_{b,2e} - C_{b,e}}{1 - \exp[n_1\ell/2]} + C_{b,e} \quad (A19)$$

$$C_{a,2m} = \frac{C_{b,i} - C_{b,2e}}{1 - \exp[n_2\ell/2]} + C_{b,2e} \quad (A20)$$

A.1 Mean concentration difference in region 1

Integration of Eq. (A13) from $x = 0$ to x results in

$$C_{a,1} - C_{b,1m} = (C_{a,i}^0 - C_{b,1m}) \exp\left[\frac{-n_1 x}{(1+R)2L}\right] \quad (A21)$$

Integration of Eq. (A21) from $x=0$ to $x=L$ and rearrangement results in

$$C_{a,1m} - C_{b,1m} = \left[\frac{2(1+R)}{n_1}\right] (C_{a,i}^0 - C_{b,1m}) \times \left\{1 - \exp\left[\frac{-n_1}{2(1+R)}\right]\right\} \quad (A22)$$

Combination of Eqs. (A17) and (A22) to eliminate $(C_{a,i}^0 - C_{b,1m}) \times \{1 - \exp[-n_1/2(1+R)]\}$ gives

$$C_{a,1m} - C_{b,1m} = \left[\frac{2(1+R)}{n_1}\right] (C_{a,i}^0 - C_{a,e}) \quad (A23)$$

Consequently, the mean concentration difference in region 1 defined in Eq. (13) may be determined with the use of Eq. (A23) as

$$(\Delta C)_{1,m} = C_{a,1m} - C_{b,1m} \quad (\text{A24})$$

A.2 Mean concentration difference in region 2

Integration of Eq. (A14) from $x = 0$ to x results in

$$C_{a,2} - C_{b,2m} = (C'_{a,i} - C_{b,2m}) \exp \left[\frac{n_2 x}{2RL} \right] \quad (\text{A25})$$

Integration of Eq. (A25) from $x = 0$ to $x = L$, one has

$$C_{a,2m} - C_{b,2m} = \left[\frac{2R}{n_2} \right] (C'_{a,e} - C_{b,2m}) \times \left\{ \exp \left[\frac{n_2}{2R} \right] - 1 \right\} \quad (\text{A26})$$

Substitution of Eq. (A18) into Eq. (A26) to eliminate $(C_{a,e} - C_{b,2m}) \times \{\exp[n_2/2R] - 1\}$ results in

$$C_{a,2m} - C_{b,2m} = \left[\frac{2R}{n_2} \right] (C_{a,e} - C'_{a,e}) \quad (\text{A27})$$

With the use of Eq. (A27), the mean concentration difference in region 2 defined in Eq. (14) is

$$(\Delta C)_{2,m} = C_{a,2m} - C_{b,2m} \quad (\text{A28})$$

A.3 Outlet concentrations

Substitution of Eqs. (A17) and (A19) into Eq. (A23) to eliminate $C_{a,1m}$ and $C_{b,1m}$ coupled with the use of Eqs. (A8)-(A12) yields Eq. (19). By substituting Eqs. (A18) and (A20) into Eq. (A27) to eliminate $C_{a,2m}$ and $C_{b,2m}$ coupled with the use of Eqs. (A8)-(A12), we have Eq. (20)

From Eqs. (A8) and (A9), one obtains

$$C_{a,i}^0 = \frac{C_{b,i} \xi_{a,e} + C_{a,e}}{1 + \xi_{a,e}} \quad (\text{A29})$$

$$C'_{a,i} = C_{a,e} + \left(\frac{C_{b,i} \xi_{a,e} + C_{a,e}}{1 + \xi_{a,e}} - C_{b,i} \right) \xi'_{a,e} \quad (\text{A30})$$

Substitution of Eqs. (A29) and (A30) into Eq. (15) results in Eq. (18) for the outlet concentration

Nomenclature

Define the following average concentrations and dimensionless groups

C_a, C_b	bulk solute concentration in phase a, phase b (kgmol/m ³)
$C_{a,1}, C_{a,2}$	C_a in sub-channel a ₁ , sub-channel a ₂ (kgmol/m ³)
$C_{a,i}^0$	mixed inlet concentration (kgmol/m ³)
$C'_{a,e}$	C_a in the outlet reflux stream (kgmol/m ³)
$(\Delta C)_m$	mean concentration difference between phases a and b (kgmol/m ³)
D	solute diffusivity (m ² /s)
h	height of flow channel, $h = h_a = h_b$ (m)
I	improvement of separation defined by Eq. (29)
K_0	overall mass transfer coefficient in the single-pass device without internal reflux (m/s)
k_a, k_b, k_m	mass transfer coefficient in phase a, phase b, membrane (m/s)
L	effective length of a dialyzer (m)
M	total mass transfer rate (kgmol/s)
M_0	total mass transfer rate in the single-pass device without internal reflux (kgmol/s)
Q_a, Q_b	volume flow rate in retentate phase, dialysate phase (m ³ /s)
R	reflux ratio
t	thickness of membrane sheet (m)
W	membrane width (m)
x	axis along the flow direction of retentate phase (m)
y	rectangular coordinate dialysate phase (m)

Greek Symbols

ε	porosity of membrane
$\xi_{a,e}, \xi'_{a,e}$	dimensionless outlet concentrations
τ	pore tortuosity of membrane

Subscript

1, 2	sub-channel a ₁ , sub-channel a ₂
e	at the outlet
i	at the inlet