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Anisotropic, non-uniform misfit strain in a thin film bonded on a plate substrate

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Abstract. Current methodologies used for the inference of thin film stresses through curvature measurements are strictly restricted to stress and curvature states which are assumed to remain uniform over the entire film/substrate system. These methodologies have recently been extended to non-uniform stress and curvature states for the thin film subject to non-uniform, isotropic misfit strains. In this paper we study the same thin film/substrate system but subject to non-uniform, anisotropic misfit strains. The film stresses and system curvatures are both obtained in terms of the non-uniform, anisotropic misfit strains. For arbitrarily non-uniform, anisotropic misfit strains, it is shown that a direct relation between film stresses and system curvatures cannot be established. However, such a relation exists for uniform or linear anisotropic misfit strains, or for the average film stresses and average system curvatures when the anisotropic misfit strains are arbitrarily non-uniform.

Keywords: anisotropic film misfit strains and stresses; non-uniform film stresses and system curvatures; stress-curvature relations; non-local effects; interfacial shear.

1. Introduction

Stoney (1909) used a plate system composed of a thin film, of thickness h_f, deposited on a relatively thick substrate, of thickness h_s, and derived a simple relation between the curvature, κ, of the system and the stress, $\sigma^{(f)}$, of the film as follows:

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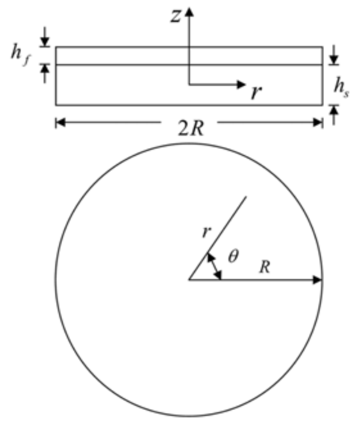

Fig. 1 A schematic diagram of the thin film/substrate system, showing the cylindrical coordinates (r, θ, z)

$$\sigma^{(f)} = \frac{E_s h_s^2 \kappa}{6 h_f (1 - v_s)} \tag{1}$$

In the above the subscripts "f" and "s" denote the thin film and substrate, respectively, and E and v are the Young's modulus and Poisson's ratio. Eq. (1) is called the Stoney formula, and it has been extensively used in the literature to infer film stress changes from experimental measurement of system curvature changes (Freund and Suresh 2004).

The Stoney formula was derived for an isotropic "thin" solid film of uniform thickness deposited on a much "thicker" plate substrate based on a number of assumptions. The assumptions include the following: (1) Both the film thickness h_f and the substrate thickness h_s are uniform and $h_f << h_s << R$, where R represents the characteristic length in the lateral direction (e.g. system radius R shown in Fig. 1); (2) The strains and rotations of the plate system are infinitesimal; (3) Both the film and substrate are homogeneous, isotropic, and linearly elastic; (4) The film stress states are in-plane isotropic or equi-biaxial (two equal stress components in any two, mutually orthogonal in-plane directions) while the out-of-plane direct stress and all shear stresses vanish; (5) The system's curvature components are equi-biaxial (two equal direct curvatures) while the twist curvature vanishes in all directions; and (6) All surviving stress and curvature components are spatially constant over the plate system's surface, a situation which is often violated in practice.

The assumption of equi-biaxial ($\kappa_{xx} = \kappa_{yy} = \kappa$, $\kappa_{xy} = \kappa_{yx} = 0$) and spatially constant curvature (κ independent of position) is equivalent to assuming that the plate system would deform spherically under the action of the film stresses. If this assumption were to be true, a rigorous application of the Stoney formula would indeed furnish a single film stress value. This value represents the common magnitude of each of the two direct stresses in any two, mutually orthogonal directions (i.e. $\sigma_{xx} = \sigma_{yy} = \sigma^{(f)}$, $\sigma_{xy} = \sigma_{yx} = 0$, $\sigma^{(f)}$ independent of position). This is the uniform stress for the entire film and it is derived from measurement of a single uniform curvature value which fully characterizes the system provided the deformation is indeed spherical.

Despite the explicitly stated assumptions of spatial stress and curvature uniformity, the Stoney formula is often, arbitrarily, applied to cases of practical interest where these assumptions are violated. This is typically done by applying the Stoney formula pointwise and thus extracting a local value of stress from a local measurement of the curvature of the system. This approach of inferring film stresses clearly violates the uniformity assumptions of the analysis and, as such, its accuracy as an approximation is expected to deteriorate as the levels of curvature non-uniformity become more severe.

Following the initial formulation by Stoney, a number of extensions have been derived by various researchers who have relaxed some of the other assumptions (other than the assumption of uniformity) made by his analysis. Such extensions of the initial formulation include relaxation of the assumption of equi-biaxiality as well as the assumption of small deformations/deflections. A biaxial form of Stoney formula, appropriate for anisotropic film stresses, including different stress values at two different directions and non-zero, in-plane shear stresses, was derived by relaxing the assumption of curvature equi-biaxiality (Freund and Suresh 2004). Related analyses treating discontinuous films in the form of bare periodic lines (Wikstrom *et al.* 1999a) or composite films with periodic line structures (e.g. bare or encapsulated periodic lines) have also been derived (Shen *et al.* 1996, Wikstrom *et al.* 1999b, Park and Suresh 2000). These latter analyses have also removed the assumption of equi-biaxiality and have allowed the existence of three independent curvature and stress components in the form of two, non-equal, direct components and one shear or twist component. However, the uniformity assumption of all of these quantities over the entire plate system was retained. In addition to the above, single, multiple and graded films and substrates have been treated in various "large" deformation analyses (Masters and Salamon 1993, Salamon and Masters 1995 Finot *et al.* 1997, Freund 2000). These analyses have removed both the restrictions of an equi-biaxial curvature state as well as the assumption of infinitesimal deformations. They have allowed for the prediction of kinematically nonlinear behavior and bifurcations in curvature states. These bifurcations are transformations from an initially equi-biaxial to a subsequently biaxial curvature state that may be induced by an increase in film stresses beyond a critical level. This critical level is intimately related to the system's aspect ratio, i.e., the ratio of in-plane to thickness dimension and the elastic stiffness. These analyses also retain the assumption of spatial curvature and stress uniformity across the system. However, they allow for deformations to evolve from an initially spherical shape to an energetically favored shape (e.g. ellipsoidal, cylindrical or saddle shapes) which features three different, still spatially constant, curvature components (Lee *et al.* 2001).

None of the above-discussed extensions of the Stoney methodology have relaxed the most restrictive of Stoney's original assumption of spatial uniformity which does not allow either film stress or curvature components to vary across the plate surface. This crucial assumption is often violated in practice since film stresses and the associated system curvatures are non-uniformly distributed over the plate area. Huang and Rosakis (2005) and Huang *et al.* (2005) have recently made progress to remove the two restrictive assumptions of the Stoney analysis relating to spatial uniformity and equi-biaxiality. They have studied the cases of thin film/substrate systems subject to non-uniform but axisymmetric temperature distribution $T(r)$ and misfit strain $\varepsilon_m(r)$, respectively. Their results show that the relations between film stresses and system curvatures feature not only a "local part which involves a direct dependence of stresses on curvatures at the same point, but also a "non-local part which reflects of the effect of curvatures at other points on the location of scrutiny. The "non-local effect comes into play in the axisymmetric analysis via the average curvature in the thin film. The "non-local" analysis has been extended to general non-uniform temperature (Huang and Rosakis 2007) and misfit strains (Ngo *et al.* 2006), thin film with non-uniform thickness (Ngo *et al.* 2007) or different radius from the substrate radius (Feng *et al.* 2006). The X-ray diffraction and coherent gradient sensing experiments have verified the non-local analysis (Brown *et al.* 2006, 2007).

The main purpose of the present paper is to extend the nonl-local analysis for the general case of a thin film/substrate system subject to arbitrary anisotropic misfit strain distribution $\varepsilon_{ij}^m(r,\theta)$. Our goal is to relate film stresses and system curvatures to the misfit strain distribution, and explore a relation between the film stresses and the system curvatures for general anisotropic misfit strain distributions.

2. Governing equations

A thin film of radius R and thickness h_f is deposited on a substrate of the same radius and thickness h_s, and $h_f << h_s << R$. The Young's modulus and Poisson's ratio of the film and substrate are denoted by E_f, ν_f, E_s and ν_s, respectively. The thin film is subject to arbitrary anisotropic and non-uniform misfit strains $\varepsilon_{ij}^m(r,\theta)$ in the film plane, where r and θ are polar coordinates (Fig. 1). For convenience we use $\varepsilon_\Sigma^m = \frac{1}{2}(\varepsilon_{rr}^m + \varepsilon_{\theta\theta}^m) = \frac{1}{2}(\varepsilon_{xx}^m + \varepsilon_{yy}^m)$, $\varepsilon_\Delta^m = \frac{1}{2}(\varepsilon_{rr}^m - \varepsilon_{\theta\theta}^m) = \frac{1}{2}(\varepsilon_{xx}^m - \varepsilon_{yy}^m)\cos 2\theta + \varepsilon_{xy}^m \sin 2\theta$, and $\gamma^m = 2\varepsilon_{r\theta}^m = 2\varepsilon_{xy}^m \cos 2\theta - (\varepsilon_{xx}^m - \varepsilon_{yy}^m)\sin 2\theta$, where x and y are the Cartesian coordinates. For uniform misfit strains ε_{xx}^m, ε_{yy}^m, and ε_{xy}^m = constants in the Cartesian coordinates (Freund and Suresh 2004), ε_Σ^m is also uniform, but ε_Δ^m and γ^m become linear combinations of $\cos 2\theta$ and $\sin 2\theta$.

The thin film is modeled as a membrane that has no resistance against bending due to its small thickness $h_f << h_s$. Let $u_r^{(f)}$ and $u_\theta^{(f)}$ denote the displacements in the radial (r) and circumferential (θ) directions. The strains in the thin film are $\varepsilon_{rr} = \frac{\partial u_r^{(f)}}{\partial r}$, $\varepsilon_{\theta\theta} = \frac{u_r^{(f)}}{r} + \frac{1}{r}\frac{\partial u_\theta^{(f)}}{\partial \theta}$, and $\gamma_{r\theta} = \frac{1}{r}\frac{\partial u_r^{(f)}}{\partial \theta} + \frac{\partial u_\theta^{(f)}}{\partial r} - \frac{u_\theta^{(f)}}{r}$.

The stresses in the thin film can be obtained from the linear elastic constitutive model as

$$
\begin{aligned}
\sigma_{rr} + \sigma_{\theta\theta} &= \frac{E_f}{1-\nu_f}\left(\frac{\partial u_r^{(f)}}{\partial r} + \frac{u_r^{(f)}}{r} + \frac{1}{r}\frac{\partial u_\theta^{(f)}}{\partial \theta} - 2\varepsilon_\Sigma^m\right) \\
\sigma_{rr} - \sigma_{\theta\theta} &= \frac{E_f}{1+\nu_f}\left(\frac{\partial u_r^{(f)}}{\partial r} - \frac{u_r^{(f)}}{r} - \frac{1}{r}\frac{\partial u_\theta^{(f)}}{\partial \theta} - 2\varepsilon_\Delta^m\right) \\
\sigma_{r\theta} &= \frac{E_f}{2(1+\nu_f)}\left(\frac{1}{r}\frac{\partial u_r^{(f)}}{\partial \theta} + \frac{\partial u_\theta^{(f)}}{\partial r} - \frac{u_\theta^{(f)}}{r} - \gamma^m\right)
\end{aligned}
\tag{2}
$$

The membrane forces in the thin film are $N_r^{(f)} = h_f\,\sigma_{rr}$, $N_\theta^{(f)} = h_f\,\sigma_{\theta\theta}$, and $N_{r\theta}^{(f)} = h_f\,\sigma_{r\theta}$.

For non-uniform misfit strains distribution, the normal stress traction σ_{zz} still vanishes, but the shear stresses σ_{rz} and $\sigma_{\theta z}$ at the interface do not vanish anymore, and are denoted by τ_r and τ_θ, respectively. The equilibrium equations for the thin film, accounting for the effect of interface shear stresses τ_r and τ_θ, become

$$
\begin{aligned}
\frac{\partial N_r^{(f)}}{\partial r} + \frac{N_r^{(f)} - N_\theta^{(f)}}{r} + \frac{1}{r}\frac{\partial N_{r\theta}^{(f)}}{\partial \theta} - \tau_r &= 0 \\
\frac{\partial N_{r\theta}^{(f)}}{\partial r} + \frac{2}{r}N_{r\theta}^{(f)} + \frac{1}{r}\frac{\partial N_\theta^{(f)}}{\partial \theta} - \tau_\theta &= 0
\end{aligned}
\tag{3}
$$

The substitution of Eq. (2) into (3) yields the following governing equations for $u_r^{(f)}$, $u_\theta^{(f)}$, τ_r and τ_θ

$$
\begin{aligned}
&\frac{\partial^2 u_r^{(f)}}{\partial r^2} + \frac{1}{r}\frac{\partial u_r^{(f)}}{\partial r} - \frac{u_r^{(f)}}{r^2} + \frac{1-\nu_f}{2}\frac{1}{r^2}\frac{\partial^2 u_r^{(f)}}{\partial \theta^2} + \frac{1+\nu_f}{2}\frac{1}{r}\frac{\partial^2 u_\theta^{(f)}}{\partial r \partial \theta} - \frac{3-\nu_f}{2}\frac{1}{r^2}\frac{\partial u_\theta^{(f)}}{\partial \theta} \\
&= \frac{1-\nu_f^2}{E_f h_f}\tau_r + \left[(1+\nu_f)\frac{\partial \varepsilon_\Sigma^m}{\partial r} + (1-\nu_f)\frac{\partial \varepsilon_\Delta^m}{\partial r} + (1-\nu_f)\frac{2}{r}\varepsilon_\Delta^m + \frac{1-\nu_f}{2}\frac{1}{r}\frac{\partial \gamma^m}{\partial \theta}\right]
\end{aligned}
\tag{4a}
$$

$$\frac{1+\nu_f}{2}\frac{1}{r}\frac{\partial^2 u_r^{(f)}}{\partial r\partial\theta}+\frac{3-\nu_f}{2}\frac{1}{r^2}\frac{\partial u_r^{(f)}}{\partial\theta}+\frac{1-\nu_f}{2}\left(\frac{\partial^2 u_\theta^{(f)}}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta^{(f)}}{\partial r}-\frac{u_\theta^{(f)}}{r^2}\right)+\frac{1}{r^2}\frac{\partial^2 u_\theta^{(f)}}{\partial\theta^2}$$

$$=\frac{1-\nu_f^2}{E_f h_f}\tau_\theta+\left[(1+\nu_f)\frac{1}{r}\frac{\partial\varepsilon_\Sigma^m}{\partial\theta}-(1-\nu_f)\frac{1}{r}\frac{\partial\varepsilon_\Delta^m}{\partial\theta}+\frac{1-\nu_f}{2}\frac{\partial\gamma^m}{\partial r}+(1-\nu_f)\frac{\gamma^m}{r}\right] \tag{4b}$$

Let $u_r^{(s)}$ and $u_\theta^{(s)}$ denote the displacements in the radial (r) and circumferential (θ) directions at the neutral axis ($z=0$) of the substrate, and w the displacement in the normal (z) direction. It is important to consider w since the substrate can be subject to bending and is modeled as a plate. The strains in the substrate are given by $\varepsilon_{rr}=\frac{\partial u_r^{(s)}}{\partial r}-z\frac{\partial^2 w}{\partial r^2}$, $\varepsilon_{\theta\theta}=\frac{u_r^{(s)}}{r}+\frac{1}{r}\frac{\partial u_\theta^{(s)}}{\partial\theta}-z\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)$, and $\gamma_{r\theta}=\frac{1}{r}\frac{\partial u_r^{(s)}}{\partial\theta}+\frac{\partial u_\theta^{(s)}}{\partial r}-\frac{u_\theta^{(s)}}{r}+-2z\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}\right)$ The stresses in the substrate can then be obtained from the linear elastic constitutive model as

$$\sigma_{rr}=\frac{E_s}{1-\nu_s^2}\left\{\frac{\partial u_r^{(s)}}{\partial r}+\nu_s\left(\frac{u_r^{(s)}}{r}+\frac{1}{r}\frac{\partial u_\theta^{(s)}}{\partial\theta}\right)-z\left[\frac{\partial^2 w}{\partial r^2}+\nu_s\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)\right]\right\}$$

$$\sigma_{\theta\theta}=\frac{E_s}{1-\nu_s^2}\left[\nu_s\frac{\partial u_r^{(s)}}{\partial r}+\frac{u_r^{(s)}}{r}+\frac{1}{r}\frac{\partial u_\theta^{(s)}}{\partial\theta}-z\left(\nu_s\frac{\partial^2 w}{\partial r^2}+\frac{\partial w}{\partial r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)\right] \tag{5}$$

$$\sigma_{r\theta}=\frac{E_s}{2(1+\nu_s)}\left[\frac{1}{r}\frac{\partial u_r^{(s)}}{\partial\theta}+\frac{\partial u_\theta^{(s)}}{\partial r}-\frac{u_\theta^{(s)}}{r}-2z\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}\right)\right]$$

The forces in the substrate are obtained by averaging the stresses over the thickness as $N_r^{(s)}=\frac{E_s h_s}{1-\nu_s^2}\left[\frac{\partial u_r^{(s)}}{\partial r}+\nu_s\left(\frac{u_r^{(s)}}{r}+\frac{1}{r}\frac{\partial u_\theta^{(s)}}{\partial\theta}\right)\right]$, $N_\theta^{(s)}=\frac{E_s h_s}{1-\nu_s^2}\left[\nu_s\frac{\partial u_r^{(s)}}{\partial r}+\frac{u_r^{(s)}}{r}+\frac{1}{r}\frac{\partial u_\theta^{(s)}}{\partial\theta}\right]$, and $N_{r\theta}^{(s)}=\frac{E_s h_s}{2(1+\nu_s)}\left(\frac{1}{r}\frac{\partial u_r^{(s)}}{\partial\theta}+\frac{\partial u_\theta^{(s)}}{\partial r}-\frac{u_\theta^{(s)}}{r}\right)$.

The moments in the substrate are obtained from $-\int_{h_s/2}^{h_s/2} z\sigma_{ij}dz$ as $M_r=\frac{E_s h_s^3}{12(1-\nu_s^2)}\left[\frac{\partial^2 w}{\partial r^2}+\nu_s\left(\frac{1}{r}\frac{\partial w}{\partial r}\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)\right]$, $M_\theta=\frac{E_s h_s^3}{12(1-\nu_s^2)}=\left(\nu_s\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)$ and $M_{r\theta}=\frac{E_s h_s^3}{12(1+\nu_s)}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}\right)$.

The shear stresses τ_r and τ_θ at the thin film/substrate interface are equivalent to the distributed forces τ_r in the radial direction and τ_θ in the circumferential direction, and bending moments $h_s/2\tau_r$ and $h_s/2\tau_\theta$ applied at the neutral axis ($z=0$) of the substrate. The in-plane force equilibrium equations of the substrate then become

$$\frac{\partial N_r^{(s)}}{\partial r}+\frac{N_r^{(s)}-N_\theta^{(s)}}{r}+\frac{1}{r}+\frac{\partial N_{r\theta}^{(s)}}{\partial\theta}+\tau_r=0 \tag{6}$$

$$\frac{\partial N_{r\theta}^{(s)}}{\partial r}+\frac{2}{r}N_{r\theta}^{(s)}+\frac{1}{r}\frac{\partial N_{\theta}^{(s)}}{\partial \theta}+\tau_\theta=0$$

The substitution of $N_r^{(s)}, N_\theta^{(s)}$, and $N_{r\theta}^{(s)}$ in terms of the displacements into the above equation yields the following governing equations for $u_r^{(s)}, u_\theta^{(s)}, \tau_r$, and τ_θ

$$\frac{\partial^2 u_r^{(s)}}{\partial r^2}+\frac{1}{r}\frac{\partial u_r^{(s)}}{\partial r}-\frac{u_r^{(s)}}{r^2}+\frac{1-\nu_s}{2}\frac{1}{r^2}\frac{\partial^2 u_r^{(s)}}{\partial \theta^2}+\frac{1+\nu_s}{2}\frac{1}{r}\frac{\partial^2 u_\theta^{(s)}}{\partial r\partial \theta}-\frac{3-\nu_s}{2}\frac{1}{r^2}\frac{\partial u_\theta^{(s)}}{\partial \theta}=-\frac{1-\nu_s^2}{E_s h_s}\tau_r \tag{7a}$$

$$\frac{1+\nu_s}{2}\frac{1}{r}\frac{\partial^2 u_r^{(s)}}{\partial r\partial \theta}+\frac{3-\nu_s}{2}\frac{1}{r^2}\frac{\partial u_r^{(s)}}{\partial \theta}+\frac{1-\nu_s}{2}\left(\frac{\partial^2 u_\theta^{(s)}}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta^{(s)}}{\partial r}-\frac{u_\theta^{(s)}}{r^2}\right)+\frac{1}{r^2}\frac{\partial^2 u_\theta^{(s)}}{\partial \theta^2}=-\frac{1-\nu_s^2}{E_s h_s}\tau_\theta \tag{7b}$$

The out-of-plane moment and force equilibrium equations are given by

$$\begin{aligned}&\frac{\partial M_r}{\partial r}+\frac{M_r-M_\theta}{r}+\frac{1}{r}\frac{\partial M_{r\theta}}{\partial \theta}+Q_r-\frac{h_s}{2}\tau_r=0\\&\frac{\partial M_{r\theta}}{\partial r}+\frac{2}{r}M_{r\theta}+\frac{1}{r}\frac{\partial M_\theta}{\partial \theta}+Q_\theta-\frac{h_s}{2}\tau_\theta=0\end{aligned} \tag{8}$$

$$\frac{\partial Q_r}{\partial r}+\frac{Q_r}{r}+\frac{1}{r}\frac{\partial Q_\theta}{\partial \theta}=0 \tag{9}$$

where Q_r and Q_θ are the shear forces normal to the neutral axis. Elimination of Q_r and Q_θ from the above two equations in conjunction with the moments-displacement relation, give the following governing equations for w, τ_r and τ_θ

$$\nabla^2(\nabla^2 w)=\frac{6(1-\nu_s^2)}{E_s h_s^2}\left(\frac{\partial \tau_r}{\partial r}+\frac{\tau_r}{r}+\frac{1}{r}\frac{\partial \tau_\theta}{\partial \theta}\right) \tag{10}$$

where $\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$

The continuity of displacements across the thin film/substrate interface requires

$$u_r^{(f)}=u_r^{(s)}-\frac{h_s}{2}\frac{\partial w}{\partial r},\qquad u_\theta^{(f)}=u_\theta^{(s)}-\frac{h_s}{2}\frac{1}{r}\frac{\partial w}{\partial \theta} \tag{11}$$

Eqs. (4), (7), (10) and (11) constitute seven ordinary differential equations for $u_r^{(f)}$, $u_\theta^{(f)}$, $u_r^{(s)}$, $u_\theta^{(s)}$, w, τ_r and τ_θ. Under the limit $h_f << h_s$ these seven equations can be decoupled to solve $u_r^{(s)}$, $u_\theta^{(s)}$ first, followed by w, then $u_r^{(f)}$ and $u_\theta^{(f)}$, and finally τ_r and τ_θ, as discussed in the following.

(i) Elimination of τ_r and τ_θ from Eqs. (4) and (7) yields two equations for $u_r^{(f)}$, $u_\theta^{(f)}$, $u_r^{(s)}$, and $u_\theta^{(s)}$. For $h_f << h_s$, $u_r^{(f)}$ and $u_\theta^{(f)}$ disappear in these two equations which give the following governing equations for $u_r^{(s)}$ and $u_\theta^{(s)}$ only,

$$\begin{aligned}&\frac{\partial^2 u_r^{(s)}}{\partial r^2}+\frac{1}{r}\frac{\partial u_r^{(s)}}{\partial r}-\frac{u_r^{(s)}}{r^2}+\frac{1-\nu_s}{2}\frac{1}{r^2}\frac{\partial^2 u_r^{(s)}}{\partial \theta^2}+\frac{1+\nu_s}{2}\frac{1}{r}\frac{\partial^2 u_\theta^{(s)}}{\partial r\partial \theta}-\frac{3-\nu_s}{2}\frac{1}{r^2}\frac{\partial u_\theta^{(s)}}{\partial \theta}\\&=\frac{E_f h_f}{1-\nu_f^2}\frac{1-\nu_s^2}{E_s h_s}\left[(1+\nu_f)\frac{\partial \varepsilon_\Sigma^m}{\partial r}+(1-\nu_f)\frac{\partial \varepsilon_\Delta^m}{\partial r}+(1-\nu_f)\frac{2}{r}\varepsilon_\Delta^m+\frac{1-\nu_f}{2}\frac{1}{r}\frac{\partial \gamma^s}{\partial \theta}\right]\end{aligned} \tag{12a}$$

$$\frac{1+\nu_s}{2}\frac{1}{r}\frac{\partial^2 u_r^{(s)}}{\partial r\partial \theta}+\frac{3-\nu_s}{2}\frac{1}{r^2}\frac{\partial u_r^{(s)}}{\partial \theta}+\frac{1-\nu_s}{2}\left(\frac{\partial^2 u_\theta^{(s)}}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta^{(s)}}{\partial r}-\frac{u_\theta^{(s)}}{r^2}\right)+\frac{1}{r^2}\frac{\partial^2 u_\theta^{(s)}}{\partial \theta^2}$$

$$= \frac{E_f h_f}{1-\nu_f^2}\frac{1-\nu_s^2}{E_s h_s}\left[(1+\nu_f)\frac{1}{r}\frac{\partial \varepsilon_\Sigma^m}{\partial\theta} - (1-\nu_f)\frac{1}{r}\frac{\partial \varepsilon_\Delta^m}{\partial\theta} + \frac{1-\nu_f}{2}\frac{\partial \gamma^m}{\partial r} + (1-\nu_f)\frac{\gamma^m}{r}\right] \tag{12b}$$

(ii) Elimination of $u_r^{(f)}$ and $u_\theta^{(f)}$ from Eqs. (4) and (11) gives τ_r and τ_θ in terms of $u_r^{(s)}$, $u_\theta^{(s)}$ and w (and $\varepsilon_\Sigma^m, \varepsilon_\Delta^m, \gamma^m$).

(iii) The substitution of τ_r and τ_θ in (ii) into Eq. (10) yields the following governing equation for the normal displacement w. For $h_f << h_s$, the governing equation becomes

$$\nabla^2(\nabla^2 w) = -6\frac{E_f h_f}{1-\nu_f^2}\frac{1-\nu_s^2}{E_s h_s^2}\begin{bmatrix} (1+\nu_f)\left(\frac{\partial^2 \varepsilon_\Sigma^m}{\partial r^2} + \frac{1}{r}\frac{\partial \varepsilon_\Sigma^m}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varepsilon_\Sigma^m}{\partial \theta^2}\right) \\ +(1-\nu_f)\left(\frac{\partial^2 \varepsilon_\Delta^m}{\partial r^2} + \frac{3}{r}\frac{\partial \varepsilon_\Delta^m}{\partial r} - \frac{1}{r^2}\frac{\partial^2 \varepsilon_\Delta^m}{\partial \theta^2}\right) \\ +(1-\nu_f)\left(\frac{1}{r}\frac{\partial^2 \gamma^m}{\partial r \partial\theta} + \frac{1}{r^2}\frac{\partial \gamma^m}{\partial\theta}\right) \end{bmatrix} \tag{13}$$

This biharmonic equation can be solved analytically, which gives the substrate displacement w.

(iv) The displacements $u_r^{(f)}$ and $u_\theta^{(f)}$ are obtained from Eq. (11). The leading terms of the interface shear stresses τ_r and τ_θ are then obtained from Eq. (4) as

$$\tau_r = -\frac{E_f h_f}{1-\nu_f^2}\left[(1+\nu_f)\frac{\partial \varepsilon_\Sigma^m}{\partial r} + (1-\nu_f)\frac{\partial \varepsilon_\Delta^m}{\partial r} + (1-\nu_f)\frac{2}{r}\varepsilon_\Delta^m + \frac{1-\nu_f}{2}\frac{1}{r}\frac{\partial \gamma^m}{\partial\theta}\right]$$

$$\tau_\theta = -\frac{E_f h_f}{1-\nu_f^2}\left[(1+\nu_f)\frac{1}{r}\frac{\partial \varepsilon_\Sigma^m}{\partial\theta} - (1-\nu_f)\frac{1}{r}\frac{\partial \varepsilon_\Delta^m}{\partial\theta} + \frac{1-\nu_f}{2}\frac{\partial \gamma^m}{\partial\theta} + (1-\nu_f)\frac{\gamma^m}{r}\right] \tag{14}$$

These are remarkable results that hold regardless of boundary conditions at the edge $r = R$. Therefore the interface shear stresses are proportional to the gradients of misfit strains. For uniform misfit strain ε_{xx}^m, ε_{yy}^m, and ε_{xy}^m=constants in the Cartesian coordinates (Freund and Suresh 2004), the interface shear stresses do NOT vanish unless $\varepsilon_{xx}^m = \varepsilon_{yy}^m$=constant and ε_{xy}^m=0 (i.e. the isotropic Stoney formula).

We expand the arbitrary non-uniform misfit strain distributions $\varepsilon_\Sigma^m(r,\theta)$, $\varepsilon_\Delta^m(r,\theta)$ and $\gamma^m(r,\theta)$ to the Fourier series in order to solve the above partial differential equations. analytically

$$\varepsilon_\Sigma^m(r,\theta) = \sum_{n=0}^{\infty}\varepsilon_{\Sigma c}^{m(n)}(r)\cos n\theta + \sum_{n=1}^{\infty}\varepsilon_{\Sigma s}^{m(n)}(r)\sin n\theta$$

$$\varepsilon_\Delta^m(r,\theta) = \sum_{n=0}^{\infty}\varepsilon_{\Delta c}^{m(n)}(r)\cos n\theta + \sum_{n=1}^{\infty}\varepsilon_{\Delta s}^{m(n)}(r)\sin n\theta$$

$$\gamma^m(r,\theta) = \sum_{n=0}^{\infty}\gamma_c^{m(n)}(r)\cos n\theta + \sum_{n=1}^{\infty}\gamma_s^{m(n)}(r)\sin n\theta \tag{15}$$

where $\varepsilon_{\Sigma c}^{m(0)}(r) = \frac{1}{2\pi}\int_0^{2\pi}\varepsilon_\Sigma^m(r,\theta)d\theta$, $\varepsilon_{\Delta c}^{m(0)}(r) = \frac{1}{2\pi}\int_0^{2\pi}\varepsilon_\Delta^m(r,\theta)d\theta$, $\gamma_c^{m(0)}(r) = \frac{1}{2\pi}\int_0^{2\pi}\gamma^m(r,\theta)d\theta$, $\varepsilon_{\Sigma c}^{m(n)}(r) =$

$\frac{1}{\pi}\int_0^{2\pi}\varepsilon_\Sigma^m(r,\theta)\cos n\theta d\theta$, $\varepsilon_{\Delta c}^{m(n)}(r)=\frac{1}{\pi}\int_0^{2\pi}\varepsilon_\Delta^m(r,\theta)\cos n\theta d\theta$, $\gamma_c^{m(n)}(r)=\frac{1}{\pi}\int_0^{2\pi}\gamma^m(r,\theta)\ \cos n\ \theta d\theta\ \ (n\geq 1)$,

$\varepsilon_{\Sigma s}^{m(n)}\ (r)=\frac{1}{\pi}\int_0^{2\pi}\varepsilon_\Sigma^m(r,\theta)\sin n\theta d\theta$, $\varepsilon_{\Delta s}^{m(n)}(r)=\frac{1}{\pi}\int_0^{2\pi}\varepsilon_\Delta^m(r,\theta)\sin n\theta d\theta$ and $\gamma_s^{m(n)}(r)=\frac{1}{\pi}\int_0^{2\pi}\gamma^m(r,\theta)\sin n\theta d\theta$

($n\geq 1$). Without losing generality, we focus on the $\cos n\theta$ term in $\varepsilon_\Sigma^m(r,\theta)$ and $\varepsilon_\Delta^m(r,\theta)$ and $\sin n\theta$ term in $\gamma^m(r,\theta)$ in the following. The corresponding displacements and interface shear stresses can be expressed as

$$u_r^{(s)}=u_r^{(sn)}(r)\cos n\theta,\ u_\theta^{(s)}=u_\theta^{(sn)}(r)\sin n\theta,\ w=w^{(n)}(r)\cos n\theta \tag{16}$$

Eq. (12) then gives two ordinary differential equations for $u_r^{(sn)}$ and $u_\theta^{(sn)}$, which have the general solution

$$u_r^{(sn)}=\frac{1}{8}\frac{E_f h_f}{1-v_f^2}\frac{1+v_s}{E_s h_s}\left\{\begin{array}{l}4(1+v_f)(1-v_s)\left[r^{-(1+n)}\int_0^r\eta^{n+1}\varepsilon_{\Sigma c}^{m(n)}d\eta+r^{n-1}\int_R^r\eta^{1-n}\varepsilon_{\Sigma c}^{m(n)}d\eta\right]\\ +(1-v_f)[2(1-v_s)-(1+v_s)n]r^{1+n}\int_R^r\eta^{-(n+1)}(2\varepsilon_{\Delta c}^{m(n)}+\gamma_s^{m(n)})d\eta\\ +(1-v_f)[2(1-v_s)+(1+v_s)n]r^{1-n}\int_0^r\eta^{n-1}(2\varepsilon_{\Delta c}^{m(n)}-\gamma_s^{m(n)})d\eta\\ -(1-v_f)(1+v_s)n\left[\begin{array}{l}r^{-(1+n)}\int_0^r\eta^{n+1}(2\varepsilon_{\Delta c}^{m(n)}-\gamma_s^{m(n)})d\eta\\ -r^{n-1}\int_R^r\eta^{1-n}(2\varepsilon_{\Delta c}^{m(n)}+\gamma_s^{m(n)})d\eta\end{array}\right]\end{array}\right\}$$

$$+\left(1-v_s-\frac{1+v_s}{2}n\right)A_0r^{1+n}-D_0r^{n-1} \tag{17a}$$

$$u_\theta^{(sn)}=\frac{1}{8}\frac{E_f h_f}{1-v_f^2}\frac{1+v_s}{E_s h_s}\left\{\begin{array}{l}4(1+v_f)(1-v_s)\left[r^{-(1+n)}\int_0^r\eta^{n+1}\varepsilon_{\Sigma c}^{m(n)}d\eta-r^{n-1}\int_R^r\eta^{1-n}\varepsilon_{\Sigma c}^{m(n)}d\eta\right]\\ 1-v_f[(1+v_s)n+4]r^{1+n}\int_R^r\eta^{-(n+1)}(2\varepsilon_{\Delta c}^{m(n)}+\gamma_s^{m(n)})d\eta\\ +(1-v_f)[(1+v_s)n-4]r^{1-n}\int_0^r\eta^{n-1}(2\varepsilon_{\Delta c}^{m(n)}+\gamma_s^{m(n)})d\eta\\ -(1-v_f)(1+v_s)n\left[\begin{array}{l}r^{-(1+n)}\int_0^r\eta^{n+1}(2\varepsilon_{\Delta c}^{m(n)}-\gamma_s^{m(n)})d\eta\\ +r^{n-1}\int_R^r\eta^{1-n}(2\varepsilon_{\Delta c}^{m(n)}+\gamma_s^{m(n)})d\eta\end{array}\right]\end{array}\right\}$$

$$+\left(\frac{1+v_s}{2}n+2\right)(A_0r^{1+n}+D_0r^{n-1}) \tag{17b}$$

where A_0 and D_0 are constants to be determined, and the condition of finite displacements at the center $r=0$ has been used.

The normal displacement is obtained from the biharmonic Eq. (13) as

$$w^{(n)} = -\frac{3}{4}\frac{E_f h_f}{1-v_f^2}\frac{1-v_s^2}{E_s h_s^2}\left\{\begin{array}{l} (1+v_f)\frac{4}{n}\left[r^n\int_R^r \eta^{1-n}\varepsilon_{\Sigma c}^{m(n)}d\eta - r^{-n}\int_0^r \eta^{n+1}\varepsilon_{\Sigma c}^{m(n)}d\eta\right] \\ +(1-v_f)\left[r^{n+2}\int_R^r \eta^{-(n+1)}(2\varepsilon_{\Delta c}^{m(n)}+\gamma_s^{m(n)})d\eta + r^{2-n}\int_0^r \eta^{n-1}(2\varepsilon_{\Delta c}^{m(n)}-\gamma_s^{m(n)})d\eta\right] \\ -(1-v_f)\left[r^n\int_R^r \eta^{1-n}(2\varepsilon_{\Delta c}^{m(n)}+\gamma_s^{m(n)})d\eta + r^{-n}\int_0^r \eta^{n+1}(2\varepsilon_{\Delta c}^{m(n)}-\gamma_s^{m(n)})d\eta\right] \end{array}\right\}$$

$$+A_1 r^{n+2} + B_1 r^n \tag{18}$$

where A_1 and B_1 are constants to be determined, and the condition of finite w at the center $r=0$ has been used. The displacements in the thin film are obtained from the interface continuity condition (11).

It is important to point out that Eqs. (17) and (18) hold for $n>0$. For $n=0$ the displacements are given and discussed in details in Section 5.

3. Boundary conditions

The first two boundary conditions at the free edge $r=R$ require that the net forces vanish,

$$N_r^{(f)} + N_r^{(s)} = 0 \text{ and } N_{r\theta}^{(f)} + N_{r\theta}^{(s)} = 0 \text{ at } r=R \tag{19}$$

which give A_0 and D_0 as

$$A_0 = \frac{1}{4}\frac{E_f h_f}{1-v_f^2}\frac{1+v_s}{E_s h_s}\left\{\begin{array}{l} 4(1+v_f)\frac{1-v_s}{1+v_s}R^{-2(n+1)}\int_0^R \eta^{1+n}\varepsilon_{\Sigma c}^{m(n)}d\eta \\ -(1-v_f)nR^{-2(n+1)}\int_0^R \eta^{n+1}(2\varepsilon_{\Delta c}^{m(n)}-\gamma_s^{m(n)})d\eta \\ +(1-v_f)(n-1)R^{-2n}\int_0^R \eta^{n-1}(2\varepsilon_{\Delta c}^{m(n)}-\gamma_s^{m(n)})d\eta \end{array}\right\} \tag{20a}$$

$$D_0 = -\frac{1}{8}\frac{E_f h_f}{1-v_f^2}\frac{1+v_s}{E_s h_s}\left\{\begin{array}{l} 4(1+v_f)(1-v_s)(1+n)R^{-2n}\int_0^R \eta^{1+n}\varepsilon_{\Sigma c}^{m(n)}d\eta \\ +(1-v_f)(1+v_s)n^2R^{-2(n-1)}\int_0^R \eta^{n-1}(2\varepsilon_{\Delta c}^{m(n)}-\gamma_s^{m(n)})d\eta \\ -(1-v_f)(1+v_s)n(n+1)R^{-2n}\int_0^R \eta^{1+n}(2\varepsilon_{\Delta c}^{m(n)}-\gamma_s^{m(n)})d\eta \end{array}\right\} \tag{20b}$$

under the limit $h_f << h_s$. The other two boundary conditions at the free edge $r=R$ are the vanishing of net moments, i.e.,

$$M_r - \frac{h_s}{2} N_r^{(f)} = 0 \text{ and } Q_r - \frac{1}{r}\frac{\partial}{\partial\theta}\left(M_{r\theta} - \frac{h_s}{2} N_{r\theta}^{(f)}\right) = 0 \text{ at } r = R, \tag{21}$$

which give A_1 and B_1 as

$$A_1 = -\frac{3}{4}\frac{E_f}{1-\nu_f^2}\frac{h_f}{E_s}\frac{1-\nu_s^2}{h_s^2}\frac{1-\nu_s}{3+\nu_s}\left\{\begin{array}{l} 4(1+\nu_f)R^{-2(n+1)}\int_0^R \eta^{1+n}\varepsilon_{\Sigma c}^{m(n)} d\eta \\ -(1-\nu_f)(n-1)R^{-2n}\int_0^R \eta^{n-1}(2\varepsilon_{\Delta c}^{m(n)} - \gamma_s^{m(n)}) d\eta \\ +(1-\nu_f)nR^{-2(n+1)}\int_0^R \eta^{n+1}(2\varepsilon_{\Delta c}^{m(n)} - \gamma_s^{m(n)}) d\eta \end{array}\right\} \tag{22a}$$

$$B_1 = \frac{3}{4}\frac{E_f}{1-\nu_f^2}\frac{h_f}{E_s}\frac{1-\nu_s^2}{h_s^2}\left\{\begin{array}{l} 4(1+\nu_f)\frac{(1-\nu_s)}{(3+\nu_s)}\frac{n+1}{n}R^{-2n}\int_0^R \eta^{n+1}\varepsilon_{\Sigma c}^{m(n)} d\eta \\ +(1-\nu_f)\left(\frac{1-n^2}{n}\frac{1-\nu_s}{3+\nu_s} - \frac{1}{n}\frac{3+\nu_s}{1-\nu_s}\right)R^{-2(n-1)}\int_0^R \eta^{n-1}(2\varepsilon_{\Delta c}^{m(n)} - \gamma_s^{m(n)}) d\eta \\ +(1-\nu_f)\frac{(1-\nu_s)}{(3+\nu_s)}(n+1)R^{-2n}\int_0^R \eta^{n+1}(2\varepsilon_{\Delta c}^{m(n)} - \gamma_s^{m(n)}) d\eta \end{array}\right\} \tag{22b}$$

4. Thin-film stresses and system curvatures

We provide the general solution that includes both cosine and sine terms in this section. The system curvatures are

$$\kappa_{rr} = \frac{\partial^2 w}{\partial r^2},\ \kappa_{\theta\theta} = \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2},\ \kappa_{r\theta} = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}\right) \tag{23}$$

The sum of system curvatures is related to the misfit strains by

$$\kappa_{rr} + \kappa_{\theta\theta} = -3\frac{E_f}{1-\nu_f^2}\frac{h_f}{E_s}\frac{1-\nu_s^2}{h_s^2}$$

$$* \left\{\begin{array}{l} 2(1+\nu_f)\varepsilon_\Sigma^m + 2(1-\nu_f)\varepsilon_\Delta^m + 4(1-\nu_f)\int_R^r \eta^{-1}\varepsilon_{\Delta c}^{m(0)} d\eta + (1+\nu_f)\frac{1-\nu_s}{1+\nu_s}\frac{4}{R^2}\int_0^R \eta\varepsilon_{\Sigma c}^{m(0)} d\eta \\ +(1-\nu_f)\sum_{n=1}^{\infty}(n+1)r^n\left[\cos n\theta\int_R^r \eta^{-(n+1)}(2\varepsilon_{\Delta c}^{m(n)} + \gamma_s^{m(n)}) d\eta + \sin n\theta\int_R^r \eta^{-(n+1)}(2\varepsilon_{\Delta s}^{m(n)} - \gamma_c^{m(n)}) d\eta\right] \\ -(1-\nu_f)\sum_{n=1}^{\infty}(n-1)r^{-n}\left[\cos n\theta\int_0^r \eta^{n-1}(2\varepsilon_{\Delta c}^{m(n)} - \gamma_s^{m(n)}) d\eta + \sin n\theta\int_0^r \eta^{n-1}(2\varepsilon_{\Delta s}^{m(n)} + \gamma_c^{m(n)}) d\eta\right] \end{array}\right\}$$

$$-3\frac{E_f h_f}{1-v_f^2}\frac{1-v_s^2}{E_s h_s^2}\frac{1-v_s}{3+v_s}$$

$$*\left\{\begin{array}{l} 4(1+v_f)\sum_{n=1}^{\infty}(n+1)r^n\frac{r^n}{R^{2(n+1)}}\left[\cos n\theta\int_0^R\eta^{n+1}\varepsilon_{\Sigma c}^{m(n)}d\eta+\sin n\theta\int_0^R\eta^{n+1}\varepsilon_{\Sigma s}^{m(n)}d\eta\right] \\ -(1-v_f)\sum_{n=1}^{\infty}(n^2-1)\frac{r^n}{R^{2n}}\left[\cos n\theta\int_0^R\eta^{n-1}(2\varepsilon_{\Delta c}^{m(n)}-\gamma_s^{m(n)})d\eta+\sin n\theta\int_0^R\eta^{n-1}(2\varepsilon_{\Delta s}^{m(n)}+\gamma_c^{m(n)})d\eta\right] \\ +(1-v_f)\sum_{n=1}^{\infty}n(n+1)\frac{r^n}{R^{2(n+1)}}\left[\cos n\theta\int_0^R\eta^{n+1}(2\varepsilon_{\Delta c}^{m(n)}-\gamma_s^{m(n)})d\eta+\sin n\theta\int_0^R\eta^{n+1}(2\varepsilon_{\Delta s}^{m(n)}+\gamma_c^{m(n)})d\eta\right] \end{array}\right\}$$

(24)

The average curvature sum over the entire thin film $\overline{\kappa_{rr}+\kappa_{\theta\theta}}=\frac{1}{\pi R^2}\int_0^{2\pi}d\theta\int_0^R\eta(\kappa_{rr}+\kappa_{\theta\theta})d\eta$ is then obtained as

$$\overline{\kappa_{rr}+\kappa_{\theta\theta}}=-12\frac{E_f h_f}{1-v_f}\frac{1-v_s}{E_s h_s^2}\overline{\varepsilon_\Sigma^m} \tag{25}$$

where $\overline{\varepsilon_\Sigma^m}=\frac{1}{\pi R^2}\int_0^{2\pi}d\theta\int_0^R\eta\varepsilon_\Sigma^m d\eta$ is the average misfit strain sum. The subtraction of the average curvature sum from Eq. (24) gives

$$\kappa_{rr}+\kappa_{\theta\theta}-\overline{\kappa_{rr}+\kappa_{\theta\theta}}=-3\frac{E_f h_f}{1-v_f^2}\frac{1-v_s^2}{E_s h_s^2}$$

$$*\left\{\begin{array}{l} 2(1+v_f)(\varepsilon_\Sigma^m-\overline{\varepsilon_\Sigma^m})+2(1-v_f)\varepsilon_\Delta^m+4(1-v_f)\int_R^r\eta^{-1}\varepsilon_{\Delta c}^{m(0)}d\eta \\ +(1-v_f)\sum_{n=1}^{\infty}(n+1)r^n\left[\cos n\theta\int_R^r\eta^{-(n+1)}(2\varepsilon_{\Delta c}^{m(n)}+\gamma_s^{m(n)})d\eta+\sin n\theta\int_R^r\eta^{-(n+1)}(2\varepsilon_{\Delta s}^{m(n)}-\gamma_c^{m(n)})d\eta\right] \\ -(1-v_f)\sum_{n=1}^{\infty}(n-1)r^{-n}\left[\cos n\theta\int_0^r\eta^{n-1}(2\varepsilon_{\Delta c}^{m(n)}-\gamma_s^{m(n)})d\eta+\sin n\theta\int_0^r\eta^{n-1}(2\varepsilon_{\Delta s}^{m(n)}+\gamma_c^{m(n)})d\eta\right] \end{array}\right\}$$

$$-3\frac{E_f h_f}{1-v_f^2}\frac{1-v_s^2}{E_s h_s^2}\frac{1-v_s}{3+v_s}$$

$$* \left\{ \begin{array}{l} 4(1+\nu_f)\sum_{n=1}^{\infty}(n+1)\frac{r^n}{R^{2(n+1)}}\left[\cos n\theta\int_0^R \eta^{n+1}\varepsilon_{\Sigma c}^{m(n)}d\eta + \sin n\theta\int_0^R \eta^{n+1}\varepsilon_{\Sigma s}^{m(n)}d\eta\right] \\ -(1-\nu_f)\sum_{n=1}^{\infty}(n^2-1)\frac{r^n}{R^{2n}}\left[\cos n\theta\int_0^R \eta^{n-1}(2\varepsilon_{\Delta c}^{m(n)} - \gamma_s^{m(n)})d\eta + \sin n\theta\int_0^R \eta^{n-1}(2\varepsilon_{\Delta s}^{m(n)} + \gamma_c^{m(n)})d\eta\right] \\ +(1-\nu_f)\sum_{n=1}^{\infty}n(n+1)\frac{r^n}{R^{2(n+1)}}\left[\cos n\theta\int_0^R \eta^{n+1}(2\varepsilon_{\Delta c}^{m(n)} - \gamma_s^{m(n)})d\eta + \sin n\theta\int_0^R \eta^{n+1}(2\varepsilon_{\Delta s}^{m(n)} + \gamma_c^{m(n)})d\eta\right] \end{array} \right\} \quad (26)$$

The difference between two curvatures, $\kappa_{rr} - \kappa_{\theta\theta}$, and the twist $\kappa_{r\theta}$ are given by

$$\kappa_{rr} - \kappa_{\theta\theta} = -\frac{3}{2}\frac{E_f h_f}{1-\nu_f^2}\frac{1-\nu_s^2}{E_s h_s^2}$$

$$* \left\{ \begin{array}{l} 4(1+\nu_f)\varepsilon_{\Sigma}^m + 4(1-\nu_f)\varepsilon_{\Delta}^m - (1+\nu_f)\frac{8}{r^2}\int_0^r \eta\varepsilon_{\Sigma c}^{m(0)}d\eta \\ +4(1+\nu_f)\sum_{n=1}^{\infty}(n-1)r^{n-2}\left[\cos n\theta\int_R^r \eta^{1-n}\varepsilon_{\Sigma c}^{m(n)}d\eta + \sin n\theta\int_R^r \eta^{1-n}\varepsilon_{\Sigma s}^{m(n)}d\eta\right] \\ -4(1+\nu_f)\sum_{n=1}^{\infty}(n+1)r^{-(n+2)}\left[\cos n\theta\int_0^r \eta^{n+1}\varepsilon_{\Sigma c}^{m(n)}d\eta + \sin n\theta\int_0^r \eta^{n+1}\varepsilon_{\Sigma s}^{m(n)}d\eta\right] \\ +(1-\nu_f)\sum_{n=1}^{\infty}n(n+1)r^{n}\left[\cos n\theta\int_R^r \eta^{-(n+1)}(2\varepsilon_{\Delta c}^{m(n)} + \gamma_s^{m(n)})d\eta + \sin n\theta\int_R^r \eta^{-(n+1)}(2\varepsilon_{\Delta s}^{m(n)} - \gamma_c^{m(n)})d\eta\right] \\ +(1-\nu_f)\sum_{n=1}^{\infty}n(n-1)r^{-n}\left[\cos n\theta\int_0^r \eta^{n-1}(2\varepsilon_{\Delta c}^{m(n)} - \gamma_s^{m(n)})d\eta + \sin n\theta\int_0^r \eta^{n-1}(2\varepsilon_{\Delta s}^{m(n)} + \gamma_c^{m(n)})d\eta\right] \\ -(1-\nu_f)\sum_{n=1}^{\infty}n(n-1)r^{n-2}\left[\cos n\theta\int_R^r \eta^{1-n}(2\varepsilon_{\Delta c}^{m(n)} + \gamma_s^{m(n)})d\eta + \sin n\theta\int_R^r \eta^{1-n}(2\varepsilon_{\Delta s}^{m(n)} - \gamma_c^{m(n)})d\eta\right] \\ -(1-\nu_f)\sum_{n=1}^{\infty}n(n+1)r^{-(n+2)}\left[\cos n\theta\int_0^r \eta^{n+1}(2\varepsilon_{\Delta c}^{m(n)} - \gamma_s^{m(n)})d\eta + \sin n\theta\int_0^r \eta^{n+1}(2\varepsilon_{\Delta s}^{m(n)} + \gamma_c^{m(n)})d\eta\right] \end{array} \right\} \quad (27)$$

$$-\frac{3}{2}\frac{E_f h_f}{1-\nu_f^2}\frac{1-\nu_s^2}{E_s h_s^2}\frac{1-\nu_s}{3+\nu_s}$$

$$
*\left\{\begin{array}{l}
4(1+\nu_f)\sum_{n=1}^{\infty} n(n+1)\dfrac{r^n}{R^{2(n+1)}}\left[\cos n\theta\int_0^R \eta^{n+1}\varepsilon_{\Sigma c}^{m(n)}d\eta+\sin n\theta\int_0^R \eta^{n+1}\varepsilon_{\Sigma s}^{m(n)}d\eta\right]\\
-(1-\nu_f)\sum_{n=1}^{\infty} n(n^2-1)\dfrac{r^n}{R^{2n}}\left[\cos n\theta\int_0^R \eta^{n-1}(2\varepsilon_{\Delta c}^{m(n)}-\gamma_s^{m(n)})d\eta+\sin n\theta\int_0^R \eta^{n-1}(2\varepsilon_{\Delta s}^{m(n)}+\gamma_c^{m(n)})d\eta\right]\\
+(1-\nu_f)\sum_{n=1}^{\infty} n^2(n+1)\dfrac{r^n}{R^{2(n+1)}}\left[\cos n\theta\int_0^R \eta^{n+1}(2\varepsilon_{\Delta c}^{m(n)}-\gamma_s^{m(n)})d\eta+\sin n\theta\int_0^R \eta^{n+1}(2\varepsilon_{\Delta s}^{m(n)}+\gamma_c^{m(n)})d\eta\right]\\
-4(1+\nu_f)\sum_{n=1}^{\infty} (n^2-1)\dfrac{r^{n-2}}{R^{2n}}\left[\cos n\theta\int_0^R \eta^{n+1}\varepsilon_{\Sigma c}^{m(n)}d\eta+\sin n\theta\int_0^R \eta^{n+1}\varepsilon_{\Sigma s}^{m(n)}d\eta\right]\\
+(1-\nu_f)\sum_{n=1}^{\infty} (n-1)\left[(n^2-1)+\left(\dfrac{3+\nu_s}{1-\nu_s}\right)^2\right]\dfrac{r^{n-2}}{R^{2(n-1)}}\left[\begin{array}{c}\cos n\theta\int_0^R \eta^{n-1}(2\varepsilon_{\Delta c}^{m(n)}-\gamma_s^{m(n)})d\eta\\ +\sin n\theta\int_0^R \eta^{n-1}(2\varepsilon_{\Delta s}^{m(n)}+\gamma_c^{m(n)})d\eta\end{array}\right]\\
-(1-\nu_f)\sum_{n=1}^{\infty} n(n^2-1)\dfrac{r^{n-2}}{R^{2n}}\left[\cos n\theta\int_0^R \eta^{n+1}(2\varepsilon_{\Delta c}^{m(n)}-\gamma_s^{m(n)})d\eta+\sin n\theta\int_0^R \eta^{n+1}(2\varepsilon_{\Delta s}^{m(n)}+\gamma_c^{m(n)})d\eta\right]
\end{array}\right\}
\tag{27}
$$

$$
\kappa_{r\theta}=\frac{3}{4}\frac{E_f h_f}{1-\nu_f^2}\frac{1-\nu_s^2}{E_s h_s^2}
$$

$$
*\left\{\begin{array}{l}
4(1+\nu_f)\sum_{n=1}^{\infty}(n-1)r^{n-2}\left[\sin n\theta\int_R^r \eta^{1-n}\varepsilon_{\Sigma c}^{m(n)}d\eta-\cos n\theta\int_R^r \eta^{1-n}\varepsilon_{\Sigma s}^{m(n)}d\eta\right]\\
+4(1+\nu_f)\sum_{n=1}^{\infty}(n+1)r^{-(n+2)}\left[\sin n\theta\int_0^r \eta^{n+1}\varepsilon_{\Sigma c}^{m(n)}d\eta-\cos n\theta\int_0^r \eta^{n+1}\varepsilon_{\Sigma s}^{m(n)}d\eta\right]\\
+(1-\nu_f)\sum_{n=1}^{\infty}n(n+1)r^{n}\left[\sin n\theta\int_R^r \eta^{-(n+1)}(2\varepsilon_{\Delta c}^{m(n)}+\gamma_s^{m(n)})d\eta-\cos n\theta\int_R^r \eta^{-(n+1)}(2\varepsilon_{\Delta s}^{m(n)}-\gamma_c^{m(n)})d\eta\right]\\
-(1-\nu_f)\sum_{n=1}^{\infty}n(n-1)r^{-n}\left[\sin n\theta\int_0^r \eta^{n-1}(2\varepsilon_{\Delta c}^{m(n)}-\gamma_s^{m(n)})d\eta-\cos n\theta\int_0^r \eta^{n-1}(2\varepsilon_{\Delta s}^{m(n)}+\gamma_c^{m(n)})d\eta\right]\\
-(1-\nu_f)\sum_{n=1}^{\infty}n(n-1)r^{n-2}\left[\sin n\theta\int_R^r \eta^{n-1}(2\varepsilon_{\Delta c}^{m(n)}+\gamma_s^{m(n)})d\eta-\cos n\theta\int_R^r \eta^{n-1}(2\varepsilon_{\Delta s}^{m(n)}-\gamma_c^{m(n)})d\eta\right]\\
+(1-\nu_f)\sum_{n=1}^{\infty}n(n+1)r^{-(n+2)}\left[\sin n\theta\int_0^r \eta^{n+1}(2\varepsilon_{\Delta c}^{m(n)}-\gamma_s^{m(n)})d\eta-\cos n\theta\int_0^r \eta^{n+1}(2\varepsilon_{\Delta s}^{m(n)}+\gamma_c^{m(n)})d\eta\right]
\end{array}\right\}
$$

$$+\frac{3}{4}\frac{E_f h_f}{1-\nu_f^2}\frac{1-\nu_s^2}{E_s h_s^2}\frac{1-\nu_s}{3+\nu_s}$$

$$*\left\{\begin{array}{l}
4(1+\nu_f)\sum_{n=1}^{\infty} n(n+1)\frac{r^n}{R^{2(n+1)}}\left[\sin n\theta\int_0^R \eta^{n+1}\varepsilon_{\Sigma c}^{m(n)}d\eta-\cos n\theta\int_0^R \eta^{n+1}\varepsilon_{\Sigma s}^{m(n)}d\eta\right] \\
-(1-\nu_f)\sum_{n=1}^{\infty} n(n^2-1)\frac{r^n}{R^{2n}}\left[\sin n\theta\int_0^R \eta^{n-1}(2\varepsilon_{\Delta c}^{m(n)}-\gamma_s^{m(n)})d\eta-\cos n\theta\int_0^R \eta^{n-1}(2\varepsilon_{\Delta s}^{m(n)}+\gamma_c^{m(n)})d\eta\right] \\
+(1-\nu_f)\sum_{n=1}^{\infty} n^2(n+1)\frac{r^n}{R^{2(n+1)}}\left[\sin n\theta\int_0^R \eta^{n+1}(2\varepsilon_{\Delta c}^{m(n)}-\gamma_s^{m(n)})d\eta-\cos n\theta\int_0^R \eta^{n+1}(2\varepsilon_{\Delta s}^{m(n)}+\gamma_c^{m(n)})d\eta\right] \\
-4(1+\nu_f)\sum_{n=1}^{\infty} (n^2-1)\frac{r^{n-2}}{R^{2n}}\left[\sin n\theta\int_0^R \eta^{n+1}\varepsilon_{\Sigma c}^{m(n)}d\eta-\cos n\theta\int_0^R \eta^{n+1}\varepsilon_{\Sigma s}^{m(n)}d\eta\right] \\
+(1-\nu_f)\sum_{n=1}^{\infty} (n-1)\left(\left[(n^2-1)+\left(\frac{3+\nu_s}{1-\nu_s}\right)^2\right]\frac{r^{n-2}}{R^{2(n-1)}}\left[\begin{array}{l}\sin n\theta\int_0^R \eta^{n-1}(2\varepsilon_{\Delta c}^{m(n)}-\gamma_s^{m(n)})d\eta \\ -\cos n\theta\int_0^R \eta^{n-1}(2\varepsilon_{\Delta s}^{m(n)}+\gamma_c^{m(n)})d\eta\end{array}\right]\right) \\
-(1-\nu_f)\sum_{n=1}^{\infty} n(n^2-1)\frac{r^{n-2}}{R^{2n}}\left[\sin n\theta\int_0^R \eta^{n+1}(2\varepsilon_{\Delta c}^{m(n)}-\gamma_s^{m(n)})d\eta-\cos n\theta\int_0^R \eta^{n+1}(2\varepsilon_{\Delta s}^{m(n)}+\gamma_c^{m(n)})d\eta\right]
\end{array}\right\} \quad (28)$$

The stresses in the thin film are obtained from Eq. (2). Specifically, the sum of stresses $\sigma_{rr}^{(f)}+\sigma_{\theta\theta}^{(f)}$ is related to the misfit strains by

$$\sigma_{rr}^{(f)}+\sigma_{\theta\theta}^{(f)}=\frac{E_f}{1-\nu_f}(-2\varepsilon_{\Sigma}^{m}) \quad (29)$$

The difference between stresses, $\sigma_{rr}^{(f)}-\sigma_{\theta\theta}^{(f)}$, and shear stress $\sigma_{r\theta}^{(f)}$ are given by

$$\sigma_{rr}^{(f)}-\sigma_{\theta\theta}^{(f)}=\frac{E_f}{1+\nu_f}(-2\varepsilon_{\Delta}^{m}) \quad (30)$$

$$\sigma_{r\theta}^{(f)}=\frac{E_f}{2(1+\nu_f)}(-\gamma^{m}) \quad (31)$$

5. Limiting cases

We present a few limit cases to further illustrate the thin film stresses and system curvatures in Section 4.

5.1 Uniform misfit strains in the Cartesian coordinates

Freund and Suresh (2004) obtained the solution for arbitrarily anisotropic but uniform misfit strains in the Cartesian coordinates, ε_{xx}^{m}, ε_{yy}^{m} and $\gamma_{xy}^{m}=2\varepsilon_{xy}^{m}$ constants. For this case $\varepsilon_{\Sigma}^{m} = \frac{1}{2}(\varepsilon_{xx}^{m}+\varepsilon_{yy}^{m})$ is a constant, but $\varepsilon_{\Delta}^{m} = \frac{1}{2}(\varepsilon_{xx}^{m}-\varepsilon_{yy}^{m})\cos 2\theta+\frac{1}{2}\gamma_{xy}^{m}\sin 2\theta$ and $\gamma^{m} = \gamma_{xy}^{m}\cos 2\theta-(\varepsilon_{xx}^{m}-\varepsilon_{yy}^{m})\sin 2\theta$ depend on θ. These give the non-vanishing coefficients of the Fourier series of the misfit strains as $\varepsilon_{\Sigma c}^{m(0)} = \frac{1}{2}(\varepsilon_{xx}^{m}+\varepsilon_{yy}^{m})$, $\varepsilon_{\Delta c}^{m(2)} = -\frac{1}{2}\gamma_{s}^{m(2)} = \frac{1}{2}(\varepsilon_{xx}^{m}-\varepsilon_{yy}^{m})$ and $\varepsilon_{\Delta c}^{m(2)} = \frac{1}{2}\gamma_{c}^{m(2)} = \frac{1}{2}\gamma_{xy}^{m}$. Eqs. (24)-(28) give the system curvatures, which can be transformed to curvatures in the Cartesian coordinates as

$$\begin{gathered}\kappa_{xx}+\kappa_{yy} = -6\frac{E_f h_f}{1-\nu_f}\frac{1-\nu_s}{E_s h_s^2}(\varepsilon_{xx}^{m}+\varepsilon_{yy}^{m})\\ \begin{Bmatrix}\kappa_{xx}-\kappa_{yy}\\ \kappa_{xy}\end{Bmatrix} = -6\frac{E_f h_f}{1+\nu_f}\frac{1+\nu_s}{E_s h_s^2}\begin{Bmatrix}\varepsilon_{xx}^{m}-\varepsilon_{yy}^{m}\\ \frac{\gamma_{xy}^{m}}{2}\end{Bmatrix}\end{gathered} \tag{32}$$

which are also constant curvatures. The thin-film stresses in the Cartesian coordinates can be obtained from Eqs. (29)-(31) as

$$\sigma_{xx}^{(f)}+\sigma_{yy}^{(f)} = -\frac{E_f}{1-\nu_f}(\varepsilon_{xx}^{m}+\varepsilon_{yy}^{m}) \tag{33}$$

$$\begin{Bmatrix}\sigma_{xx}^{(f)}-\sigma_{yy}^{(f)}\\ \sigma_{xy}^{(f)}\end{Bmatrix} = -\frac{E_f}{1+\nu_f}\begin{Bmatrix}\varepsilon_{xx}^{m}-\varepsilon_{yy}^{m}\\ \frac{\gamma_{xy}^{m}}{2}\end{Bmatrix}$$

which are constant stresses in the thin film. Elimination of misfit strains from Eqs. (32) and (33) gives the relation between thin film stresses and system curvatures

$$\begin{gathered}\sigma_{xx}^{(f)}+\sigma_{yy}^{(f)} = -\frac{E_f\, h_f^2}{6(1-\nu_s)h_f}(\kappa_{xx}+\kappa_{yy})\\ \begin{Bmatrix}\sigma_{xx}^{(f)}-\sigma_{yy}^{(f)}\\ \sigma_{xy}^{(f)}\end{Bmatrix} = \frac{E_s\, h_s^2}{6(1+\nu_s)h_f}\begin{Bmatrix}\kappa_{xx}-\kappa_{yy}\\ \kappa_{xy}\end{Bmatrix}\end{gathered} \tag{34}$$

which is identical to Freund and Suresh (2004).

5.2 Axisymmetric normal misfit strains

We consider the axisymmetric normal misfit strains $\varepsilon_{rr}^m = \varepsilon_{rr}^m(r)$, $\varepsilon_{\theta\theta}^m = \varepsilon_{\theta\theta}^m(r)$ and $\gamma_{r\theta}^m = 0$, which give $\varepsilon_\Sigma^m = \frac{1}{2}[\varepsilon_{rr}^m(r) + \varepsilon_{\theta\theta}^m(r)]$ and $\varepsilon_\Delta^m = \frac{1}{2}[\varepsilon_{rr}^m(r) - \varepsilon_{\theta\theta}^m(r)]$. The non-vanishing displacement in the substrate is

$$u_r^{(s)} = \frac{E_f h_f}{1-\nu_f^2}\frac{1-\nu_s^2}{E_s h_s}\left[\frac{1+\nu_f}{r}\int_0^r \eta\varepsilon_\Sigma^m d\eta + (1-\nu_f) r\int_R^r \frac{\varepsilon_\Delta^m}{\eta} d\eta + (1+\nu_f)\frac{1-\nu_s}{1+\nu_s}\frac{r}{R^2}\int_0^R \eta\varepsilon_\Sigma^m d\eta\right] \tag{35}$$

The normal displacement is given by

$$\frac{dw}{dr} = 6\frac{E_f h_f}{1-\nu_f^2}\frac{1-\nu_s^2}{E_s h_s^2}\left\{\frac{1+\nu_f}{r}\int_0^r \eta\varepsilon_\Sigma^m d\eta + (1-\nu_f) r\int_R^r \frac{\varepsilon_\Delta^m}{\eta} d\eta + (1+\nu_f)\frac{1-\nu_s}{1+\nu_s}\frac{r}{R^2}\int_0^R \eta\varepsilon_\Sigma^m d\eta\right\} \tag{36}$$

which gives the non-vanishing system curvatures as

$$\kappa_{rr} + \kappa_{\theta\theta} = -6\frac{E_f h_f}{1-\nu_f^2}\frac{1-\nu_s^2}{E_s h_s^2}\left\{\begin{array}{c}(1+\nu_f)\varepsilon_\Sigma^m + (1-\nu_f)\varepsilon_\Delta^m + 2(1-\nu_f)\int_R^r \frac{\varepsilon_\Delta^m}{\eta} d\eta \\ +(1+\nu_f)\frac{1-\nu_s}{1+\nu_s}\frac{2}{R^2}\int_0^R \eta\varepsilon_\Sigma^m d\eta\end{array}\right\}$$

$$\kappa_{rr} - \kappa_{\theta\theta} = -6\frac{E_f h_f}{1-\nu_f^2}\frac{1-\nu_s^2}{E_s h_s^2}\left\{(1+\nu_f)\varepsilon_\Sigma^m + (1-\nu_f)\varepsilon_\Delta^m - \frac{2(1-\nu_f)}{r^2}\int_0^r \eta\varepsilon_\Sigma^m d\eta\right\} \tag{37}$$

The non-zero stresses in the thin film are $\sigma_{rr}^{(f)} + \sigma_{\theta\theta}^{(f)} = \frac{E_f}{1-\nu_f}(-2\varepsilon_\Sigma^m)$ and $\sigma_{rr}^{(f)} - \sigma_{\theta\theta}^{(f)} = \frac{E_f}{1+\nu_f}(-2\varepsilon_\Delta^m)$. Eq. (37) seems to provide two equations to determine ε_Σ^m and ε_Δ^m (and therefore the thin-film stresses) in terms of curvatures. However, these two equations are NOT independent, as to be shown in the following.

The average curvature sum over the entire thin film $\overline{\kappa_{rr} + \kappa_{\theta\theta}} = \frac{1}{R^2}\int_0^R \eta(\kappa_{rr} + \kappa_{\theta\theta}) d\eta$ can be obtained from Eq. (37) as

$$\overline{\kappa_{rr} + \kappa_{\theta\theta}} = -12\frac{E_f h_f}{1-\nu_f}\frac{1-\nu_s}{E_s h_s^2}\overline{\varepsilon_\Sigma^m} = \frac{6(1-\nu_s)h_f}{E_s h_s^2}\overline{\sigma_{rr}^{(f)} + \sigma_{\theta\theta}^{(f)}} \tag{38}$$

where $\overline{\varepsilon_\Sigma^m}$ and $\overline{\sigma_{rr}^{(f)} + \sigma_{\theta\theta}^{(f)}}$ are the average of ε_Σ^m and $\sigma_{rr}^{(f)} + \sigma_{\theta\theta}^{(f)}$, respectively. It is clear that $\overline{\sigma_{rr}^{(f)} + \sigma_{\theta\theta}^{(f)}}$ and $\overline{\kappa_{rr} + \kappa_{\theta\theta}}$ satisfy the Stoney formula. The subtraction of Eq. (38) from Eq. (37) yields

$$\kappa_{rr}+\kappa_{\theta\theta}-\overline{\kappa_{rr}+\kappa_{\theta\theta}} = -6\frac{E_f h_f}{1-\nu_f^2}\frac{1-\nu_s^2}{E_s h_s^2}\left\{\begin{array}{c}(1+\nu_f)\varepsilon_\Sigma^m+(1-\nu_f)\varepsilon_\Delta^m+2(1-\nu_f)\int_R^r\frac{\varepsilon_\Delta^m}{\eta}d\eta \\ -(1+\nu_f)\frac{2}{R^2}\int_0^R\eta\varepsilon_\Sigma^m d\eta\end{array}\right\}$$

$$\kappa_{rr}-\kappa_{\theta\theta} = -6\frac{E_f h_f}{1-\nu_f^2}\frac{1-\nu_s^2}{E_s h_s^2}\left\{(1+\nu_f)\varepsilon_\Sigma^m+(1-\nu_f)\varepsilon_\Delta^m-\frac{2(1+\nu_f)}{r^2}\int_0^r\eta\varepsilon_\Sigma^m d\eta\right\} \tag{39}$$

It can be shown that, if the misfit strains satisfy $(1-\nu_f)\varepsilon_\Delta^m = \frac{2(1+\nu_f)}{r^2}\int_0^r\eta\varepsilon_\Sigma^m d\eta-(1+\nu_f)\varepsilon_\Sigma^m$, the right sides of both Eq. (39) vanish. The curvatures then become uniform and equi-biaxial, $\kappa_{rr} = \kappa_{\theta\theta} = -6\frac{E_f h_f}{1-\nu_f}\frac{1-\nu_s}{E_s h_s^2}\overline{\varepsilon_\Sigma^m}$ but the stresses are still non-uniform and non-equibiaxial given by $\sigma_{rr}^{(f)} = \frac{E_f}{1-\nu_f}\left(-\frac{2}{r^2}\int_0^r\eta\ \varepsilon_\Sigma^m d\eta\right)$ and $\sigma_{\theta\theta}^{(f)} = \frac{E_f}{1-\nu_f}\left(-2\varepsilon_\Sigma^m+\frac{2}{r^2}\int_0^r\eta\varepsilon_\Sigma^m d\eta\right)$. Therefore, for axisymmetric misfit strains, the thin-film stresses may not be expressed in terms of the system curvatures. This point will become clearer in the next section 35.

5.3 Axisymmetric shear misfit strain

We consider the axisymmetric shear misfit strain $\varepsilon_{rr}^m = \varepsilon_{\theta\theta}^m = 0$ and $\gamma_{r\theta}^m = \gamma^m(r)$. The substrate displacement in the radial direction vanishes, $u_r^{(s)} = 0$, and that in the circumferential direction is given by

$$u_r^{(s)} = \frac{E_f h_f}{1-\nu_f}\frac{1+\nu_s}{E_s h_s^2}r\int_R^r\frac{\gamma^m}{\eta}d\eta \tag{40}$$

The normal displacement also vanishes $w = 0$, which gives vanishing system curvatures

$$\kappa_{rr} = \kappa_{\theta\theta} = \kappa_{r\theta} = 0 \tag{41}$$

The normal stresses in the thin film are also zero, but the shear stress does not vanish

$$\sigma_{rr}^{(f)} = \sigma_{\theta\theta}^{(f)} = 0,\ \sigma_{r\theta}^{(f)} = \frac{E_f}{2(1+\nu_f)}(-\gamma^m) \tag{42}$$

It is clear that, for axisymmetric shear misfit strain, the non-vanishing thin-film stresses cannot be expressed in terms of the vanishing curvatures.

6. Extension of Stoney formula for nonuniform anisotrpic misfit strains

Freund and Suresh (2004) obtained the anisotropic relation between thin film stresses and system curvatures for uniform misfit strains. In this section we extend it to nonuniform, linearly distributed misfit strains, i.e.,

$$\begin{Bmatrix} \varepsilon_{xx}^m \\ \varepsilon_{yy}^m \\ \varepsilon_{xy}^m \end{Bmatrix} = \begin{Bmatrix} \overline{\varepsilon_{xx}^m} \\ \overline{\varepsilon_{yy}^m} \\ \overline{\varepsilon_{xy}^m} \end{Bmatrix} (1 + ax + by) \tag{43}$$

where a and b are constants, and $\overline{\varepsilon_{ij}^m}$ are the average misfit strains, which can be related to the average system curvatures by

$$\begin{aligned} \overline{\kappa_{xx} + \kappa_{yy}} &= -6 \frac{E_f h_f}{1 - \nu_f} \frac{1 - \nu_s}{E_s h_s^2} \overline{\varepsilon_{xx}^m + \varepsilon_{yy}^m} \\ \begin{Bmatrix} \overline{\kappa_{xx} - \kappa_{yy}} \\ \overline{\kappa_{xy}} \end{Bmatrix} &= -6 \frac{E_f h_f}{1 + \nu_f} \frac{1 + \nu_s}{E_s h_s^2} \begin{Bmatrix} \overline{\varepsilon_{xx}^m - \varepsilon_{yy}^m} \\ \overline{\varepsilon_{xy}^m} \end{Bmatrix} \end{aligned} \tag{44}$$

The constants a and b in Eq. (43) can be obtained by averaging $x(\kappa_{xx} + \kappa_{yy})$ and $y(\kappa_{xx} + \kappa_{yy})$ over the entire thin film as

$$\begin{aligned} a &= \frac{2(3+\nu_s)}{R^2} \frac{\left[(1+\nu_s)\overline{(\kappa_{xx} + \kappa_{yy})} - \frac{(1-\nu_s)}{2}\overline{(\kappa_{xx} - \kappa_{yy})} \right] \overline{x(\kappa_{xx} + \kappa_{yy})} - (1-\nu_s)\overline{\kappa_{xy}}\,\overline{y(\kappa_{xx} + \kappa_{yy})}}{[(1+\nu_s)\overline{(\kappa_{xx} + \kappa_{yy})}]^2 - \left[\frac{(1-\nu_s)}{2}\overline{(\kappa_{xx} - \kappa_{yy})} \right]^2 - [(1-\nu_s)\overline{\kappa_{xy}}]^2} \\ b &= \frac{2(3+\nu_s)}{R^2} \frac{\left[(1+\nu_s)\overline{(\kappa_{xx} + \kappa_{yy})} + \frac{(1-\nu_s)}{2}\overline{(\kappa_{xx} - \kappa_{yy})} \right] \overline{y(\kappa_{xx} + \kappa_{yy})} - (1-\nu_s)\overline{\kappa_{xy}}\,\overline{x(\kappa_{xx} + \kappa_{yy})}}{[(1+\nu_s)\overline{(\kappa_{xx} + \kappa_{yy})}]^2 - \left[\frac{(1-\nu_s)}{2}\overline{(\kappa_{xx} - \kappa_{yy})} \right]^2 - [(1-\nu_s)\overline{\kappa_{xy}}]^2} \end{aligned} \tag{45}$$

where $\overline{x(\kappa_{xx} + \kappa_{yy})}$ and $\overline{y(\kappa_{xx} + \kappa_{yy})}$ are the average of $x(\kappa_{xx} + \kappa_{yy})$ and $y(\kappa_{xx} + \kappa_{yy})$, respectively.

The thin-film stresses in the Cartesian coordinates can be obtained from Eqs. (29)-(31) as

$$\begin{aligned} \sigma_{xx}^{(f)} + \sigma_{yy}^{(f)} &= -\frac{E_f}{1 - \nu_f}(\varepsilon_{xx}^m + \varepsilon_{yy}^m) \\ \begin{Bmatrix} \sigma_{xx}^{(f)} - \sigma_{yy}^{(f)} \\ \sigma_{xy}^{(f)} \end{Bmatrix} &= -\frac{E_f}{1 + \nu_f} \begin{Bmatrix} \varepsilon_{xx}^m - \varepsilon_{yy}^m \\ \varepsilon_{xy}^m \end{Bmatrix} \end{aligned} \tag{46}$$

Elimination of misfit strains from Eqs. (44) and (46) gives the relation between thin film stresses and system curvatures

$$\begin{Bmatrix} \sigma_{xx}^{(f)} + \sigma_{yy}^{(f)} \\ \sigma_{xx}^{(f)} - \sigma_{yy}^{(f)} \\ \sigma_{xy}^{(f)} \end{Bmatrix} = \frac{E_s h_s^2}{6(1-v_s^2)h_f} \begin{Bmatrix} (1+v_s)\overline{(\kappa_{xx}+\kappa_{yy})} \\ (1-v_s)\overline{(\kappa_{xx}-\kappa_{yy})} \\ (1-v_s)\overline{\kappa_{xy}} \end{Bmatrix} (1+ax+by) \tag{47}$$

7. Concluding remarks and discussion

The stresses and curvatures are given in terms of anisotropic misfit strains in Section 4. For uniform misfit strains in Cartesian coordinates, the direct relation (34) between the thin-film stresses and system curvatures is established, and it is identical to Freund and Suresh (2004). However, for axisymmetric normal and shear misfit strains in Sections 34 and 35, such a film stress-curvature relation cannot be established because some components of anisotropic misfit strains give vanishing system curvatures but non-vanishing film stresses. This observation of no direct relation between film stresses and system curvatures also holds for non-uniform, anisotropic misfit strains. It is somewhat puzzling why a direction relation can be established for uniform, anisotropic misfit strains (in Cartesian coordinates) as in Eq. (34) but not for non-uniform misfit strains.

The average curvatures in Cartesian coordinates provide an explanation. The average curvature sum over the entire thin film in Eq. (38) can be rewritten in terms of the Cartesian components as

$$\overline{(\kappa_{xx}+\kappa_{yy})} = -6\frac{E_f h_f}{1-v_f}\frac{1-v_s}{E_s h_s^2}\overline{\varepsilon_{xx}^m + \varepsilon_{yy}^m} \tag{48}$$

The curvature components $\kappa_{xx} - \kappa_{yy}$ and κ_{xy} in Cartesian coordinates can be obtained from $\kappa_{rr} - \kappa_{\theta\theta}$ and $\kappa_{r\theta}$ in Eqs. (27) and (28), and their average over the entire thin film gives

$$\begin{Bmatrix} \overline{\kappa_{xx}-\kappa_{yy}} \\ \overline{\kappa_{xy}} \end{Bmatrix} = -6\frac{E_f h_f}{1+v_f}\frac{1+v_s}{E_s h_s^2}\begin{Bmatrix} \overline{\varepsilon_{xx}^m - \varepsilon_{yy}^m} \\ \overline{\varepsilon_{xy}^m} \end{Bmatrix} \tag{49}$$

Eqs. (43) and (44) suggest that the average misfit strains (and average film stresses) can be linked directly to the average curvatures. In fact, they become identical to Eq. (32) if the average misfit strains are replaced by uniform misfit strains.

The subtraction of curvatures by their averages gives $\kappa_{xx} + \kappa_{yy} - \overline{\kappa_{xx} + \kappa_{yy}}$, $\kappa_{xx} - \kappa_{yy} - \overline{\kappa_{xx} - \kappa_{yy}}$ and $\kappa_{xy} - \overline{\kappa_{xy}}$ in terms of $\varepsilon_{xx}^m + \varepsilon_{yy}^m - \overline{\varepsilon_{xx}^m + \varepsilon_{yy}^m}$, $\varepsilon_{xx}^m - \varepsilon_{yy}^m - \overline{\varepsilon_{xx}^m - \varepsilon_{yy}^m}$ and $\varepsilon_{xy}^m - \overline{\varepsilon_{xy}^m}$. However, these relations cannot be inverted to express the misfit strain deviation $\varepsilon_{\alpha\beta}^m - \overline{\varepsilon_{\alpha\beta}^m}$ in terms of the curvature deviation $\kappa_{\alpha\beta} - \overline{\kappa_{\alpha\beta}}$. This is because all curvatures are related to the same displacement w such that their derivatives are not independent. For example, for axisymmetric misfit strains in Section 34, the derivatives of curvatures satisfy $\frac{d}{dr}[r^2(\kappa_{rr} - \kappa_{\theta\theta})] = r^2\frac{d}{dr}(\kappa_{rr} + \kappa_{\theta\theta})$. This relation becomes trivial for uniform curvatures. For non-uniform curvatures, however, it indicates that the derivatives of curvatures, or equivalently the curvature deviation $\kappa_{\alpha\beta} - \overline{\kappa_{\alpha\beta}}$, are not independent. This is the reason that the misfit strain deviation $\varepsilon_{\alpha\beta}^m - \overline{\varepsilon_{\alpha\beta}^m}$ cannot be solved from the curvature deviation $\kappa_{\alpha\beta} - \overline{\kappa_{\alpha\beta}}$.

However, for linear misfit strain distributions, the direct relation between the thin film stresses and system curvatures can be established.

The interface shear stresses are related to the gradient of misfit strains via Eq. (14), and cannot be given in terms of curvatures directly.

References

Brown, M.A., Park, T.S., Rosakis, A.J., Ustundag, E., Huang, Y., Tamura, N. and Valek, B. (2006), "A comparison of X-ray microdiffraction and coherent gradient sensing in measuring discontinuous curvatures in thin film: substrate systems", *J. Appl. Mech.*, **73**, 723-729.

Brown, M.A., Rosakis, A.J., Feng, X., Huang, Y. and Ustundag, E. (2007), "Thin film/substrate systems featuring arbitrary film thickness and misfit strain distributions: Part II. Experimental validation of the non-local stress-curvature relations", *Int. J. Solids Struct*., **44**, 1755-1767.

Feng, X., Huang, Y., Jiang, H., Ngo, D. and Rosakis, A.J. (2006), "The effect of thin film/substrate radii on the Stoney formula for thin film/substrate subjected to non-uniform axisymmetric misfit strain and temperature". *J. Mech. Mater. Struct.*, **1**, 1041-1054.

Finot, M., Blech, I.A., Suresh, S. and Fijimoto, H. (1997), "Large deformation and geometric instability of substrates with thin-film deposits" *J. Appl. Phys*., **81**, 3457-3464.

Freund, L.B. (2000), "Substrate curvature due to thin film mismatch strain in the nonlinear deformation range", *J. Mech. Phys. Solids*., **48**, 1159.

Freund, L.B. and Suresh, S. (2004), *Thin Film Materials; Stress, Defect Formation and Surface Evolution*. Cambridge University Press, Cambridge, U.K..

Huang, Y. and Rosakis, A.J. (2005), "Extension of Stoney's formula to non-uniform misfit strain distributions in thin film/substrate systems. The case of radial symmetry", *J. Mech. Phys. Solids.*, **53**, 2483-2500.

Huang, Y., Ngo, D. and Rosakis, A.J. (2005), "Non-uniform, axisymmetric misfit strain in thin films bonded on plate substrates/substrate systems: The relation between non-uniform film stresses and system curvatures", *Acta Mechanica Sinica.*, **21**, 362-370.

Huang, Y. and Rosakis, A.J. (2007), "Extension of Stoney's formula to arbitrary temperature distributions in thin film/substrate systems", *J. Appl. Mech*., **74**, 1225-1233.

Lee, H., Rosakis, A.J. and Freund, L.B. (2001), "Full field optical measurement of curvatures in ultra-thin film/ substrate systems in the range of geometrically nonlinear deformations", *J. Appl. Phys*., **89**, 6116-6129

Master, C.B. and Salamon, N.J. (1993), "Geometrically nonlinear stress-deflection relations for thin film/substrate systems", *Int. J. Engrg. Sci.*, **31**, 915-925.

Ngo, D., Feng, X., Huang, Y., Rosakis, A.J. and Brown, M.A. (2007), "Thin film/substrate systems featuring arbitrary film thickness and misfit strain distributions: Part I. Analysis for obtaining film stress from nonlocal curvature information", *Int. J. Solids Struct*., **44**, 1745-1754.

Ngo, D., Huang, Y., Rosakis, A.J. and Feng, X. (2006), "Spatially non-uniform, isotropic misfit strain in thin films bonded on plate substrates: the relation between non-uniform stresses and system curvatures", *Thin Solid Films*., **515**, 2220-2229.

Park, T.S. and Suresh, S. (2000), "Effects of line and passivation geometry on curvature evolution during processing and thermal cycling in copper interconnect lines", *Acta Materialia*., **48**, 3169-3175.

Salamon, N.J. and Masters, C.B. (1995), "Bifurcation in isotropic thin film/substrate plates", *Int. J. Solids Struct*., **32**, 473-481.

Shen, Y.L., Suresh, S. and Blech, I.A. (1996), "Stresses, curvatures, and shape changes arising from patterned lines on silicon wafers", *J. Appl. Phys*., **80**, 1388-1398.

Stoney, G.G. (1909), "The tension of metallic films deposited by electrolysis", *Proc. R. Soc. Lond*., A**82**, 172-175.

Wikstrom, A., Gudmundson, P. and Suresh, S. (1999a), "Thermoelastic analysis of periodic thin lines deposited on a substrate", *J. Mech. Phys. Solids.*, **47**, 1113-1130.

Wikstrom, A., Gudmundson, P. and Suresh, S. (1999b), "Analysis of average thermal stresses in passivated metal interconnects", *J. Appl. Phys*., **86**, 6088-6095.