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Electro-magneto-thermoelastic surface waves in non-homogeneous orthotropic granular half space

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Abstract. The effect of various parameters on the propagation of surface waves in electro-magneto thermoelastic orthotropic granular non-homogeneous medium subjected to gravity and initial compression has been studied. All material coefficients are obeyed the same exponent-law dependence on the depth of the granular elastic half space. Some special cases investigated by earlier researchers have also been deduced. Dispersion curves are computed numerically and presented graphically.

Keywords: inhomogeneous; granular media; gravity, initial stress; variable density; surface waves

1. Introduction

Seismic surface waves are a complex form of vibratory movement transmitted through a medium because in solid media there are two types of resistance to the external mechanical action for the propagation of these waves; the resistance to the changes of volume, or sizes of elements of the medium and the resistance to shear. For the frequencies of the seismic vibrations usually observed, liquids and gases only exhibit the formal type of resistance. The propagation of vibrations then reduces to transmission of cubical deformations through a medium and these are called longitudinal waves. In solids, these vibrations becomes complicated because at the boundaries of, for instant, layers in the Earth, an incident longitudinal wave will produce four waves in a general case, namely, a longitudinal reflected and a transverse reflected wave, and two analogous refracted waves. The same holds for an incident non-polarized transverse wave which produces a longitudinal reflected and transverse reflected wave and a similar pair of refracted waves. This is the main cause of seismic waves.

Surface waves which are the combination of compression and shear waves are called Rayleigh waves. These waves propagate with slightly lesser speed as compared to bulk shear waves. Whereas, the Stoneley arises due to interaction of waves with curved or plane interface between solid and liquid media, so Stoneley waves are also known as interfacial waves. Stoneley waves are a combination of compression and shear waves. The study of surface waves in granular medium is useful in the field of soil mechanics.

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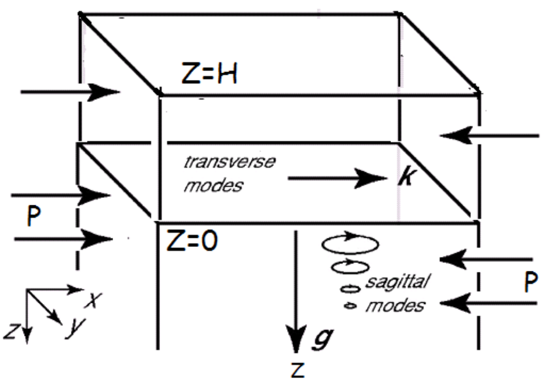

Fig. 1 Schematic of the problem

Waves in granular media are more complex due to the heterogeneous nature of these media. The study of granular medium is useful for soil mechanics, mining engineering, and so forth. Their traveling speed is slightly smaller than bulk shear waves. Paria (1960) was the first who developed the theory of surface waves in granular media. Bromwich (1898) considered gravity as a body force in his research. The work of Bromwich was further used by Love (1911) to show the effect of gravity on Stoneley waves. The literature on surface waves is available in the books of Jeffreys (1959), Stoneley (1924) and Bullen (1965). Britan and Ben-Dor (2006) studied the dynamical behavior of granular materials. Also, the concept of gravity was introduced by Ahmed (1999, 2005) in his study of surface waves in granular medium. Sharma *et al*. (2007) discussed effects of inviscid fluid loadings on Stoneley waves in microstretch thermoelastic continua. El-Maghraby (2008) presented a paper on 2-D generalized thermoelasticity problem for a half-space under the action of a body force.

Hou *et al*. (2013) gave the general solution for 3-D steady-state isotropic thermoelastic materials with applications. Cicco and Ieşan (2013) studied thermal effects in anisotropic porous elastic rods. Kakar and Gupta (2012) studied propagation of Love waves in a non-homogeneous orthotropic layer under '*P*' overlying semi-infinite non-homogeneous medium. Kakar (2013) discussed the effect of impulsive line source and non-homogeneity on the propagation of SH-wave in elastic medium. Recently, Kakar and Gupta (2013) investigated torsional surface waves in a non-homogeneous isotropic layer over viscoelastic half-space.

In this paper, the combined effect of magnetic field, electric field, gravity, temperature, initial compression and non- homogeneity on the propagation of surface waves in a granular half space is studied. We consider the medium consists of large or small grains and is discontinuous in nature. These grains possess both translational and rotational motion (Fig. 1). The friction is produced by the movement of these grains; therefore we have taken the concept of friction in the governing equations of the problem.

2. Formulation of the problem

Let M_1 and M_2 be two non-homogeneous electro-magneto-thermoelastic orthotropic granular media. They are connected in such a manner that there is no relative motion between them. z-axis is taken vertically downwards into the medium M_1. Here we assume the free surface and interface of orthotropic electro-magneto-thermoelastic granular layer lying on non-homogeneous electro-magneto-thermoelastic orthotropic granular half space of thickness $z = H$. Surface wave is assumed to propagate in the positive direction of x-axis. For 2 D surface wave propagation, the displacement components along x and z-directions are non-zero i.e., u_1 and u_3 are non-zero while u_2 is zero. Due to rotation of grains, there exists non-symmetric stress tensor and stress couple given by $\sigma_{ij} \neq \sigma_{ji}$ and $M_{ij} \neq M_{ji}$. The stress tensor σ_{ij} can be expressed in symmetric and anti-symmetric tensors τ_{ij} and τ'_{ij} given by

$$\sigma_{ij} = \tau_{ij} + \tau'_{ij}, \tag{1}$$

where

$$\tau_{ij} = \frac{1}{2}\left(\sigma_{ij} + \sigma_{ji}\right) \text{ and } \tau'_{ij} = \frac{1}{2}\left(\sigma_{ij} - \sigma_{ji}\right). \tag{2}$$

Further symmetric strain tensor are given by relation

$$e_{ij} = e_{ji} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{3}$$

The anti-symmetric stresses τ'_{ij} are given by (Ahmed 2005)

$$\tau'_{23} = -F\frac{\partial \xi}{\partial t}, \quad \tau'_{31} = -F\frac{\partial \eta}{\partial t}, \quad \tau'_{12} = -F\frac{\partial \zeta}{\partial t}, \quad \tau'_{11} = \tau'_{22} = \tau'_{33} = 0, \tag{4}$$

where, F is the co-efficient of friction and the rotation vector of grain about its C.G. is given by (ξ, η, ζ).

The stress couple M_{ij} is given by (Bhattacharya 1969)

$$M_{ij} = M v_{ij}, \tag{5}$$

where M be the third elastic constant

$$v_{11} = \frac{\partial \xi}{\partial x}, \quad v_{22} = 0, \quad v_{33} = \frac{\partial \xi}{\partial z}, \quad v_{23} = 0, \quad v_{31} = \frac{\partial \xi}{\partial z}, \quad v_{12} = \frac{\partial}{\partial x}\left(\omega_2 + \eta\right),$$

$$v_{32} = \frac{\partial}{\partial z}\left(\omega_2 + \eta\right), \quad v_{13} = \frac{\partial \zeta}{\partial x}, \quad v_{21} = 0,$$

where

$$\omega_1 = \frac{1}{2}\left(u_{3,y} - u_{2,z}\right), \quad \omega_2 = \frac{1}{2}\left(u_{1,z} - u_{3,x}\right), \quad \omega_3 = \frac{1}{2}\left(u_{2,x} - u_{1,y}\right) \tag{6}$$

Let g be the acceleration due to gravity and ρ be the density of the material, The state of initial stresses are given by

$$\sigma_{ij} \begin{array}{l} = \sigma;\ i = j \\ = 0;\ i \neq j \end{array} \quad \text{where } i, j = 1, 2, 3$$

where, σ is a function of z. The equation in terms of initial stresses can be written as

$$\sigma_{,x} = 0; \quad \sigma_{,z} = -\rho g = 0, \tag{7}$$

For magneto elastic problems, the basic equations will be electromagnetism and elasticity. Therefore Maxwell's equations are

$$\vec{\nabla} \cdot \vec{E} = 0, \quad \vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \vec{\nabla} \times \vec{B} = \mu_e \varepsilon_e \frac{\partial \vec{E}}{\partial t}. \tag{8a}$$

where, $\vec{E}$, $\vec{B}$, μ_e and ε_e are electric field, magnetic field, permeability and permittivity of the medium.

And

$$\vec{H}(0, 0, H) = \vec{H}_0 + \vec{H}_i \tag{8b}$$

Suppose granular elastic media is under the influence of constant primary magnetic field $\vec{H}_0$ acting on the y-axis, gravity g, perturbation $\vec{H}_i$ and an initial stress P along the x-axis. Then the dynamical equations of motion in the x and z dimensions of granular medium under gravity are (Bhattacharya 1969)

$$\frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{13}}{\partial z} - \rho g \frac{\partial u_3}{\partial x} + P\left(\frac{\partial \omega_3}{\partial y} - \frac{\partial \omega_2}{\partial z}\right) + \frac{\partial \tau'_{31}}{\partial z} = \rho \frac{\partial^2 u_1}{\partial t^2}, \tag{9a}$$

$$\frac{\partial \sigma_{12}}{\partial x} + \frac{\partial \sigma_{23}}{\partial z} + P\left(\frac{\partial \omega_3}{\partial x}\right) + \frac{\partial \tau'_{12}}{\partial x} - \frac{\partial \tau'_{23}}{\partial z} = 0, \tag{9b}$$

$$\frac{\partial \sigma_{13}}{\partial x} + \frac{\partial \sigma_{33}}{\partial z} + \rho g \frac{\partial u_1}{\partial x} + P\left(\frac{\partial \omega_2}{\partial x}\right) - \frac{\partial \tau'_{31}}{\partial x} = \rho \frac{\partial^2 u_1}{\partial t^2}, \tag{9c}$$

and

$$\tau'_{23} + \sigma_{23} - \sigma_{32} + \frac{\partial M_{11}}{\partial x} + \frac{\partial M_{31}}{\partial z} = 0, \tag{10a}$$

$$\tau'_{31} + \sigma_{31} - \sigma_{13} + \frac{\partial M_{12}}{\partial x} + \frac{\partial M_{32}}{\partial z} = 0, \tag{10b}$$

$$\tau'_{12} + \sigma_{12} - \sigma_{21} + \frac{\partial M_{13}}{\partial x} + \frac{\partial M_{33}}{\partial z} = 0. \tag{10c}$$

3. Solution of problem for Stoneley waves

Further the stress components in presence of electric, magnetic and thermal field are given by

$$\sigma_{11} = (C_{11} + P)\frac{\partial u_1}{\partial x} + (C_{13} + P)\frac{\partial u_3}{\partial z} + \Delta\mu_e H_0^2 + \Delta\varepsilon_e E_0^2 - \Xi T, \tag{11a}$$

$$\sigma_{33} = C_{13}\frac{\partial u_1}{\partial x} + C_{33}\frac{\partial u_3}{\partial z} + \Delta\mu_e H_0^2 + \Delta\varepsilon_e E_0^2 - \Xi T, \tag{11b}$$

$$\sigma_{13} = C_{44}\left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x}\right). \tag{11c}$$

Inserting Eqs. (11), (5), (6) and (4) in Eqs. (9a)-(9c); we get

$$\begin{aligned}
&(C_{11} + P)\frac{\partial^2 u_1}{\partial x^2} + (C_{13} + P)\frac{\partial^2 u_3}{\partial x \partial z} + \mu_e H_0^2 \frac{\partial \Delta}{\partial x} + \varepsilon_e E_0^2 \frac{\partial \Delta}{\partial x} \\
&+ C_{44}\frac{\partial}{\partial z}\left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x}\right) + \left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x}\right)\frac{\partial}{\partial z}(C_{44}) - \Xi \frac{\partial T}{\partial x} \\
&- \frac{P}{2}\frac{\partial}{\partial z}\left(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}\right) - \rho g \frac{\partial u_3}{\partial x} + \frac{\partial}{\partial z}\left(-F\frac{\partial \eta}{\partial t}\right) = \rho \frac{\partial^2 u_1}{\partial t^2},
\end{aligned} \tag{12a}$$

$$\frac{\partial}{\partial z}\left(F\frac{\partial \xi}{\partial t}\right) - \frac{\partial}{\partial x}\left(F\frac{\partial \zeta}{\partial t}\right) = 0, \tag{12b}$$

$$\begin{aligned}
&C_{44}\frac{\partial}{\partial x}\left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(C_{13}\frac{\partial u_1}{\partial x} + C_{33}\frac{\partial u_3}{\partial z}\right) - \Xi\frac{\partial T}{\partial z} + \frac{\partial}{\partial z}\left(\mu_e H_0^2 \Delta\right) \\
&+ \frac{\partial}{\partial z}\left(\varepsilon_e E_0^2 \Delta\right) - \frac{P}{2}\frac{\partial}{\partial x}\left(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}\right) + \rho g \frac{\partial u_1}{\partial x} + \frac{\partial}{\partial x}\left(F\frac{\partial \eta}{\partial t}\right) = \rho\frac{\partial^2 u_3}{\partial t^2},
\end{aligned} \tag{12c}$$

$$-F\frac{\partial \xi}{\partial t} + M\nabla^2 \xi + \frac{\partial \xi}{\partial z}\frac{\partial (M)}{\partial z} = 0, \tag{12d}$$

$$-F\frac{\partial \eta}{\partial t} + M\nabla^2(\eta + \omega_2) + \frac{\partial}{\partial z}(\omega_2 + \eta)\frac{\partial M}{\partial z} = 0, \tag{12e}$$

$$-F\frac{\partial \zeta}{\partial t} + M\nabla^2 \zeta + \frac{\partial \zeta}{\partial z}\frac{\partial (M)}{\partial z} = 0, \tag{12f}$$

where, $\Delta = \dfrac{\partial u_1}{\partial x} + \dfrac{\partial u_3}{\partial z}$.

We assume the non-homogeneities of the granular half-space as

$$\begin{aligned}
&C_{ij} = e_{ij}\, e^{mz}, \quad \rho = \rho_0\, e^{mz}, \qquad F = F_0\, e^{mz}, \qquad M = M_0\, e^{mz}, \\
&P = P_0\, e^{mz}, \quad \mu_e = (\mu_e)_0\, e^{mz}, \quad \varepsilon_e = (\varepsilon_e)_0\, e^{mz}, \quad \Xi = \Xi_0\, e^{mz}
\end{aligned}$$

where, e_{ij}, ρ_0, F_0, M_0, P_0, $(\mu_e)_0$, $(\varepsilon_e)_0$, Ξ_0 are constants.

Inserting inhomogeneities in Eqs. (11a)-(11c), we get

$$(e_{11} + P_0)\frac{\partial^2 u_1}{\partial x^2} + (\mu_e)_0 H_0^2 \frac{\partial}{\partial x}\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_3}{\partial z}\right) + (\varepsilon_e)_0 E_0^2 \frac{\partial}{\partial x}\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_3}{\partial z}\right) \tag{13a}$$

$$\begin{aligned}&+e_{44}\frac{\partial}{\partial z}\left(\frac{\partial u_1}{\partial z}+\frac{\partial u_3}{\partial x}\right)-\Xi_0\frac{\partial T}{\partial x}+\left(\frac{\partial u_1}{\partial z}+\frac{\partial u_3}{\partial x}\right)m(e_{44})-\frac{P_0}{2}\frac{\partial}{\partial z}\left(\frac{\partial u_1}{\partial z}-\frac{\partial u_3}{\partial x}\right)\\&+(e_{13}+P_0)\frac{\partial^2 u_3}{\partial x\partial z}-\rho_0 g\frac{\partial u_3}{\partial x}-F_0\frac{\partial}{\partial t}\left(\frac{\partial \eta}{\partial z}\right)-F_0 m\frac{\partial \eta}{\partial t}=\rho_0\frac{\partial^2 u_1}{\partial t^2},\end{aligned} \tag{13a}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \xi}{\partial z}-\frac{\partial \zeta}{\partial x}\right)+m\left(\frac{\partial \xi}{\partial t}\right)=0, \tag{13b}$$

$$\begin{aligned}&e_{44}\frac{\partial}{\partial x}\left(\frac{\partial u_1}{\partial z}+\frac{\partial u_3}{\partial x}\right)+e_{13}\frac{\partial^2 u_1}{\partial z\partial x}+me_{13}\frac{\partial u_1}{\partial x}+e_{33}\frac{\partial^2 u_3}{\partial z^2}+e_{33}m\frac{\partial u_3}{\partial z}\\&+(\varepsilon_e)_0 E_0^2\left[m\left(\frac{\partial u_1}{\partial x}+\frac{\partial u_3}{\partial z}\right)+\frac{\partial}{\partial z}\left(\frac{\partial u_1}{\partial x}+\frac{\partial u_3}{\partial z}\right)\right]\\&+(\mu_e)_0 H_0^2\left[m\left(\frac{\partial u_1}{\partial x}+\frac{\partial u_3}{\partial z}\right)+\frac{\partial}{\partial z}\left(\frac{\partial u_1}{\partial x}+\frac{\partial u_3}{\partial z}\right)\right]\\&+\rho_0 g\frac{\partial u_1}{\partial x}+F_0\frac{\partial}{\partial t}\left(\frac{\partial \eta}{\partial x}\right)-\Xi_0\left(\frac{\partial T}{\partial z}+mT\right)-\frac{P_0}{2}\frac{\partial}{\partial x}\left(\frac{\partial u_1}{\partial z}-\frac{\partial u_3}{\partial x}\right)=\rho_0\frac{\partial^2 u_3}{\partial t^2},\end{aligned} \tag{13c}$$

$$-F_0\frac{\partial \xi}{\partial t}+M_0\nabla^2\xi+\xi_{,z}\frac{\partial}{\partial z}(M_0)=0, \tag{13d}$$

$$-F_0\frac{\partial \eta}{\partial t}+M_0\nabla^2(\eta+\omega_2)+\frac{\partial}{\partial z}(\omega_2+\eta)\frac{\partial (M_0)}{\partial z}=0, \tag{13e}$$

$$-F_0\frac{\partial z}{\partial t}+M_0\nabla^2\zeta+\zeta_{,z}\frac{\partial (M_0)}{\partial z}=0, \tag{13f}$$

We introduce displacement potentials in terms of displacement components given by

$$u_1=\frac{\partial \phi}{\partial x}-\frac{\partial \psi}{\partial z} \qquad u_3=\frac{\partial \phi}{\partial z}+\frac{\partial \psi}{\partial x}. \tag{14}$$

Introducing Eq. (14) into Eqs. (13a)-(13f), we get

$$\begin{aligned}&\left(e_{11}+P_0+(\mu_e)_0 H_0^2+(\varepsilon_e)_0 E_0^2\right)\frac{\partial^2 \phi}{\partial x^2}+\left(e_{13}+2e_{44}+P_0+(\mu_e)_0 H_0^2+(\varepsilon_e)_0 E_0^2\right)\frac{\partial^2 \phi}{\partial z^2}\\&+2me_{44}\frac{\partial \phi}{\partial z}-(\rho_0 g+me_{44})\frac{\partial \phi}{\partial z}-\Xi_0 T=\rho_0\frac{\partial^2 \phi}{\partial t^2},\end{aligned} \tag{15a}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \xi}{\partial z}-\frac{\partial \zeta}{\partial x}\right)+m\left(\frac{\partial \xi}{\partial t}\right)=0, \tag{15b}$$

$$\left(e_{44}+\frac{P_0}{2}\right)\frac{\partial^2\psi}{\partial x^2}+\left(e_{33}-e_{13}-e_{44}+\frac{P_0}{2}\right)\frac{\partial^2\psi}{\partial z^2}+e_{33}m\frac{\partial\psi}{\partial z}$$
$$+\left(me_{13}+(\mu_e)_0H_0^2+(\varepsilon_e)_0E_0^2+\rho_0 g\right)\frac{\partial\phi}{\partial x}+F_0\frac{\partial\eta}{\partial t}=\rho_0\frac{\partial^2\psi}{\partial t^2}, \tag{15c}$$

$$\nabla^2\xi-a\frac{\partial\xi}{\partial t}+m\frac{\partial\xi}{\partial z}=0, \tag{15d}$$

$$\nabla^2\eta-a\frac{\partial\eta}{\partial t}+m\left[\frac{\partial\eta}{\partial z}-\nabla^2\left(\frac{\partial\psi}{\partial z}\right)\right]=0, \tag{15e}$$

$$\nabla^2\zeta-a\frac{\partial\zeta}{\partial t}+m\frac{\partial\zeta}{\partial z}=0. \tag{15f}$$

where

$$a=\frac{F_0}{M_0} \tag{16}$$

We use Fourier's law of heat conduction for determination of T i.e.

$$p\nabla^2T=C_v\frac{\partial T}{\partial t}+T_0G_L\frac{\partial}{\partial t}\left(\nabla^2\phi\right), \tag{17}$$

where, K is the thermal conductivity and C_v is the specific heat of the body at constant volume, let Eq. (17) obeys the law as given by $K = K_0\, e^{mz}$, $p = K_0 / \rho_0$.

Therefore, Eq. (17) can be written as

$$K_0\nabla^2T=\rho_0C_v\frac{\partial T}{\partial t}+\Xi_0T_0\nabla^2\frac{\partial\phi}{\partial t} \tag{18}$$

Eliminating T from Eq. (17) and Eq. (15a); we get

$$\left(\nabla^2-\frac{\partial}{\partial t}\left(\frac{\rho_0C_v}{K_0}\right)\right)\begin{bmatrix}\left(e_{11}+P_0+(\mu_e)_0H_0^2+(\varepsilon_e)_0E_0^2\right)\dfrac{\partial^2\phi}{\partial x^2}\\+\left(e_{13}+2e_{44}+P_0+(\mu_e)_0H_0^2+(\varepsilon_e)_0E_0^2\right)\dfrac{\partial^2\phi}{\partial z^2}\\+2me_{44}\dfrac{\partial\phi}{\partial z}-(me_{44}-\rho_0 g)\dfrac{\partial\psi}{\partial x}-\rho_0\dfrac{\partial^2\phi}{\partial t^2}\end{bmatrix}-\nabla^2\left(\frac{\Xi_0^2T_0}{\rho_0\Theta}\right)\left(\frac{\partial\phi}{\partial t}\right)=0,$$

or

$$\left(\nabla^2-\frac{1}{\Theta}\frac{\partial}{\partial t}\right)\begin{bmatrix}\left(e_{11}+P_0+(\mu_e)_0H_0^2+(\varepsilon_e)_0E_0^2\right)\dfrac{\partial^2\phi}{\partial x^2}\\+\left(e_{13}+2e_{44}+P+(\mu_e)_0H_0^2+(\varepsilon_e)_0E_0^2\right)\dfrac{\partial^2\phi}{\partial z^2}\\+2me_{44}\dfrac{\partial\phi}{\partial z}+\left(me_{44}-\rho_0 g\right)\dfrac{\partial\psi}{\partial x}-\rho_0\dfrac{\partial^2\phi}{\partial t^2}\end{bmatrix}-\Lambda\nabla^2\frac{\partial\phi}{\partial t}=0, \tag{19}$$

where

$$\Theta=\frac{K_0}{\rho_0 C_v},\qquad \Lambda=\frac{\Xi_0^2 T_0}{\rho_0\Theta}. \tag{20}$$

Eliminating η from Eqs. (15c) and (15e); we get

$$\left(\nabla^2-a\frac{\partial}{\partial t}+m\frac{\partial}{\partial z}\right)\begin{bmatrix}\left(e_{44}+\dfrac{P_0}{2}\right)\dfrac{\partial^2\psi}{\partial x^2}\\+\left(e_{33}-e_{13}-e_{44}+\dfrac{P_0}{2}\right)\dfrac{\partial^2\psi}{\partial z^2}\\+\begin{pmatrix}me_{13}+(\mu_e)_0H_0^2\\+(\varepsilon_e)_0E_0^2+\rho_0 g\end{pmatrix}\dfrac{\partial\phi}{\partial x}\\+e_{33}m\dfrac{\partial\psi}{\partial z}-\rho_0\dfrac{\partial^2\psi}{\partial t^2}\end{bmatrix}+F_0\nabla^4\left(\frac{\partial\psi}{\partial t}\right)+mF_0\nabla^4\left(\frac{\partial^2\psi}{\partial z\partial t}\right)=0. \tag{21}$$

To solve Eqs. (15a)-(15f), we assume that

$$\begin{gathered}\phi(x,z,t)=\phi_1(z)\,e^{1(lx-bt)},\quad \psi(x,z,t)=\psi_1(z)\,e^{1(lx-bt)},\quad \xi(x,z,t)=\xi_1(z)\,e^{1(lx-bt)},\\ \eta(x,z,t)=\eta_1(z)\,e^{i(lx-bt)},\quad \zeta(x,z,t)=\zeta_1(z)\,e^{i(lx-bt)}.\end{gathered} \tag{22}$$

Put Eq. (22) in Eqs. (19) and (21), we get

$$\left[A\left(\frac{d}{dz}\right)^4+B\left(\frac{d}{dz}\right)^3+D\left(\frac{d}{dz}\right)^2+E\left(\frac{d}{dz}\right)+G\right]\phi_1+\left[I\left(\frac{d}{dz}\right)^2+J\right]\psi_1=0, \tag{23}$$

$$\left[L\left(\frac{d}{dz}\right)^4+N\left(\frac{d}{dz}\right)^3+O\left(\frac{d}{dz}\right)^2+Q\left(\frac{d}{dz}\right)+S\right]\psi_1+\left[T\left(\frac{d}{dz}\right)^2+U\left(\frac{d}{dz}\right)+V\right]\phi_1=0. \tag{24}$$

where, $A=(e_{13}+2e_{44}+P_0+(\mu_e)_0\,H_0^2+(\varepsilon_e)_0\,E_0^2)$, $B=2me_{44}$

$$D=-\left(e_{11}+P_0+(\mu_e)_0 H_0^2+(\varepsilon_e)_0 E_0^2\right)l^2-2me_{44}l^2+\rho_0 b^2$$
$$+\frac{ib\left(e_{11}+P_0+(\mu_e)_0 H_0^2+(\varepsilon_e)_0 E_0^2\right)}{\Theta}+ib\Lambda,$$

$$E=-2me_{44}l^2+\frac{2imbe_{44}}{\Theta},\ I=il(me_{44}-\rho_0 g),\ J=(me_{44}-\rho_0 g)\left(-il^3-\frac{bl}{\Theta}\right)$$

$$G=\left(e_{13}+2e_{44}+P_0+(\mu_e)_0 H_0^2+(\varepsilon_e)_0 E_0^2\right)l^4-\rho_0 b^2 l^2+\frac{i\rho_0 b^3}{\Theta}$$
$$-ib\Lambda l^2-\frac{ib^2\left(e_{13}+2e_{44}+P_0+(\mu_e)_0 H_0^2+(\varepsilon_e)_0 E_0^2\right)l^2}{\Theta}$$

$$L=\left(e_{33}-e_{13}-e_{44}+\frac{P_0}{2}\right)ibF_0,\ N=me_{33}+m\left(e_{33}-e_{13}-e_{44}+\frac{P_0}{2}\right)-ibmF_0,$$

$$O=-l^2 e_{44}\frac{P_0}{2}-l^2-l^2\left(e_{33}-e_{13}-e_{44}+\frac{P_0}{2}\right)$$
$$+\rho_0 b^2+iab\left(e_{33}-e_{13}-e_{44}+\frac{P_0}{2}\right)+me_{33}+2ibl^2F_0, \tag{25}$$

$$Q=-ml^2e_{33}+imae_{33}-ml^2\left(e_{44}+\frac{P_0}{2}\right)+\rho_0 b^2 m+2ibF_0 l^2,$$

$$S=\left(e_{44}+\frac{P_0}{2}\right)l^4-\rho_0 b^2 l^2-ibal^2+\left(e_{44}+\frac{P_0}{2}\right)ib^3 a\rho_0-ibF_0 l^4,$$

$$T=il\left(me_{13}+m(\mu_e)_0 H_0^2+m(\varepsilon_e)_0 E_0^2+\rho_0 g\right),\ U=iml\left(me_{13}+m(\mu_e)_0 H_0^2+m(\varepsilon_e)_0 E_0^2+\rho_0 g\right),$$

$$V=-il^3\left(me_{13}+m(\mu_e)_0 H_0^2+m(\varepsilon_e)_0 E_0^2+\rho_0 g\right)-alb\left(me_{13}+m(\mu_e)_0 H_0^2+m(\varepsilon_e)_0 E_0^2+\rho_0 g\right)$$

Therefore the solutions of Eq. (23) and Eq. (24) is of the form

$$\phi_1=A_j e^{-\lambda_j z}, \tag{26}$$

$$\psi_1=B_j e^{-\lambda_j z},\quad j=3,4,5,6 \tag{27}$$

where , λ_j (j = 3, 4, 5, 6) are the real roots of the following equation

$$a_1D^8+a_2D^7+a_3D^6+a_4D^5+a_5D^4+a_6D^3+a_7D^2+a_8D+a_9=0 \tag{28}$$

where

$$a_1=AL,\ a_{21}=AN+BL,\quad a_3=AO+BN+DL,\quad a_4=AQ+BO+DN+EL, \tag{29}$$

$$a_5 = AS + BQ + DO + EN + GL - IT, \quad a_6 = BS + DQ + EO + GN - IU,$$
$$a_7 = DS + EQ + GI - IV - JT, \quad a_8 = ES + GQ - JU, \; a_9 = GS - JV. \tag{29}$$

where A, B, D, E, G, I, J, L, N, O, Q, S, T, U and V are given by Eq. (25). Further the constants A_j, B_j (j = 3, 4, 5, 6) are related by means of Eq. (23). Equating the coefficients of $e^{-\lambda jz}$ (j = 3, 4, 5, 6) to zero and using Eqs. (23) and (24); we get

$$A_j = n_j B_j \quad (j = 3, 4, 5), \tag{30}$$

$$n_j = \frac{-I\lambda_j^2 - J}{A\lambda_j^4 - B\lambda_j^3 + D\lambda_j^2 - E\lambda_j + G} \quad (j = 3, 4, 5, 6) \tag{31}$$

using Eqs. (15e), (25) and (27), we get

$$\eta_1 = \Upsilon_j \left(B_j e^{-\lambda_j z} \right) \quad (j = 3, 4, 5, 6) \tag{32}$$

where

$$\Upsilon_j = \frac{\lambda_j^4 - m\lambda_j^3 - 2l^2\lambda_j^2 + ml^2\lambda_j + l^4}{\lambda_j^2 - m\lambda_j + iba - l^2} \tag{33}$$

using Eqs. (15a), (22) and (26), we get

$$T = P_j \left(A_j e^{-\lambda_j z} \right) \quad (j = 3, 4, 5, 6), \tag{34}$$

where

$$P_j = \frac{1}{\Xi_0} \begin{bmatrix} -l^2 \left(e_{33} - e_{13} - e_{44} + \dfrac{P_0}{2} \right) + \left(me_{13} + m(\mu_e)_0 H_0^2 + m(\varepsilon_e)_0 E_0^2 \right) \lambda_j^2 \\ -2me_{44}\lambda_j + il(me_{44} - \rho_0 g) \dfrac{1}{\Upsilon_j} + b^2 \rho_0 \end{bmatrix} \tag{35}$$

Further substituting Eq. (22) into Eqs. (15b), (15d) and (15f), we get

$$(D + m)\xi_1 - il\zeta_1 = 0, \tag{36}$$

$$\left(D^2 + m + \left(iab - l^2 \right)^2 \right) \xi_1 = 0, \tag{37}$$

$$\left[\left(D^2 + m + \left(iab - l^2 \right)^2 \right) \right] \zeta_1 = 0. \tag{38}$$

The solutions of Eqs. (37) and (38) are given by

$$\xi_1 = A_1 e^{\alpha z} + A_1 z \, e^{-\beta z}, \tag{39}$$

$$\zeta_1 = B_1 e^{\alpha z} + B_2 z\, e^{-\beta z}, \tag{40}$$

where

$$\alpha = \frac{-m + \sqrt{m^2 - 4(iab - l^2)^2}}{2}, \quad \beta = \frac{m + \sqrt{m^2 - 4(iab - l^2)^2}}{2}, \quad m^2 - 4(iab - l^2)^2 > 0. \tag{41}$$

Substituting Eq. (39), Eq. (40) into Eq. (36), we get

$$(A_1\,\alpha + A_1\, m)e^{\alpha z} + (-A_2\,\beta + m\,A_2)e^{-\beta z} = il(B_1\, e^{\alpha z} + B_2\, e^{-\beta z}). \tag{42}$$

Equating the co-efficients of $e^{\alpha z}$ and $e^{-\beta z}$ to zero in Eq. (42), we get

$$A_1 = \frac{ilB_1}{\alpha + m}, \quad A_2 = \frac{ilB_2}{m - \beta}. \tag{43}$$

Let λ_0, μ_0, ρ_0, F_0, M_0 are the characteristics of layer and $\bar{\lambda}_0$, $\bar{\mu}_0$, $\bar{\rho}_0$, $\bar{F}_0$, $\bar{M}_0$ are the characteristics of half-space, also for the lower half-space and description of surface wave propagation ξ_1, ϕ_1, η_1, ψ_1, ζ_1 goes to zero as $z \to \infty$, also the non-homogeneity constant m is replaced by constant $\bar{m}$ for lower granular half-space also it is assumed that the real parts of (j = 3, 4, 5, 6) are positive.

Thus for lower half-space

$$\bar{\phi}_1 = \bar{A}_j e^{-\bar{\lambda}_j z}, \tag{44}$$

$$\bar{\psi}_1 = \bar{B}_j e^{-\bar{\lambda}_j z}, \tag{45}$$

$$\bar{\eta}_1 = \bar{\Upsilon}_j\left(\bar{B}_j e^{-\bar{\lambda}_j z}\right), \tag{46}$$

$$\bar{T} = \bar{P}_j\left(\bar{A}_j e^{-\bar{\lambda}_j z}\right), \tag{47}$$

$$\bar{\xi}_1 = \frac{il}{\bar{m} - \bar{\beta}}\bar{B}_2 e^{-\bar{\beta} z}, \tag{48}$$

$$\bar{\zeta}_1 = \bar{B}_2 e^{-\bar{\beta} z} \quad (j = 3, 4, 5, 6). \tag{49}$$

4. Boundary conditions and dispersion equation

Case - I

The boundary conditions on interface $z = H$ are

$$
\begin{array}{ll}
\text{(i)} & u_1 = \bar{u}_1, \\
\text{(ii)} & u_3 = \bar{u}_3, \\
\text{(iii)} & \xi = \bar{\xi}, \\
\text{(iv)} & \eta = \bar{\eta}, \\
\text{(v)} & \zeta = \bar{\zeta}, \\
\text{(vi)} & M_{33} = \bar{M}_{33}, \\
\text{(vii)} & M_{31} = \bar{M}_{31}, \\
\text{(viii)} & M_{32} = \bar{M}_{32}, \\
\text{(ix)} & \sigma_{33} = \bar{\sigma}_{33}, \\
\text{(x)} & \sigma_{31} = \bar{\sigma}_{31}, \\
\text{(xi)} & \sigma_{32} = \bar{\sigma}_{32}. \\
\text{(xii)} & T = \bar{T} \\
\text{(xiii)} & \dfrac{\partial T}{\partial z} + \theta T = \dfrac{\partial \bar{T}}{\partial z} + \theta \bar{T}
\end{array}
\tag{50}
$$

Case - II

The boundary conditions on free surface $z = 0$ are

$$
\begin{array}{ll}
\text{(xiv)} & M_{33} = 0, \\
\text{(xv)} & M_{31} = 0, \\
\text{(xvi)} & M_{32} = 0, \\
\text{(xvii)} & \sigma_{33} = 0, \\
\text{(xviii)} & \sigma_{31} = 0, \\
\text{(xix)} & \sigma_{32} = 0,
\end{array}
\tag{51}
$$

where

$$M_{33} = M\frac{\partial \xi}{\partial z}, \quad M_{32} = M\frac{\partial}{\partial z}\left(\eta - \nabla^2\psi\right), \quad M_{31} = M\frac{\partial \xi}{\partial z},$$

$$\sigma_{33} = C_{13}\frac{\partial^2\phi}{\partial x^2} + C_{33}\frac{\partial^2\phi}{\partial z^2} + \left(C_{13} - C_{33}\right)\frac{\partial^2\phi}{\partial x\partial z} + \mu_e H_0^2\nabla^2\phi + \Delta\varepsilon_e E_0^2\nabla^2\phi - \Xi T,$$

$$\sigma_{32} = -F\frac{\partial \xi}{\partial t}, \quad \sigma_{31} = C_{44}\left(\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial z^2} + 2\frac{\partial^2\psi}{\partial x\partial z}\right) - F\frac{\partial \eta}{\partial t}.$$

where, θ is the ratio of the coefficients of heat transfer to the thermal conductivity.

From the boundary conditions (iii), (v), (vi) and (vii), we get

$$\frac{B_1}{\alpha + m}e^{\alpha H} + \frac{B_2}{m - \beta}e^{-\beta H} = \frac{\bar{B}_2}{\bar{m} - \bar{\beta}}e^{-\bar{\beta} H}, \tag{52}$$

$$B_1\, e^{\alpha H} + B_2\, e^{-\beta H} = \overline{B}_2\, e^{-\overline{\beta} H}, \tag{53}$$

$$M_0\, e^{mH}\left[B_1\, \alpha\, e^{\alpha H} - B_2\, \beta\, e^{-\beta H}\right] = -\overline{M}_0\, e^{-\overline{m} H}\, \overline{\beta}\, \overline{B}_2\, e^{-\beta H}, \tag{54}$$

$$M_0\, e^{mH}\left[\frac{B_1\, \alpha\, e^{\alpha H}}{\alpha + m} - \frac{B_2\, \beta\, e^{-\beta H}}{m - \beta}\right] = -\overline{M}_0\, e^{-\overline{m} H}\, \frac{\overline{\beta}\, \overline{B}_2}{\overline{m} - \overline{\beta}}\, e^{-\beta H}. \tag{55}$$

From Eqs. (52) to (55), we have

$$B_1 = B_2 = \overline{B}_2 = 0 \tag{56}$$

i.e.,

$$\xi = \zeta = \overline{\zeta} = \overline{\xi} = 0. \tag{57}$$

The other boundary conditions give the following relations

$$\begin{aligned}
&\text{(i)}\ \ (il\, n_j - \lambda_j)B_j = (il\, \overline{n}_j - \overline{\lambda}_j)\overline{B}_j,\\
&\text{(ii)}\ \ (il - n_j\lambda_j)B_j = (il - \overline{n}_j\overline{\lambda}_j)\overline{B}_j,\\
&\text{(iii)}\ \ \Upsilon_j(B_j) = \overline{\Upsilon}_j(\overline{B}_j)\\
&\text{(viii)}\ \ M_0\left[(l^2 + \Upsilon_j)\lambda_j - \lambda_j^3\right](B_j) = \overline{M}_0\left[(l^2 + \overline{\Upsilon}_j)\overline{\lambda}_j - \overline{\lambda}_j^3\right](\overline{B}_j),
\end{aligned} \tag{58}$$

$$\begin{aligned}
&\text{(ix)}\ \ \begin{bmatrix}(e_{33}\lambda_j^2 - e_{13}l^2)n_j - il(e_{33} - e_{13})\lambda_j - \Xi_0 P_j\eta_j\\ +(\mu_e)_0\, H_0^2 n_j(\lambda_j^2 - l^2) + (\varepsilon_e)_0\, E_0^2 n_j(\lambda_j^2 - l^2)\end{bmatrix}B_j\\
&\qquad = \begin{bmatrix}(e_{33}\overline{\lambda}_j^2 - e_{13}l^2)\overline{n}_j - il(e_{33} - e_{13})\overline{\lambda}_j - \Xi_0\overline{P}_j\overline{\eta}_j\\ +(\overline{\mu}_e)_0\, H_0^2\overline{n}_j(\overline{\lambda}_j^2 - l^2) + (\overline{\varepsilon}_e)_0\, E_0^2\overline{n}_j(\lambda_j^2 - l^2)\end{bmatrix}\overline{B}_j,\\
&\text{(x)}\ \ \left[(e_{44}(l^2 + \lambda_j^2 + 2il n_j\lambda_j) - ibF_0\Upsilon_j\right]B_j = \left[(e_{44}(l^2 + \overline{\lambda}_j^2 + 2il\overline{n}_j\overline{\lambda}_j) - ib\overline{F}_0\overline{\Upsilon}_j\right]\overline{B}_j\\
&\text{(xi)}\ \ P_j\eta_j B_j = \overline{P}_j\overline{\eta}_j\overline{B}_j,\\
&\text{(xii)}\ \ -P_j\eta_j\lambda_j B_j + \theta\eta_j\lambda_j P_j B_j = -\overline{P}_j\overline{\eta}_j\overline{\lambda}_j\overline{B}_j + \overline{\theta}\overline{\eta}_j\overline{\lambda}_j\overline{P}_j\overline{B}_j.
\end{aligned} \tag{58}$$

Eliminating $\overline{B}_j$, B_j (j = 3, 4, 5, 6), we get 8 × 8 determinant, which gives wave velocity equation

$$\begin{vmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18}\\
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28}\\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38}\\
a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48}\\
a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58}\\
a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68}\\
a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & a_{78}\\
a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & a_{88}
\end{vmatrix} = 0 \tag{59}$$

where

$$a_{11}=i\ln_3-\lambda_3,\quad a_{12}=i\ln_4-\lambda_4,\quad a_{13}=i\ln_5-\lambda_5,\quad a_{14}=i\ln_6-\lambda_6,$$
$$a_{15}=i\overline{\ln}_3-\overline{\lambda}_3,\quad a_{16}=i\overline{\ln}_4-\overline{\lambda}_4,\quad a_{17}=i\overline{\ln}_5-\overline{\lambda}_5,\quad a_{18}=i\overline{\ln}_6-\overline{\lambda}_6,$$

$$a_{21}=i\ln_3-n_3\lambda_3,\quad a_{22}=i\ln_4-n_4\lambda_4,\quad a_{23}=i\ln_5-n_5\lambda_5,\quad a_{24}=i\ln_6-n_6\lambda_6,$$
$$a_{25}=i\overline{\ln}_3-\overline{n}_3\overline{\lambda}_3,\quad a_{26}=i\overline{\ln}_4-\overline{n}_4\overline{\lambda}_4,\quad a_{27}=i\overline{\ln}_5-\hat{n}_5\overline{\lambda}_5,\quad a_{28}=i\overline{\ln}_6-\overline{n}_6\overline{\lambda}_6,$$
$$a_{31}=\Upsilon_3,\quad a_{32}=\Upsilon_4,\quad a_{33}=\Upsilon_5,\quad a_{34}=\Upsilon_6,$$
$$a_{35}=\overline{\Upsilon}_3,\quad a_{36}=\overline{\Upsilon}_4,\quad a_{37}=\overline{\Upsilon}_5,\quad a_{38}=\overline{\Upsilon}_6,$$

$$a_{41}=M_0\left[l^2+\Upsilon_3-\lambda_3^2\right]\lambda_3,\quad a_{42}=M_0\left[l^2+\Upsilon_4-\lambda_4^2\right]\lambda_4,$$
$$a_{43}=M_0\left[l^2+\Upsilon_5-\lambda_5^2\right]\lambda_5,\quad a_{44}=M_0\left[l^2+\Upsilon_6-\lambda_6^2\right]\lambda_6,$$
$$a_{45}=M_0\left[l^2+\overline{\Upsilon}_3-\overline{\lambda}_3^2\right]\overline{\lambda}_3,\quad a_{46}=M_0\left[l^2+\overline{\Upsilon}_4-\overline{\lambda}_4^2\right]\overline{\lambda}_4,$$
$$a_{47}=M_0\left[l^2+\overline{\Upsilon}_5-\overline{\lambda}_5^2\right]\overline{\lambda}_5,\quad a_{48}=M_0\left[l^2+\overline{\Upsilon}_6-\overline{\lambda}_6^2\right]\overline{\lambda}_6,$$

$$a_{51}=\left(e_{33}\lambda_3^2-e_{13}l^2-\Xi_0+\left(\mu_e\right)_0H_0^2\lambda_3^2+\left(\varepsilon_e\right)_0E_0^2\lambda_3^2+\left(\mu_e\right)_0H_0^2l^2\lambda_3^2+\left(\varepsilon_e\right)_0E_0^2l^2\lambda_3^2\right)\eta_3-il(e_{33}-e_{13})\lambda_3,$$
$$a_{52}=\left(e_{33}\lambda_4^2-e_{13}l^2-\Xi_0+\left(\mu_e\right)_0H_0^2\lambda_4^2+\left(\varepsilon_e\right)_0E_0^2\lambda_4^2+\left(\mu_e\right)_0H_0^2l^2\lambda_4^2+\left(\varepsilon_e\right)_0E_0^2l^2\lambda_4^2\right)\eta_4-il(e_{33}-e_{13})\lambda_4,$$
$$a_{53}=\left(e_{33}\lambda_5^2-e_{13}l^2-\Xi_0+\left(\mu_e\right)_0H_0^2\lambda_5^2+\left(\varepsilon_e\right)_0E_0^2\lambda_5^2+\left(\mu_e\right)_0H_0^2l^2\lambda_5^2+\left(\varepsilon_e\right)_0E_0^2l^2\lambda_5^2\right)\eta_5-il(e_{33}-e_{13})\lambda_5,\tag{60}$$
$$a_{54}=\left(e_{33}\lambda_6^2-e_{13}l^2-\Xi_0+\left(\mu_e\right)_0H_0^2\lambda_6^2+\left(\varepsilon_e\right)_0E_0^2\lambda_6^2+\left(\mu_e\right)_0H_0^2l^2\lambda_6^2+\left(\varepsilon_e\right)_0E_0^2l^2\lambda_6^2\right)\eta_6-il(e_{33}-e_{13})\lambda_6,$$
$$a_{55}=\left(e_{33}\overline{\lambda}_3^2-e_{13}l^2-\overline{\Xi}_0+\left(\overline{\mu}_e\right)_0H_0^2\overline{\lambda}_3^2+\left(\overline{\varepsilon}_e\right)_0E_0^2\overline{\lambda}_3^2+\left(\overline{\mu}_e\right)_0H_0^2l^2\overline{\lambda}_3^2+\left(\overline{\varepsilon}_e\right)_0E_0^2l^2\overline{\lambda}_3^2\right)\overline{\eta}_3-il(e_{33}-e_{13})\overline{\lambda}_3,$$
$$a_{56}=\left(e_{33}\overline{\lambda}_4^2-e_{13}l^2-\overline{\Xi}_0+\left(\overline{\mu}_e\right)_0H_0^2\overline{\lambda}_4^2+\left(\overline{\varepsilon}_e\right)_0E_0^2\overline{\lambda}_4^2+\left(\overline{\mu}_e\right)_0H_0^2l^2\overline{\lambda}_4^2+\left(\overline{\varepsilon}_e\right)_0E_0^2l^2\overline{\lambda}_4^2\right)\overline{\eta}_4-il(e_{33}-e_{13})\overline{\lambda}_4,$$
$$a_{57}=\left(e_{33}\overline{\lambda}_5^2-e_{13}l^2-\overline{\Xi}_0+\left(\overline{\mu}_e\right)_0H_0^2\overline{\lambda}_5^2+\left(\overline{\varepsilon}_e\right)_0E_0^2\overline{\lambda}_5^2+\left(\overline{\mu}_e\right)_0H_0^2l^2\overline{\lambda}_5^2+\left(\overline{\varepsilon}_e\right)_0E_0^2l^2\overline{\lambda}_5^2\right)\overline{\eta}_5-il(e_{33}-e_{13})\overline{\lambda}_5,$$
$$a_{58}=\left(e_{33}\overline{\lambda}_6^2-e_{13}l^2-\overline{\Xi}_0+\left(\overline{\mu}_e\right)_0H_0^2\overline{\lambda}_6^2+\left(\overline{\varepsilon}_e\right)_0E_0^2\overline{\lambda}_6^2+\left(\overline{\mu}_e\right)_0H_0^2l^2\overline{\lambda}_6^2+\left(\overline{\varepsilon}_e\right)_0E_0^2l^2\overline{\lambda}_6^2\right)\overline{\eta}_6-il(e_{33}-e_{13})\overline{\lambda}_6,$$

$$a_{61}=e_{44}(l^2+\lambda_3^2+2iln_3\lambda_3)-ibF_0\Upsilon_3,a_{62}=e_{44}(l^2+\lambda_4^2+2iln_4\lambda_4)-ibF_0\Upsilon_4,$$
$$a_{63}=e_{44}(l^2+\lambda_5^2+2iln_5\lambda_5)-ibF_0\Upsilon_5,a_{64}=e_{44}(l^2+\lambda_6^2+2iln_6\lambda_6)-ibF_0\Upsilon_6,$$
$$a_{65}=e_{44}(l^2+\overline{\lambda}_3^2+2il\overline{n}_3\overline{\lambda}_3)-ib\overline{F}_0\overline{\Upsilon}_3,a_{66}=e_{44}(l^2+\overline{\lambda}_4^2+2il\overline{n}_4\overline{\lambda}_4)-ib\overline{F}_0\overline{\Upsilon}_4,$$
$$a_{67}=e_{44}(l^2+\overline{\lambda}_5^2+2il\overline{n}_5\overline{\lambda}_5)-ib\overline{F}_0\overline{\Upsilon}_5,a_{68}=e_{44}(l^2+\overline{\lambda}_6^2+2il\overline{n}_6\overline{\lambda}_6)-ib\overline{F}_0\overline{\Upsilon}_6,$$

$$a_{71}=\mathrm{P}_1,a_{72}=\mathrm{P}_2,a_{73}=\mathrm{P}_3,a_{74}=\mathrm{P}_4,a_{75}=\mathrm{P}_5,a_{76}=\mathrm{P}_6,a_{77}=\mathrm{P}_7,a_{78}=\mathrm{P}_8,$$

$$a_{81}=(-\lambda_3+\theta)n_3\mathrm{P}_3,a_{82}=(-\lambda_4+\theta)n_4\mathrm{P}_4,a_{83}=(-\lambda_5+\theta)n_5\mathrm{P}_5,a_{84}=(-\lambda_6+\theta)n_6\mathrm{P}_6,$$
$$a_{85}=(-\overline{\lambda}_3+\theta)\overline{n}_3\overline{\mathrm{P}}_3,a_{86}=(-\overline{\lambda}_4+\overline{\theta})\overline{n}_4\overline{\mathrm{P}}_4,a_{87}=(-\overline{\lambda}_5+\overline{\theta})\overline{n}_5\overline{\mathrm{P}}_5,a_{88}=(-\overline{\lambda}_6+\overline{\theta})\overline{n}_6\overline{\mathrm{P}}_6.$$

5. Discussion

Eq. (59) is the frequency equation of Stoneley waves in a thermo-electro-magneto non-homogeneous orthotropic granular medium under the influence of gravity and initial stress. The wave velocity $c = b / l$ depends on the gravity field, various non-homogeneities of the material medium and the granular rotations. The Eq. (59) also depends on the particular values of λ_j and $\overline{\lambda}_j$, which means that the general wave form is dispersive in nature.

From Eqs. (29), (31) and (59), we can conclude that when l is large, i.e., the wavelength is small, the effect of gravity is sufficiently small. On the other hand if the wave length of the wave is large then the effect of gravity is no longer negligible and plays an important role on the determination of the wave velocity c.

6. Particular cases

(1) If we neglect the gravity field, magnetic field and electric field, we get the wave velocity equation for Stoneley waves in a non-homogeneous orthotropic thermo granular medium under the effect of initial stress which is the same equation as Eq. (59) with

$$n_j = \frac{-I'\lambda_j^2 - J'}{A'\lambda_j^4 - B'\lambda_j^3 + D'\lambda_j^2 - E'\lambda_j + G'} \tag{61}$$

and, λ_j are the real roots of the following equation

$$a_1'D^8 + a_2'D^7 + a_3'D^6 + a_4'D^5 + a_5'D^4 + a_6'D^3 + a_7'D^2 + a_8'D + a_9' = 0 \tag{62}$$

where

$$\begin{aligned}
a_1' &= A'L',\\
a_2' &= A'N' + B'L',\\
a_3' &= A'O' + B'N' + D'L',\\
a_4' &= A'Q' + B'O' + D'N' + E'L',\\
a_5' &= A'S' + B'Q' + D'O' + E'N' + G'L' - I'T',\\
a_6' &= B'S' + D'Q' + E'O' + G'N' - I'U',\\
a_7' &= D'S' + E'Q' + G'O' + I'N' - J'T',\\
a_8' &= E'S' + G'Q' - J'U',\\
a_9' &= G'S' - J'V'.
\end{aligned} \tag{63}$$

and

$$\begin{gathered}
A' = (e_{13} + 2e_{44} + P_0),\ B' = 2e_{44},\ D' = -(e_{11} + P_0)l^2 - 2e_{44}l^2 + \rho_0 b^2 + \frac{ib(e_{11} + P_0)}{\Theta} + ib\Lambda,\\
E' = -2e_{44}l^2 + \frac{2imbe_{44}}{\Theta},\ I' = il(me_{44}),\ J' = (me_{44})\left(-il^3 - \frac{bl}{\Theta}\right),
\end{gathered} \tag{64}$$

$$G' = (e_{13} + 2e_{44} + P_0)l^4 - \rho_0 b^2 l^2 + \frac{i\rho_0 b^3}{\Theta} - ib\Lambda l^2 - \frac{ib^2(e_{13} + 2e_{44} + P_0)l^2}{\Theta},$$

$$L' = \left(e_{33} - e_{13} - e_{44} + \frac{P_0}{2}\right) - ibF_0,\ N' = me_{33} + m\left(e_{33} - e_{13} - e_{44} + \frac{P_0}{2}\right) - ibmF_0,$$

$$O' = -l^2 e_{44}\frac{P_0}{2} - l^2 - l^2\left(e_{33} - e_{13} - e_{44} + \frac{P_0}{2}\right) + \rho_0 b^2$$

$$+ iab\left(e_{33} - e_{13} - e_{44} + \frac{P_0}{2}\right) + me_{33} + 2ibl^2 F_0, \tag{64}$$

$$Q' = -ml^2 e_{33} + imabe_{33} - ml^2\left(e_{44} + \frac{P_0}{2}\right) + \rho_0 b^2 m + 2ibF_0 l^2,$$

$$S' = \left(e_{44} + \frac{P_0}{2}\right)l^4 - \rho_0 b^2 l^2 - ibal^2 + \left(e_{44} + \frac{P_0}{2}\right)ib^3 a\rho_0 - ibF_0 l^4,$$

$$T' = il(me_{13}),\ U' = iml(me_{13}),\ V' = -il^3(me_{13}) - alb(me_{13})$$

where A', B', D', E', G', I', J', L', N', O', Q', S', T', U' and V' are given by Eq. (64).

(2) If $\Xi = 0$, $F_0 = 0$, $M_0 = 0$ i.e., both the media are elastic and having no thermal fields, then by using Eqs. (15a), (15c) and (30), we get

$$n_j = \frac{il(\rho_0 g - me_{44})}{\begin{array}{c}(e_{13} + 2e_{44} + P_0 + (\mu_e)_0 H_0^2 + (\varepsilon_e)_0 E_0^2)\lambda_j^2 - 2me_{44}\lambda_j \\ + e_{11} + (b^2\rho_0 - l^2 P_0 - l^2(\mu_e)_0 H_0^2 - l^2(\varepsilon_e)_0 E_0^2)\end{array}} \quad (j = 3,\ 4) \tag{65}$$

and, λ_j are the real roots of the following equation

$$\begin{aligned}&\left[\left(e_{13} + 2e_{44} + P_0 + (\mu_e)_0 H_0^2 + (\varepsilon_e)_0 E_0^2\right)\left(e_{33} - e_{13} - e_{44} + \frac{P_0}{2}\right)\right]D^4 \\ &+ \left[\left(e_{13} + 2e_{44} + P_0 + (\mu_e)_0 H_0^2 + (\varepsilon_e)_0 E_0^2\right)me_{33}\left(e_{33} - e_{13} - e_{44} + \frac{P_0}{2}\right)\right]D^3 \\ &+ \left[\begin{array}{l}\left(\rho_0 b^2 - l^2 e_{44} - l^2\frac{P_0}{2}\right)\left(e_{13} + 2e_{44} + P_0 + (\mu_e)_0 H_0^2 + (\varepsilon_e)_0 E_0^2\right) \\ + 2m^2 e_{44} e_{33}\left(\rho_0 b^2 - l^2 e_{11} - l^2(\mu_e)_0 H_0^2 - l^2(\varepsilon_e)_0 E_0^2\right)\left(e_{33} - e_{13} - e_{44} + \frac{P_0}{2}\right)\end{array}\right]D\end{aligned} \tag{66}$$

and the dispersion equation is

$$
\begin{vmatrix}
i l n_3 - \lambda_3 & i l n_4 - \lambda_4 & i l \overline{n}_3 - \overline{\lambda}_3 & i l \overline{n}_4 - \overline{\lambda}_4 \\
il - n_3\lambda_3 & il - n_4\lambda_4 & il - \overline{n}_3\overline{\lambda}_3 & il - \overline{n}_4\overline{\lambda}_4 \\
\begin{pmatrix} e_{33}\lambda_3^2 - e_{13}l^2 \\ +(\mu_e)_0 H_0^2\lambda_3^2 \\ +(\varepsilon_e)_0 E_0^2\lambda_3^2 \\ +(\mu_e)_0 H_0^2 l^2\lambda_3^2 \\ +(\varepsilon_e)_0 E_0^2 l^2\lambda_3^2 \end{pmatrix} &
\begin{pmatrix} e_{33}\lambda_4^2 - e_{13}l^2 \\ +(\mu_e)_0 H_0^2\lambda_4^2 \\ +(\varepsilon_e)_0 E_0^2\lambda_4^2 \\ +(\mu_e)_0 H_0^2 l^2\lambda_4^2 \\ +(\varepsilon_e)_0 E_0^2 l^2\lambda_4^2 \end{pmatrix} &
\begin{pmatrix} e_{33}\lambda_3^2 - e_{13}l^2 \\ +(\mu_e)_0 H_0^2\overline{\lambda}_3^2 \\ +(\varepsilon_e)_0 E_0^2\overline{\lambda}_3^2 \\ +(\mu_e)_0 H_0^2 l^2\overline{\lambda}_3^2 \\ +(\varepsilon_e)_0 E_0^2 l^2\overline{\lambda}_3^2 \end{pmatrix} &
\begin{pmatrix} e_{33}\lambda_4^2 - e_{13}l^2 \\ +(\mu_e)_0 H_0^2\overline{\lambda}_4^2 \\ +(\varepsilon_e)_0 E_0^2\overline{\lambda}_4^2 \\ +(\mu_e)_0 H_0^2 l^2\overline{\lambda}_4^2 \\ +(\varepsilon_e)_0 E_0^2 l^2\overline{\lambda}_4^2 \end{pmatrix} \\
\eta_3 - il(e_{33}-e_{13})\lambda_3 & \eta_4 - il(e_{33}-e_{13})\lambda_4 & \overline{\eta}_3 - il(e_{33}-e_{13})\overline{\lambda}_3 & \overline{\eta}_4 - il(e_{33}-e_{13})\overline{\lambda}_4 \\
e_{44}\begin{pmatrix} l^2 + \lambda_3^2 \\ +2il\, n_3\lambda_3 \end{pmatrix} &
e_{44}\begin{pmatrix} l^2 + \lambda_4^2 \\ +2il\, n_4\lambda_4 \end{pmatrix} &
e_{44}\begin{pmatrix} l^2 + \overline{\lambda}_3^2 \\ +2il\, n_3\overline{\lambda}_3 \end{pmatrix} &
e_{44}\begin{pmatrix} l^2 + \overline{\lambda}_4^2 \\ +2il\, n_4\overline{\lambda}_4 \end{pmatrix}
\end{vmatrix} = 0 \qquad (67)
$$

Eq. (67) is the frequency equation of Stoneley waves in an electro-magneto non-homogeneous orthotropic granular medium under the influence of gravity and initial stress. The wave velocity $c = b / l$ depends on the gravity field, various non-homogeneities of the material medium and the granular rotations. The Eq. (67) also depends on the particular values of λ_j and $\overline{\lambda}_j$, which means that the general wave form is dispersive in nature. The real part gives the velocity of Stoneley waves and its imaginary part determines the attenuation of the waves due to granular nature of the medium.

(3) If $\Xi = 0$, $F_0 = 0$, $M_0 = 0$, $E = 0$ i.e., both the media are elastic and having no thermal and electric fields, then by using Eqs. (15a), (15c) and (30), we get

$$
n_j = \frac{il(\rho_0 g - m e_{44})}{\begin{array}{c}(e_{13} + 2e_{44} + P_0 + (\mu_e)_0 H_0^2)\lambda_j^2 - 2m e_{44}\lambda_j \\ + e_{11} + (b^2\rho_0 - l^2 P_0 - l^2(\mu_e)_0 H_0^2)\end{array}} \quad (j = 3,\ 4) \qquad (68)
$$

and, λ_j are the real roots of the following equation

$$
\begin{aligned}
&\left[\left(e_{13} + 2e_{44} + P_0 + (\mu_e)_0 H_0^2\right)\left(e_{33} - e_{13} - e_{44} + \frac{P_0}{2}\right)\right] D^4 \\
&+\left[\left(e_{13} + 2e_{44} + P_0 + (\mu_e)_0 H_0^2\right) m e_{33}\left(e_{33} - e_{13} - e_{44} + \frac{P_0}{2}\right)\right] D^3
\end{aligned} \qquad (69)
$$

$$
\begin{aligned}
&+\left[\begin{array}{l}\left(\rho_0 b^2 - l^2 e_{44} - l^2 \dfrac{P_0}{2}\right)\left(e_{13} + 2e_{44} + P_0 + (\mu_e)_0 H_0^2\right) \\ + 2m^2 e_{44} e_{33}\left(\rho_0 b^2 - l^2 e_{11} - l^2 (\mu_e)_0 H_0^2\right)\left(e_{33} - e_{13} - e_{44} + \dfrac{P_0}{2}\right)\end{array}\right] D \\
&+\left[\begin{array}{l}\left(\rho_0 b^2 - l^2 e_{44} - l^2 \dfrac{P_0}{2}\right)\left(\rho_0 b^2 - l^2 e_{11} + l^2 (\mu_e)_0 H_0^2\right) \\ + l^2\left(m e_{13} + \rho_0 g + m(\mu_e)_0 H_0^2\right)\left(m e_{44} - \rho_0 g\right)\end{array}\right] = 0
\end{aligned} \tag{69}
$$

and the dispersion equation is

$$
\begin{vmatrix}
i\ln_3 - \lambda_3 & i\ln_4 - \lambda_4 & i l\overline{n}_3 - \overline{\lambda}_3 & i l\overline{n}_4 - \overline{\lambda}_4 \\
i l - n_3\lambda_3 & i l - n_4\lambda_4 & i l - \overline{n}_3\overline{\lambda}_3 & i l - \overline{n}_4\overline{\lambda}_4 \\
\begin{pmatrix} e_{33}\lambda_3^2 - e_{13}l^2 \\ +(\mu_e)_0 H_0^2 \lambda_3^2 \\ +(\mu_e)_0 H_0^2 l^2 \lambda_3^2 \end{pmatrix} &
\begin{pmatrix} e_{33}\lambda_4^2 - e_{13}l^2 \\ +(\mu_e)_0 H_0^2 \lambda_4^2 \\ +(\mu_e)_0 H_0^2 l^2 \lambda_4^2 \end{pmatrix} &
\begin{pmatrix} e_{33}\lambda_3^2 - e_{13}l^2 \\ +(\mu_e)_0 H_0^2 \overline{\lambda}_3^2 \\ +(\mu_e)_0 H_0^2 l^2 \overline{\lambda}_3^2 \end{pmatrix} &
\begin{pmatrix} e_{33}\lambda_4^2 - e_{13}l^2 \\ +(\mu_e)_0 H_0^2 \overline{\lambda}_4^2 \\ +(\mu_e)_0 H_0^2 l^2 \overline{\lambda}_4^2 \end{pmatrix} \\
\eta_3 - il(e_{33} - e_{13})\lambda_3 & \eta_4 - il(e_{33} - e_{13})\lambda_4 & \overline{\eta}_3 - il(e_{33} - e_{13})\overline{\lambda}_3 & \overline{\eta}_4 - il(e_{33} - e_{13})\overline{\lambda}_4 \\
e_{44}\begin{pmatrix} l^2 + \lambda_3^2 \\ + 2il\, n_3\lambda_3 \end{pmatrix} &
e_{44}\begin{pmatrix} l^2 + \lambda_4^2 \\ + 2il\, n_4\lambda_4 \end{pmatrix} &
e_{44}\begin{pmatrix} l^2 + \overline{\lambda}_3^2 \\ + 2il\, n_3\overline{\lambda}_3 \end{pmatrix} &
e_{44}\begin{pmatrix} l^2 + \overline{\lambda}_4^2 \\ + 2il\, n_4\overline{\lambda}_4 \end{pmatrix}
\end{vmatrix} = 0 \tag{70}
$$

Eq. (70) is the frequency equation of Stoneley waves in a magneto non-homogeneous orthotropic granular medium under the influence of gravity and initial stress.

(4) If $\Xi = 0$, $F_0 = 0$, $M_0 = 0$, $E = 0$, $H = 0$, $P = 0$, $g = 0$ i.e., both the media are elastic and having no thermal, electric and magnetic fields, then by using Eqs. (15a), (15c) and (30), we get

$$
n_j = \frac{il(-me_{44})}{(e_{13} + 2e_{44})\lambda_j^2 - 2me_{44}\lambda_j + e_{11} + (b^2\rho_0)}; \quad (j = 3,\ 4) \tag{71}
$$

and, λ_j are the real roots of the following equation

$$
\begin{aligned}
&\left[\left(e_{13}+2e_{44}+P_0\right)\left(e_{33}-e_{13}-e_{44}\right)\right]D^4 \\
&+\left[\left(e_{13}+2e_{44}\right)me_{33}\left(e_{33}-e_{13}-e_{44}\right)\right]D^3 \\
&+\left[\left(\rho_0 b^2-l^2 e_{44}\right)\left(e_{13}+2e_{44}\right)+2m^2 e_{44}e_{33}\left(\rho_0 b^2-l^2 e_{11}\right)\left(e_{33}-e_{13}-e_{44}\right)\right]D \\
&+\left[\left(\rho_0 b^2-l^2 e_{44}\right)\left(\rho_0 b^2-l^2 e_{11}\right)+l^2\left(me_{13}\right)\left(me_{44}\right)\right]=0
\end{aligned}
\tag{72}
$$

and the dispersion equation is

$$
\begin{vmatrix}
i\ln_3-\lambda_3 & i\ln_4-\lambda_4 & i l\overline{n}_3-\overline{\lambda}_3 & i l\overline{n}_4-\overline{\lambda}_4 \\
i1-n_3\lambda_3 & i1-n_4\lambda_4 & i1-\overline{n}_3\overline{\lambda}_3 & i1-\overline{n}_4\overline{\lambda}_4 \\
\left(e_{33}\lambda_3^2-e_{13}l^2\right) & \left(e_{33}\lambda_4^2-e_{13}l^2\right) & \left(e_{33}\lambda_3^2-e_{13}l^2\right) & \left(e_{33}\lambda_4^2-e_{13}l^2\right) \\
\eta_3-il(e_{33}-e_{13})\lambda_3 & \eta_4-il(e_{33}-e_{13})\lambda_4 & \overline{\eta}_3-il(e_{33}-e_{13})\overline{\lambda}_3 & \overline{\eta}_4-il(e_{33}-e_{13})\overline{\lambda}_4 \\
e_{44}\begin{pmatrix} l^2+\lambda_3^2 \\ +2il\,n_3\lambda_3 \end{pmatrix} & e_{44}\begin{pmatrix} l^2+\lambda_4^2 \\ +2il\,n_4\lambda_4 \end{pmatrix} & e_{44}\begin{pmatrix} l^2+\overline{\lambda}_3^2 \\ +2il\,n_3\overline{\lambda}_3 \end{pmatrix} & e_{44}\begin{pmatrix} l^2+\overline{\lambda}_4^2 \\ +2il\,n_4\overline{\lambda}_4 \end{pmatrix}
\end{vmatrix}=0
\tag{73}
$$

Eq. (73) is the frequency equation of Stoneley waves in homogeneous orthotropic granular medium. This result is in complete agreement with the results given by Ahmed (2005).

7. Solution of problem for Rayleigh waves

The stress components in presence of electric, magnetic and thermal field are given by

$$
\begin{aligned}
\sigma_{11}&=C_{11}u_{1,x}+C_{13}u_{3,z}+\Delta\mu_e H_0^2+\Delta\varepsilon_e E_0^2-\Xi T, \\
\sigma_{33}&=C_{13}u_{1,x}+C_{33}u_{3,z}+\Delta\mu_e H_0^2+\Delta\varepsilon_e E_0^2-\Xi T, \\
\sigma_{11}&=C_{44}\left(u_{3,z}+u_{1,z}\right).
\end{aligned}
\tag{74}
$$

where

$$
C_{33}=C_{11}=\lambda+2\mu,\quad C_{13}=\lambda,\quad C_{44}=\mu,
$$

The problem deals with thermo viscoelastic solid, therefore, the thermal parameters Ξ are

$$
\Xi=\left(3\lambda+2\mu\right)\alpha_t \tag{75}
$$

where α_t is coefficient of linear expansion of solid.

Substituting Eqs. (74), (5), (6) and (4) in Eq. (75); we get

$$\begin{gathered}(\lambda+2\mu+P)u_{1,xx}+\mu u_{1,zz}+(\lambda+\mu+P)u_{3,xz}+\mu u_{1,x}\frac{\partial}{\partial x}(\lambda+2\mu+P)+u_{3,z}\frac{\partial}{\partial x}(\lambda+P)\\ +(u_{3,x}+u_{1,z})\frac{\partial}{\partial z}(\mu)-\rho g u_{3,x}-F\frac{\partial}{\partial t}(\eta_z)-\frac{\partial\eta}{\partial t}\frac{\partial}{\partial z}(F)-\frac{P}{2}(u_{1,zz}-u_{3,xz})\\ +\mu_e H_0^2(2u_{1,xx}+u_{3,xz})+\varepsilon_e E_0^2(2u_{1,xx}+u_{3,xz})-\Xi\frac{\partial T}{\partial x}=\rho u_{1,tt},\end{gathered} \tag{76}$$

$$(F\xi_{,t}),z-(F\xi_{,t}),x=0, \tag{77}$$

$$\begin{gathered}(\lambda+2\mu)u_{3,zz}+\mu u_{3,xx}+(\lambda+\mu)u_{1,xz}+(u_{3,x}+u_{1,z})\frac{\partial}{\partial x}(\mu)+u_{1,x}\frac{\partial}{\partial z}(\lambda)\\ +u_{3,z}\frac{\partial}{\partial z}(\lambda+2\mu)+(u_e H_0^2+\varepsilon_e E_0^2)(u_{1,xz}-u_{3,xx})-\frac{P}{2}(u_{1,xz}-u_{3,xx})\\ +(u_{1,x}+u_{3,z})u_e H_0^2+\varepsilon_e E_0^2(u_{1,x}+u_{3,z})+\rho g u_{1,x}-\Xi\frac{\partial T}{\partial z}+F\frac{\partial}{\partial t}(\eta_{,x})\rho u_{3,tt},\end{gathered} \tag{78}$$

$$-F\frac{\partial\xi}{\partial t}+M\nabla^2\xi+\xi_{,z}\frac{\partial}{\partial z}(M)=0, \tag{79}$$

$$-F\frac{\partial\eta}{\partial t}+M\nabla^2(\eta+w_2)+\frac{\partial}{\partial z}(w_2+\eta)\frac{\partial M}{\partial z}=0, \tag{80}$$

$$-F\frac{\partial\zeta}{\partial t}+M\nabla^2\zeta+\zeta_{,z}\frac{\partial(M)}{\partial z}=0, \tag{81}$$

where, λ, μ are Lame's constants and $\Delta=\frac{\partial u_1}{\partial x}+\frac{\partial u_3}{\partial z}=u_{1,x}+u_{3,z}$.

Now we assume the non-homogeneity of the granular half-space and co-efficient of friction are given by

$$\begin{gathered}\lambda=\lambda_0 e^{mz},\quad \mu=\mu_0 e^{mz},\quad \rho=\rho_0 e^{mz},\quad F=F_0 e^{mz},\quad M=M_0 e^{mz},\\ P=P_0 e^{mz},\quad \mu_e=(\mu_e)_0 e^{mz},\quad \varepsilon_e=(\varepsilon_e)_0 e^{mz},\quad \Xi=\Xi_0 e^{mz},\end{gathered} \tag{82}$$

where, m, λ_0, μ_0, ρ_0, F_0, M_0, P_0, $(\mu_e)_0$, $(\varepsilon_e)_0$, Ξ_0 are dimensionless constants.

Inserting Eq. (82) in Eqs. (76)-(81), we get

$$\begin{gathered}(\lambda_0+2\mu_0+P_0)u_{1,xx}+\mu_0 u_{1,zz}+(\lambda_0+\mu_0+P_0)u_{3,xz}+\mu u_{1,x}\frac{\partial}{\partial x}(\lambda_0+2\mu_0+P_0)\\ +u_{3,z}\frac{\partial}{\partial x}(\lambda_0+P_0)+(u_{3,x}+u_{1,z})\frac{\partial}{\partial z}(\mu_0)-\rho_0 g u_{3,x}-F_0\frac{\partial}{\partial t}(\eta_z)-\frac{\partial\eta}{\partial t}\frac{\partial}{\partial z}(F_0)\end{gathered} \tag{83}$$

$$+(\mu_e)_0 H_0^2\left(2u_{1,xx}+u_{3,xz}\right)+(\varepsilon_e)_0 E_0^2\left(2u_{1,xx}+u_{3,xz}\right)-\frac{P_0}{2}\left(u_{1,zz}+u_{3,xz}\right)-\Xi_0\frac{\partial T}{\partial x}=\rho_0 u_{1,tt}, \quad (83)$$

$$\left(F_0\xi_{,t}\right)_{,}z-\left(F_0\zeta_{,t}\right)_{,}x=0, \quad (84)$$

$$\left(\lambda_0+2\mu_0\right)u_{3,zz}+\mu_0 u_{3,xx}+\left(\lambda_0+\mu_0\right)u_{1,xz}+\left(u_{3,x}+u_{1,z}\right)\frac{\partial}{\partial x}(\mu_0)+u_{1,x}\frac{\partial}{\partial z}(\lambda_0)-\frac{P_0}{2}\left(u_{1,xz}-u_{3,xx}\right)$$

$$+u_{3,z}\frac{\partial}{\partial z}\left(\lambda_0+2\mu_0\right)+\left(u_e\right)_0 H_0^2\left(u_{1,xz}-u_{3,xx}\right)+\left(\varepsilon_e\right)_0 E_0^2\left(u_{1,xz}-u_{3,xx}\right)+\left(u_{1,x}-u_{3,z}\right) \quad (85)$$

$$+\left(u_e\right)_0 H_0^2+\left(\varepsilon_e\right)_0 E_0^2+\rho_0 g u_{1,x}-\Xi\frac{\partial T}{\partial z}+F_0\frac{\partial}{\partial t}\left(\eta_{,x}\right)=\rho_0 u_{3,tt},$$

$$-F_0\frac{\partial\xi}{\partial t}+M_0\nabla^2\xi+\xi_{,z}\frac{\partial}{\partial z}\left(M_0\right)=0, \quad (86)$$

$$-F_0\frac{\partial\eta}{\partial t}+M_0\nabla^2\left(\eta+w_2\right)+\frac{\partial}{\partial z}\left(w_2+\eta\right)\frac{\partial M_0}{\partial z}=0, \quad (87)$$

$$-F_0\frac{\partial\zeta}{\partial t}+M_0\nabla^2\zeta+\zeta_{,z}\frac{\partial(M_0)}{\partial z}=0, \quad (88)$$

T can be calculated from Fourier's law of heat conduction

$$p\,\nabla^2 T=C_v\frac{\partial T}{\partial t}+T_0 G_L\frac{\partial}{\partial t}\left(\nabla^2\phi\right), \quad (89)$$

where, K be the thermal conductivity and obeys the law as given by $K = K_0\, e^{mz}$, $p = \dfrac{K_0}{\rho_0}$ and C_v is the specific heat of the body at constant volume.

We introduce displacement potentials in terms of displacement components are given by

$$u_1=\phi_{,x}-\psi_{,z},\quad u_3=\phi_{,z}-\psi_{,x} \quad (90)$$

Introducing Eqs. (82), (90) into Eqs. (83)-(88), we get

$$\alpha^2\nabla^2\phi-\phi_{,tt}+g\psi_{,x}+m\beta'^2\left(2\phi_{,z}+\psi_{,x}\right)-\delta^2 T=0, \quad (91)$$

$$\frac{\partial}{\partial t}\left(\xi_{,z}-\zeta_{,x}\right)+m\xi_{,t}=0, \quad (92)$$

$$\beta^2\nabla^2\psi-\psi_{,tt}+g\phi_{,x}+s_{\eta,t}+m\left(\gamma 2\phi_{,x}+2\beta^2\psi_{,z}\right)=0, \quad (93)$$

$$-F_0\xi_{,t}+M_0\nabla^2\xi+M_0 m\xi_{,z}=0, \quad (94)$$

$$-s'\eta_{,t}+\nabla^2\eta-\nabla^4\psi-m\left[\eta_{,z}-\nabla^2\left(\psi_{,z}\right)\right]=0, \tag{95}$$

$$-s'\xi_{,t}+\nabla^2\xi+m\zeta_{,z}=0, \tag{96}$$

where

$$\gamma^2=\frac{\lambda_0+(\mu_e)_0H_0^2+(\varepsilon_e)_0E_0^2}{\rho_0},\quad s=\frac{F_0}{\rho_0},\quad \delta^2=\frac{K_0}{\rho_0},\quad s'=\frac{F_0}{M_0},\quad \beta'^2=\beta^2-\frac{P_0}{2\rho_0}. \tag{97}$$

Eliminating η from Eqs. (93) and (95); we get

$$\left(\nabla^2-s'\frac{\partial}{\partial t}+m\frac{\partial}{\partial z}\right)\left[\beta^2\nabla^2\psi-\psi_{,tt}+g\phi_{,x}+m\left(\gamma^2\phi_{,x}+2\beta^2\psi_{,z}\right)\right]+s\nabla^4\left(\psi_{,t}\right)+ms\nabla^2\left(\psi_{,zt}\right)=0 \tag{98}$$

To solve Eqs. (91)-(96), we assume that

$$\begin{aligned}\phi(x,z,t)&=\phi_1(z)\,e^{i(lx-bt)},\\ \psi(x,z,t)&=\psi_1(z)\,e^{i(lx-bt)},\\ \xi(x,z,t)&=\xi_1(z)\,e^{i(lx-bt)},\\ \eta(x,z,t)&=\eta_1(z)\,e^{i(lx-bt)},\end{aligned} \tag{99}$$

putting Eq. (99) in Eqs. (91) and (98), we get

$$\left(\alpha^2D^2-A\right)\phi_1-B\psi_1=0, \tag{100}$$

$$\left(A'D^4+B'D^3+C'D^2+d'D+E\right)\psi_1+\left(E'D^2+F'\right)\phi_1=0, \tag{101}$$

where

$$\begin{gathered}D=\frac{d}{dz},\ A=\alpha^2l^2-b^2-2m\beta^2,\ B=i\,\mathrm{lg}-ilm\beta^2,\ A'=\beta^2-ibs,\ B=3m\beta^2-ilsb,\\ C'=b(b+i\beta^2s')-2l^2(B^2-ibs)+2m^2\beta^2,\\ d'=(-2\beta^2l^2m+2\beta^2is'm-ml^2\beta^2+mb^2+imsbl^2),\\ F'=(-igl^3-\mathrm{glb}\,s'-iml^3\gamma^2-ms'\gamma^2-ms'\gamma^2b^2+i\,\mathrm{lg}\,m+ilm^2l^2).\end{gathered} \tag{101a}$$

Therefore the solutions of Eqs. (100) and (101) is of the form

$$\phi_1=A_je^{\lambda_jz}+B_je^{-\lambda_jz}, \tag{102}$$

$$\psi_1=E_je^{\lambda_jz}+F_je^{-\lambda_jz},\qquad j=3,4,5 \tag{103}$$

where

$$D^6 + P_1 D^5 + P_2 D^4 + P_3 D^3 + P_4 D^2 + P_5 D + P_6 = 0, \tag{104}$$

where

$$P_1 = \frac{3m\beta^2 - imsb}{\beta^2 - ibs},$$

$$P_2 = \frac{\left[\alpha^2 b(b + i\beta s') - 2l^2\alpha^2(\beta^2 - ibs) + 2\alpha^2 m^2\beta^2 - (\beta^2 - ibs)(\alpha^2 l^2 - b^2 - 2m\beta^2)\right]}{\alpha^2(\beta^2 - ibs)},$$

$$P_3 = \frac{\alpha^2(-2\beta^2 l^2 + 2\beta^2 is' - ml^2\beta^2 + mb^2 + imbl^2) - (\alpha^2 l^2 - b^2 - 2m\beta^2)(3m\beta^2 - imsb)}{\alpha^2(\beta^2 - ibs)}, \tag{105}$$

$$P_4 = \frac{\alpha^2 E - AC' + BE'}{\alpha^2(\beta^2 - ibs)},\ P_5 = \frac{-Ad'}{\alpha^2(\beta^2 - ibs)},\ P_6 = \frac{BF' - AE}{\alpha^2(\beta^2 - ibs)},$$

where *A*', *B*', *C*', *D*', *E*', *E*, *F*, *A*, *B* are given by Eq. (101a).

Further, the constants A_j, B_j (j = 3, 4, 5) are related with constants E_j, F_j respectively by means of Eq. (100). Equating the coefficients of $e^{\lambda jz}$, $e^{-\lambda jz}$ (j = 3, 4, 5) to zero and using Eqs. (100) and (101); we get

$$A_j = \eta_j E_j \quad \text{and} \quad B_j = \eta_j F_j \quad (j = 3, 4, 5), \tag{106}$$

$$\eta_j = \frac{i\,\mathrm{lg} - ilm\beta^2}{\alpha^2\lambda j^2 - \alpha^2 l^2 + b^2 + 2m\beta^2} \quad (j = 3, 4, 5), \tag{107}$$

Now solving Eqs. (93) and (100) for η_1 and ψ_1, we get

$$\left(a_1 D^4 + a_2 D^3 + a_3 D^2 + a_4 D + a_5\right)\psi_1 - isb\left(\alpha^2 D^2 + a_6\right)\eta_1 = 0 \tag{108}$$

Now eliminating ψ_1 from Eqs. (95) and (108), we get

$$\left[q_1 D^6 + q_2 D^5 + q_3 D^4 + q_4 D^3 + q_5 D^2 + q_6 D + q_7\right]\mu = 0, \tag{109}$$

where

$$\begin{aligned}
&q_1 = a_1 - isb\alpha^6,\ q_2 = ma_1 + a_2 - isb\alpha^6 m,\ q_3 = a_1(is'b - l^2) + a_2 m + a_3 - isba_6 + 2isbl^2\alpha^2,\\
&q_4 = a_2(is'b - l^2) + a_3 m + a_4 - isbma_6 + isbml^2\alpha^2,\\
&q_5 = a_3(is'b - l^2) + a_4 m + a_5 + 2isbl^2 a_6 + \alpha^2 l^4(-isb),\ q_6 = (is'b - l^2)a_4 + ma_5 + isbml^2 a_6,\\
&q_7 = a_5(is'b - l^2) - isbl^4 a_6,
\end{aligned} \tag{110}$$

$$a_1 = \alpha^2\beta^2,\ a_2 = 2m\alpha^2\beta^2,$$
$$a_3 = \beta^2(-\alpha^2 l^2 - b^2 + 2m\beta^2) + (b^2 - l^2\beta^2)\alpha^2,\ a_4 = 2m\beta^2(-\alpha^2 l^2 + b^2 + 2m\beta^2), \tag{110}$$
$$a_5 = (b^2 - l^2\beta^2)(-\alpha^2 l^2 + b^2 + 2m\beta^2) + il(g + m\gamma^2)(i\,\mathrm{lg} - ilm\beta^2),\ a_6 = b^2 + 2m\beta^2 - \alpha^2 l^2.$$

The solution of Eq. (109) is of the form

$$\eta_1 = (E_j e^{\lambda_j z} + F_j e^{-\lambda_j z})\,\delta_j, \tag{111}$$

where, λ_j (j = 3, 4, 5) are the real roots of Eq. (94) and $\delta_j = -i / bs\ [\beta^2 (\lambda_j^2 - l^2) + b^2 + (ilg + m\ i\ l\ \gamma^2)\ n_j + 2m\beta^2 \lambda_j]$.

Further substituting Eq. (99) into Eqs. (92), (94) and (96), we get

$$(D + m)\xi_1 - il\zeta_1 = 0, \tag{112a}$$

$$(D^2 + mD + h^2)\xi = 0, \tag{112b}$$

$$(D^2 + mD + h^2)\zeta_1 = 0, \tag{112c}$$

where, $h^2 = is'b - l^2$

The solutions of Eqs. (112b) and (112c) are given by

$$\xi_1 = A_1 e^{\alpha z} + Aze^{-\beta z}, \tag{113}$$

$$\zeta_1 = B_1 e^{\alpha z} + B_2 ze^{-\beta z}, \tag{114}$$

where, $\alpha = \dfrac{-m + \sqrt{m^2 - 4h^2}}{2},\ \beta = \dfrac{m + \sqrt{m^2 - 4h^2}}{2}, m^2 - 4h^2 > 0.$

Substituting Eq. (113), Eq. (114) into Eq. (112a), we get

$$(A_1\alpha + A_1 m)e^{\alpha z} + (-A_2\beta + mA_2)e^{-\beta z} = il(B_1 e^{\alpha z} + B_2 e^{-\beta z}). \tag{115}$$

Equating the co-efficients of $e^{\alpha z}$ and $e^{-\beta z}$ to zero in Eq. (115), we get

$$A_1 = \frac{ilB_1}{\alpha + m},\quad A_2 = \frac{ilB_2}{m - \beta}. \tag{116}$$

Let λ_0, μ_0, ρ_0, F_0, M_0 are the characteristics of layer and $\bar{\lambda}_0$, $\bar{\mu}_0$, $\bar{\rho}_0$, $\bar{F}_0$, $\bar{M}_0$ are the characteristics of half-space, also for the lower half-space and description of surface wave propagation ξ_1, ϕ_1, η_1, ψ_1, ζ_1 goes to zero as $z \to \infty$, also the non-homogeneity constant m is replaced by constant $\bar{m}$ for lower granular half-space also it is assumed that the real parts of (j = 3, 4, 5) are positive.

Thus for lower half-space

$$\begin{aligned}
\bar{\phi}_1 &= \bar{\eta}_j\, \bar{F}_j\, e^{-\bar{\lambda}_j z}, \\
\bar{\psi}_1 &= \bar{F}_j\, e^{-\bar{\lambda}_j z}, \\
\bar{\eta}_1 &= \bar{\delta}_j\, \bar{F}_j\, e^{-\bar{\lambda}_j z}, \\
\bar{\xi}_1 &= \frac{il}{m-\beta}\bar{B}_2 e^{-\bar{\beta} z},
\end{aligned} \tag{117}$$

$\bar{\zeta}_1 = \bar{B}_2 e^{-\bar{\beta} z} \quad (j = 3, 4, 5),$

where

$$M_{33} = M\frac{\partial \xi}{\partial z}, \quad M_{32} = M\frac{\partial}{\partial z}\left(\eta - \nabla^2\psi\right), \quad M_{31} = M\frac{\partial \xi}{\partial z},$$

$$\sigma_{33} = C_{13}\frac{\partial^2 \phi}{\partial x^2} + C_{33}\frac{\partial^2 \phi}{\partial z^2} + \left(C_{13} - C_{33}\right)\frac{\partial^2 \phi}{\partial x \partial z} + \mu_e H_0^2 \nabla^2 \phi + \Delta\varepsilon_e E_0^2 \nabla^2 \phi - \Xi T, \tag{118}$$

$$\sigma_{32} = -F\frac{\partial \xi}{\partial z}, \quad \sigma_{31} = C_{44}\left(\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial z^2} + 2\frac{\partial^2 \psi}{\partial x \partial z}\right) - F\frac{\partial \eta}{\partial t}.M_{31}$$

From the boundary conditions (iii), (v), (vi) and (vii), we get

$$\frac{B_1}{\alpha + m}e^{\alpha H} + \frac{B_2}{m-\beta}e^{-\beta H} = \frac{\bar{B}_2}{\bar{m} - \bar{\beta}}e^{-\bar{\beta} H}, \tag{119}$$

$$B_1 e^{\alpha H} + B_2 e^{-\beta H} = \bar{B}_2 e^{-\bar{\beta} H}, \tag{120}$$

$$M_0 e^{mH}\left(B_1 e^{\alpha H} - B_2 e^{-\beta H}\right) = -\bar{M}_0 e^{\bar{m}\bar{H}}\,\bar{\beta}\,\bar{B}_2 e^{-\bar{\beta} H}, \tag{121}$$

$$M_0 e^{mH}\left[\frac{B_1 \alpha e^{\alpha H}}{\alpha + m} - \frac{B_2 \beta e^{-\beta H}}{m - \beta}\right] = -\bar{M}_0 e^{\bar{m}\bar{H}}\,\frac{\bar{\beta}\,\bar{B}_2}{\bar{m} - \beta}e^{-\bar{\beta} H}, \tag{122}$$

From Eqs. (119) to (122), we have

$$B_1 = B_2 = \bar{B}_2 = 0 \tag{123}$$

$$\xi = \zeta = \bar{\zeta} = \bar{\xi} = 0. \tag{124}$$

The other boundary conditions gives the following relations, conditions (xii) and (xiii) are identities due to Eq. (124).

(xiv) gives

i.e., $K_1E_3 + K_2E_4 + K_3E_5 - K_1E_3 - K_2E_4 - K_3E_5 = 0,$

(xv) gives

$$K_4E_3 + K_5E_4 + K_6E_5 + K_7E_3 + K_8E_4 + K_9E_3 = 0,$$

(xvi) gives

$$K_{10}E_3 + K_{11}E_4 + K_{12}E_5 + K_{13}E_3 + K_{14}E_4 + K_{15}E_5 = 0,$$

while condition (xviii) and (xi) is an identity,

(i) gives

$$K_{16}e^{\lambda_3 H}E_3 + K_{17}e^{\lambda_4 H}E_4 + K_{18}e^{\lambda_5 H}E_5 + K_{19}e^{-\lambda_3 H}F_3 + K_{20}e^{-\lambda_4 H}F_4 + K_{21}e^{-\lambda_5 H}F_5$$
$$= \overline{K}_{19}e^{-\overline{\lambda}_3 H}\overline{F}_3 + \overline{K}_{20}e^{-\overline{\lambda}_4 H}\overline{F}_4 + \overline{K}_{21}e^{-\overline{\lambda}_5 H}\overline{F}_5,$$

(ii) gives

$$(il + n_3\lambda_3)e^{\lambda_3 H}E_3 + (il + n_4\lambda_4)e^{\lambda_4 H}E_4 + (il + n_5\lambda_5)e^{\lambda_5 H}E_5$$
$$(il - n_3\lambda_6)e^{-\lambda_3 H}F_3 + (il - n_4\lambda_4)e^{-\lambda_4 H}F_4 + (il + n_5\lambda_5)e^{-\lambda_5 H}F_5$$
$$= (il + \overline{n}_3\overline{\lambda}_3)e^{-\overline{\lambda}_3 H}\overline{F}_3 + (il + \overline{n}_4\overline{\lambda}_4)e^{-\overline{\lambda}_4 H}\overline{F}_4 + (il + \overline{n}_5\overline{\lambda}_5)e^{-\overline{\lambda}_5 H}\overline{F}_5,$$

(iv) gives

$$\delta_3 e^{\lambda_3 H}E_3 + \delta_4 e^{\lambda_4 H}E_4 + \delta_5 e^{\lambda_5 H}E_5 + \delta_3 e^{-\lambda_3 H}F_3 + \delta_4 e^{-\lambda_4 H}F_4 + \delta_5 e^{-\lambda_5 H}F_5$$
$$= \overline{\delta}_3 e^{-\overline{\lambda}_3 H}\overline{F}_3 + \overline{\delta}_4 e^{-\overline{\lambda}_4 H}\overline{F}_4 + \overline{\delta}_5 e^{-\overline{\lambda}_5 H}\overline{F}_5, \tag{125}$$

(viii) gives

$$M_0 e^{mH}\left[K_1 e^{\lambda_3 H}E_3 + K_2 e^{\lambda_4 H}E_4 + K_3 e^{\lambda_5 H}E_5 - K_1 e^{-\lambda_3 H}F_3 - K_2 e^{-\lambda_4 H}F_4 - K_3 e^{-\lambda_5 H}F_5\right]$$
$$= -\overline{M}_0 e^{\overline{m}H}\left[\overline{K}_1 e^{-\overline{\lambda}_3 H}\overline{F}_3 + \overline{K}_2 e^{-\overline{\lambda}_4 H}\overline{F}_4 + \overline{K}_3 e^{-\overline{\lambda}_5 H}\overline{F}_5\right],$$

(ix) gives

$$e^{mH}\left[K_4 e^{\lambda_3 H}E_3 + K_5 e^{\lambda_4 H}E_4 + K_6 e^{\lambda_5 H}E_5 + K_7 e^{-\lambda_3 H}F_3 + K_8 e^{-\lambda_4 H}F_4 + K_9 e^{-\lambda_5 H}F_5\right]$$
$$= e^{\overline{m}H}\left[\overline{K}_7 e^{-\overline{\lambda}_3 H}\overline{F}_3 + \overline{K}_8 e^{-\overline{\lambda}_4 H}\overline{F}_4 + \overline{K}_9 e^{-\overline{\lambda}_5 H}\overline{F}_5\right],$$

(x) gives

$$e^{mH}\left[K_{10} e^{\lambda_3 H}E_3 + K_{11} e^{\lambda_4 H}E_4 + K_{12} e^{\lambda_5 H}E_5 + K_{13} e^{-\lambda_3 H}F_3 + K_{14} e^{-\lambda_4 H}F_4 + K_{15} e^{-\lambda_5 H}F_5\right]$$
$$= e^{\overline{m}H}\left[\overline{K}_{13} e^{-\overline{\lambda}_3 H}\overline{F}_3 + \overline{K}_{14} e^{-\overline{\lambda}_4 H}\overline{F}_4 + \overline{K}_{15} e^{-\overline{\lambda}_5 H}\overline{F}_5\right],$$

where

$$\begin{aligned}
K_{j-2} &= \lambda_j\left(\delta_j - \lambda_j^2 + l^2\right), & \overline{K}_{j-2} &= \overline{\lambda}_j\left(\overline{\delta}_j - \overline{\lambda}_j^2 + l^2\right),\\
K_{j+1} &= n_j\left[\left(\lambda_j + 2\mu_0\right)\lambda_j^2 - \lambda_0\, l^2\right] + 2il\mu_0\lambda_j, & \overline{K}_{j+1} &= \overline{n}_j\left[\left(\overline{\lambda}_0 + 2\overline{\mu}_0\right)\overline{\lambda}_j^2 - \overline{\lambda}_0\, l^2\right] + 2il\overline{\mu}_0\overline{\lambda}_j,\\
K_{j+4} &= n_j\left[\left(\lambda_j + 2\mu_j\right)\lambda_j^2 - \lambda_0\, l^2\right] - 2il\mu_0\lambda_j, & \overline{K}_{j+4} &= \overline{n}_j\left[\left(\overline{\lambda}_0 + 2\overline{\mu}_0\right)\overline{\lambda}_j^2 - \overline{\lambda}_0\, l^2\right] - 2il\overline{\mu}_0\overline{\lambda}_j,\\
K_{j+7} &= ibF_0\delta_j + 2il\mu_0 n_j\lambda_j - \mu_0\left(\lambda_j^2 + l^2\right), & \overline{K}_{j+7} &= ib\overline{F}_0\overline{\delta}_j + 2il\overline{\mu}_0\overline{n}_j\overline{\lambda}_j - \overline{\mu}_0\left(\overline{\lambda}_j^2 + l^2\right),
\end{aligned} \tag{126}$$

$$K_{j+10} = ibF_0\delta_j - 2il\mu_0 n_j\lambda_j - \mu_0\left(\lambda_j^2 + l^2\right), \quad \overline{K}_{j+10} = ib\overline{F}_0\overline{\delta}_j - 2il\overline{\mu}_0\overline{n}_j\overline{\lambda}_j - \overline{\mu}_0\left(\overline{\lambda}_j^2 + l^2\right),$$
$$K_{j+13} = i\ln_j - \lambda_j, \quad K_{j+16} = i\ln_j - \lambda_j, \quad \overline{K}_{j+16} = i\,l\overline{n}_j - \overline{\lambda}_j. \tag{126}$$

Eliminating E_3, E_4, E_5, F_3, F_4, F_5, $\overline{F}_3$, $\overline{F}_4$, $\overline{F}_{35}$ from Eq. (112c),
We get 9 × 9 determinant, which gives wave-velocity equation

$$\left|a_{ij}\right| = 0, \quad \text{where} \quad i, j = 1, 2, \ldots, 9 \tag{127}$$

Eq. (127) gives the dispersion equation of Rayleigh waves for a granular non-homogeneous medium under the influence of gravity. The velocity of Rayleigh waves is given by the real part of the equation and attenuation of the waves is due to granular nature of the medium given by imaginary part of the same equation.
where

$$\begin{aligned}
&a_{11} = K_1e^{-\lambda_3H},\ a_{12} = K_2e^{-\lambda_4H},\ a_{13} = K_3e^{-\lambda_5H},\ a_{14} = K_1e^{\lambda_3H},\\
&a_{15} = K_2e^{\lambda_4H},\ a_{16} = K_3e^{\lambda_5H},\ a_{17} = a_{18} = a_{19} = 0,\\
&a_{22} = K_5e^{-\lambda_4H},\ a_{23} = K_6e^{-\lambda_5H},\ a_{24} = K_7e^{\lambda_3H},\\
&a_{25} = K_8e^{\lambda_4H},\ a_{26} = K_9e^{\lambda_5H},\ a_{27} = a_{28} = a_{29} = 0,\\
&a_{31} = K_{10}e^{-\lambda_3H},\ a_{32} = K_{11}e^{-\lambda_4H},\ a_{33} = K_{12}e^{-\lambda_5H},\ a_{34} = K_{13}e^{\lambda_3H},\\
&a_{35} = K_{14}e^{\lambda_4H},\ a_{36} = K_{15}e^{\lambda_5H},\ a_{37} = a_{38} = a_{39} = 0,\\
&a_{41} = K_{16},\ a_{42} = K_{17},\ a_{43} = K_{18},\ a_{44} = K_{19},\ a_{45} = K_{20},\ a_{46} = K_{47},\\
&a_{47} = \overline{K}_{19},\ a_{48} = \overline{K}_{20},\ a_{48} = \overline{K}_{20},\ a_{49} = \overline{K}_{21},\\
&a_{51} = il + n_3\lambda_3,\ a_{52} = il + n_4\lambda_4,\ a_{53} = il + n_5\lambda_5,\\
&a_{54} = il - n_3\lambda_3,\ a_{55} = il - n_4\lambda_4,\ a_{56} = il - n_5\lambda_5,\\
&a_{57} = il - \overline{n}_3\overline{\lambda}_3,\ a_{58} = il - \overline{n}_4\overline{\lambda}_4,\ a_{59} = il - \overline{n}_5\overline{\lambda}_5,\\
&a_{61} = \delta_3,\ a_{62} = \delta_4,\ a_{63} = \delta_5,\ \delta_{64} = \delta_3,\ a_{65} = \delta_4,\ a_{66} = \delta_5,\ a_{67} = \overline{\delta}_3,\ a_{68} = \overline{\delta}_4,\ a_{69} = \overline{\delta}_5,\\
&a_{71} = M_0e^{mH}K_1,\ a_{72} = M_0e^{mH}K_2,\ a_{73} = M_0e^{mH}K_3,\\
&a_{74} = -K_1M_0e^{mH},\ a_{75} = -K_2M_0e^{mH},\ a_{76} = -K_3M_0e^{mH},\\
&a_{77} = -\overline{K}_1\overline{M}_0e^{\overline{m}H},\ a_{78} = -\overline{K}_2\overline{M}_0e^{\overline{m}H},\ a_{79} = -\overline{K}_3\overline{M}_0e^{\overline{m}H},\\
&a_{81} = K_4e^{mH},\ a_{82} = K_5e^{mH},\ a_{83} = K_6e^{mH},\ a_{84} = K_7e^{mH},\ a_{85} = K_8e^{mH},\ a_{86} = K_9e^{mH},\\
&a_{87} = \overline{K}_7e^{\overline{m}H},\ a_{88} = \overline{K}_8e^{\overline{m}H},\ a_{89} = \overline{K}_9e^{\overline{m}H},\ a_{91} = K_{10}e^{mH},\ a_{92} = K_{11}e^{mH},\ a_{93} = K_{12}e^{mH},\\
&a_{94} = K_{13}e^{mH},\ a_{95} = K_{14}e^{mH},\ a_{96} = K_{15}e^{mH},\ a_{97} = \overline{K}_{13}e^{\overline{m}H},\ a_{98} = \overline{K}_{14}e^{\overline{m}H},\ a_{99} = \overline{K}_{15}e^{\overline{m}H}.
\end{aligned} \tag{128}$$

7. Particular cases

Eq. (127) in determinant form gives the wave velocity equation of Rayleigh wave in granular non-homogeneous medium under the influence of gravity, clearly from Eq. (127) we find that wave velocity $c = b / l$ not only depends on gravity, temperature ,magnetic field, electric field, initial stress but also on the non-homogeneity of material.

(1) In the absence of granular rotations, we get

$$\underset{M\to 0}{Lt} \quad \underset{s\to 0}{Lt} \quad \lambda_j = t_j, \qquad \underset{M\to 0}{Lt} \quad \underset{s\to 0}{Lt} \quad s\delta_j = v_j, \ (j = 3, 4, 5), \tag{129}$$

where

$$v_j = -\frac{i}{b}\left[\beta^2\left(t_j^2 - l^2\right) + b^2 + \left(i\,\mathrm{lg} + mil\gamma^2\right)\eta_j + 2m\beta^2 t_j\right]$$

and t_j are the roots of the equation by using Eq. (94)
where

$$\alpha^2\beta^2 t_j^6 + \left(3m\alpha^2\beta^2\right)t_j^5 + b_1 t_j^4 + b_2 t_j^3 + b_3 t_j^2 + b_4 t_j + b_5 = 0,$$

where

$$b_1 = -3\alpha^2\beta^2 l^2 + 2m^2\alpha^2\beta^2 + b^2 + 2m\beta^4,$$

$$b_2 = -6m\alpha^2\beta^2 l^2 + mb^2(\alpha^2 + \beta^2) + 6m^2\beta^4 + 2m\beta^2 b^2,$$

$$b_3 = -l^2\left[-2l^2\alpha^2\beta^2 + 2m\beta^4 + (\alpha^2 + \beta^2)b^2\right] + 2m^2\beta^2\left(-\alpha^2 l^2 + b^2 + 2m\beta^2\right) + \left(b^2 - l^2\beta^2\right)\left(-\alpha^2 l^2 + b^2 + 2m\beta^2\right) + il\left(g + m\gamma^2\right) + \left(i\,\mathrm{lg} - ilm\beta^2\right),$$

$$b_4 = -2ml^2\beta^2\left(-\alpha^2 l^2 + b^2 + 2m\beta^2\right) + \left(mb^2 - ml^2\beta^2\right) + \left(-\alpha^2 l^2 + b^2 + 2m\beta^2\right) + iml\left(g + m\gamma^2\right)\left(i\,\mathrm{lg} - ilm\beta^2\right),$$

$$b_5 = -l^2\left[\left(b^2 - l^2\beta^2\right)\left(-\alpha^2 l^2 + b^2 + 2m\beta^2\right) + il\left(g + m\gamma^2\right)\left(i\,\mathrm{lg} - ilm\beta^2\right)\right]$$

So Eq. (127) together with relation given by Eq. (129) forms the dispersion equation for the semi-infinite elastic, isotropic and non-homogeneous medium overlain by a granular layer under the influence of gravity, magnetic field, electric field and temperature.

(2) In the absence of non-homogeneity, Eq. (127) gives the dispersion equation of Rayleigh waves for a granular medium under the influence of gravity, magnetic field, electric field and temperature.

where

$$
\begin{aligned}
&a_{71}=M_0K_1,\ a_{72}=M_0K_2,\ a_{73}=M_0K_3,\\
&a_{74}=-M_0K_1,\ a_{75}=-M_0K_2,\ a_{76}=-M_0K_3,\\
&a_{77}=-\overline{K}_1\overline{M}_0,\ a_{78}=-\overline{K}_2\overline{M}_0,\ a_{79}=-\overline{K}_3\overline{M}_0,\\
&a_{81}=K_4,\ a_{82}=K_5,\ a_{83}=K_6,\ a_{84}=K_7,\ a_{85}=K_8,\ a_{86}=K_9,\\
&a_{87}=\overline{K}_7,\ a_{88}=\overline{K},\ a_{89}=\overline{K}_9,\\
&a_{91}=K_{10},\ a_{92}=K_{11},\ a_{93}=K_{12},\ a_{94}=K_{13},\ a_{95}=K_{14},\ a_{96}=K_{15},\\
&a_{97}=\overline{K}_{13},\ a_{98}=\overline{K}_{14},\ a_{99}=\overline{K}_{15}.
\end{aligned}
\tag{130}
$$

and rest of a_{ij}'s are same as in Eq. (128).

(3) In the absence of granular rotations and non-homogeneity, we get

$$
\begin{aligned}
&\underset{m\to 0}{Lt}\ \underset{M\to 0}{Lt}\ \underset{s\to 0}{Lt}\ \lambda_j=x_j,\\
&\underset{m\to 0}{Lt}\ \underset{M\to 0}{Lt}\ \underset{s\to 0}{Lt}\ \left(s\delta_j\right)=W_j,\ (j=3,4,5),
\end{aligned}
\tag{131}
$$

where

$$
W_j=-\frac{i}{b}\left[\beta^2\left(x_j^2-l^2\right)+b^2+i\,\lg n_j\right],\quad \eta_j=\frac{i\,\lg}{\alpha^2x_j^2-\alpha^2l^2+b^2}
$$

and x_j are the roots of the equation

$$
\begin{aligned}
&\alpha^2\beta^2x_j^6+\left[(\alpha^2+\beta^2)b^2-3\alpha^2\beta^2l^2\right]x_j^4+\begin{bmatrix}2l^4\alpha^2\beta^2-b^2l^2(\alpha^2+\beta^2)\\+(b^2-l^2\beta^2)(b^2-l^2\alpha^2)-l^2g^2\end{bmatrix}x_j^2\\
&+\left[(b^2-l^2\beta^2)l^2(\alpha^2l^2-b^2)+l^4g^2\right]=0
\end{aligned}
$$

Thus the equation $|a_{ij}|=0$, where $i,j=1,2,\ldots,9$
where a_{ij}'s are given by Eq. (130) gives the dispersion equation for the semi-infinite, elastic and isotropic medium overlain by a granular layer under the influence of gravity.

(4) In the absence of gravity, magnetic field $H_0=0$, electric field $E_0=0$, temperature $T_0=0$, initial stress $P_0=0$ and non-homogeneity, we get

$$
\lambda_5^2=l^2-\frac{b^2}{\alpha^2},
$$

$$
\left(\lambda_3^2,\lambda_4^2\right)=\frac{2l^2\beta^2-b^2-ib\beta^2s'-2ibl^2s\pm b\sqrt{(b-i\beta^2s')^2-4b^2ss'}}{2(\beta^2-ibs)}
$$

so by making η_3, $\eta_4 \to 0$, the dispersion Eq. (127) reduces to

$$\left|b_{ij}\right| = 0, \quad \text{where} \quad i, j = 1, 2, \ldots, 9 \tag{132}$$

where

$$\begin{aligned}
&b_{11} = r_1 e^{-\lambda_3 H},\ b_{12} = -r_1 e^{\lambda_3 H},\ b_{13} = r_2 e^{-\lambda_4 H},\ b_{14} = -r_2 e^{\lambda_4 H},\ b_{15} = b_{16} = b_{17} = b_{18} = b_{19} = 0,\\
&b_{21} = r_3 e^{-\lambda_3 H},\ b_{22} = -r_3 e^{\lambda_3 H},\ b_{23} = r_4 e^{-\lambda_4 H},\ b_{24} = -r_4 e^{\lambda_4 H},\ b_{25} = r_5 e^{-\lambda_5 H},\ b_{26} = -r_5 e^{-\lambda_5 H},\\
&b_{27} = b_{28} = b_{29} = 0,\\
&b_{31} = r_6 e^{-\lambda_3 H},\ b_{32} = -r_6 e^{\lambda_3 H},\ b_{33} = r_7 e^{-\lambda_4 H},\\
&b_{34} = -r_7 e^{\lambda_4 H},\ b_{35} = r_8 e^{-\lambda_5 H},\ b_{36} = -r_8 e^{\lambda_5 H},\ b_{37} = b_{38} = b_{39} = 0,\\
&b_{41} = -\lambda_3,\ b_{42} = \lambda_3,\ b_{43} = -\lambda_4,\ b_{44} = \lambda_4,\ b_{45} = il,\ b_{46} = il,\ b_{47} = \overline{\lambda}_3,\ b_{48} = \overline{\lambda}_4,\ b_{49} = il,\\
&b_{51} = b_{52} = b_{53} = b_{54} = il,\ b_{55} = \lambda_5,\ b_{56} = -\lambda_5,\ b_{57} = b_{58} = il,\ b_{59} = -\overline{\lambda}_5,\\
&b_{61} = b_{62} = \delta_3,\ b_{63} = b_{64} = \delta_4,\ b_{65} = b_{66} = 0,\ b_{67} = \overline{\delta}_3,\ b_{68} = \overline{\delta}_4,\ b_{69} = 0,\\
&b_{71} = M_0 r_1,\ b_{72} = -M_0 r_1,\ b_{73} = M_0 r_2,\ b_{74} = -M_0 r_2,\\
&b_{75} = b_{76} = 0,\ b_{77} = \overline{M}_0 \overline{r}_1,\ b_{78} = \overline{M}_0 \overline{r}_2,\ b_{79} = 0,\\
&b_{81} = r_3,\ b_{82} = -r_3,\ b_{83} = r_4,\ b_{84} = -r_4,\ b_{85} = r_5 = b_{86},\ b_{87} = -\overline{r}_3,\ b_{88} = -\overline{r}_4,\ b_{89} = -\overline{r}_5,\\
&b_{91} = r_6 = b_{92},\ b_{93} = b_{94} = r_7,\ b_{95} = r_8,\ b_{96} = -r_8,\ b_{97} = \overline{r}_6,\ b_{98} = \overline{r}_7,\ b_{99} = -\overline{r}_8.
\end{aligned} \tag{133}$$

and

$$\begin{aligned}
&r_1 = \lambda_3\left(\delta_3 - \lambda_3^2 + l^2\right),\ \overline{r}_1 = \overline{\lambda}_3\left(\overline{\delta}_3 - \overline{\lambda}_3^2 + \overline{l}^2\right),\ r_5 = \lambda_4\left(\delta_4 - \lambda_4^2 + l^2\right),\ \overline{r}_2 = \overline{\lambda}_4\left(\overline{\delta}_4 - \overline{\lambda}_4^2 + \overline{l}^2\right),\\
&r_3 = 2il\mu_0\lambda_3,\ \overline{r}_3 = 2il\overline{\mu}_0\overline{\lambda}_3,\ r_4 = 2il\mu_0\lambda_4,\ \overline{r}_4 = 2il\overline{\mu}_0\overline{\lambda}_4,\ r_5 = \mu_0\left(2l^2 - \frac{b^2}{\beta^2}\right),\\
&\overline{r}_5 = \mu_0\left(2l^2 - \frac{b^2}{\beta^2}\right),\ r_6 = ibF_0\delta_3 - \mu_0\left(\lambda_3^2 + l^2\right),\ \overline{r}_6 = ib\overline{F}_0\overline{\delta}_3 - \overline{\mu}_0\left(\overline{\lambda}_3^2 + l^2\right),\\
&r_7 = ibF_0\delta_4 - \mu_0\left(\lambda_4^2 + l^2\right),\ \overline{r}_7 = ib\overline{F}_0\overline{\delta}_4 - \overline{\mu}_0\left(\overline{\lambda}_4^2 + l^2\right),\ r_8 = 2il\mu_0\lambda_5,\ \overline{r}_8 = 2il\overline{\mu}_0\overline{\lambda}_5.
\end{aligned} \tag{134}$$

Eq. (132) gives the dispersion equation of Rayleigh waves for a granular medium in the absence of gravity and non-homogeneity and is in complete agreement with that obtained by Bhattacharaya (1965).

(5) In the absence of gravity, granular rotations, magnetic field $H_0 = 0$, temperature $T_0 = 0$, electric field $E_0 = 0$, initial stress $P_0 = 0$ and non-homogeneity.
Now using Eq. (132) and (133) into Eq. (134), we get

$$\underset{m \to 0}{Lt}\ \underset{M \to 0}{Lt}\ \underset{s \to 0}{Lt}\ \left(\lambda_3^2, \lambda_4^2\right) = \left(l^2, l^2 - \frac{b^2}{\beta^2}\right) \tag{135}$$

$$
\begin{aligned}
&\underset{m\to 0}{Lt}\ \underset{M\to 0}{Lt}\ \underset{s\to 0}{Lt}\ \left(s\delta_3\right)=-ib\\
&\underset{m\to 0}{Lt}\ \underset{M\to 0}{Lt}\ \underset{s\to 0}{Lt}\ \left(s\delta_4\right)=0\\
&\underset{m\to 0}{Lt}\ \underset{M\to 0}{Lt}\ \underset{s\to 0}{Lt}\ \delta_4=-\frac{b^2}{\beta^2}\\
&\underset{m\to 0}{Lt}\ \underset{M\to 0}{Lt}\ \underset{s\to 0}{Lt}\ r_6=-\mu\left(2l^2-\frac{b^2}{\beta^2}\right)\\
&\underset{m\to 0}{Lt}\ \underset{M\to 0}{Lt}\ \underset{s\to 0}{Lt}\ r_7=-\mu\left(2l^2-\frac{b^2}{\beta^2}\right)
\end{aligned}
\tag{135}
$$

Similar results are also holds for lower medium.
Now using Eq. (135) into Eq. (134), then after some simplification we get 6 × 6 determinantal equation

$$
\left|d_{ij}\right|=0, \quad \text{where} \quad i,j=1,2,\ldots,6, \tag{136}
$$

where

$$
\begin{aligned}
&d_{11}=2l\lambda_4 e^{\lambda_4 H},\ d_{12}=\left(2l^2-\frac{b^2}{\beta^2}\right)e^{\lambda_5 H},\\
&d_{13}=-2l\lambda_4 e^{-\lambda_4 H},\ d_{14}=\left(2l^2-\frac{b^2}{\beta^2}\right)e^{-\lambda_5 H},\ d_{15}=d_{16}=0,\\
&d_{21}=\left(2l^2-\frac{b^2}{\beta^2}\right)e^{\lambda_4 H},\ d_{22}=2l\lambda_5 e^{\lambda_5 H},\\
&d_{23}=\left(2l^2-\frac{b^2}{\beta^2}\right)e^{-\lambda_4 H},\ d_{24}=-2l\lambda_5 e^{-\lambda_5 H},\ d_{25}=d_{26}=0,\\
&d_{31}=-\lambda_4,\ d_{32}=-l,\ d_{33}=\lambda_4,\ d_{34}=-l,\ d_{35}=\overline{\lambda}_4,\ d_{36}=l,\\
&d_{41}=-l,\ d_{42}=-\lambda_5,\ d_{43}=-l,\ d_{44}=\lambda_5,\ d_{45}=l,\ d_{46}=\overline{\lambda}_5,\\
&d_{51}=2l^2-\frac{b^2}{\beta^2},\ d_{52}=2l^2\lambda_5,\ d_{53}=2l^2-\frac{b^2}{\beta^2},\ d_{54}=-2l\lambda_5,\\
&d_{55}=-\frac{\overline{\mu}_0}{\mu_0}\left(2l^2-\frac{b^2}{\beta^2}\right),\ d_{56}=2l\frac{\overline{\mu}_0}{\mu_0}\lambda_5,\\
&d_{61}=2l\lambda_4,\ d_{62}=2l^2-\frac{b^2}{\beta^2},\ d_{63}=-2l^2\lambda_4,\ d_{64}=2l^2-\frac{b^2}{\beta^2},\\
&d_{65}=-2l\frac{\overline{\mu}_0}{\mu_0}\overline{\lambda}_4,\ d_{66}=-\frac{\overline{\mu}_0}{\mu_0}\left(2l^2-\frac{b^2}{\beta^2}\right).
\end{aligned}
\tag{137}
$$

Thus Eq. (136) gives the dispersion equation of Rayleigh waves for semi-infinite elastic and isotropic medium overlain by granular layer of thickness H in the absence of gravity and non-homogeneity is in complete agreement with the equation obtained by Ewing *et al*. (1957).

8. Numerical analysis

The parameters for the material are taken in Table 1.

Numerical results have been obtained graphically to show the effect non-homogeneities and phase velocity on initial stress and dimensionless wave number on surface waves. Fig. 2 represents the variation of phase velocity with dimensionless less wave number at different values of initial stress and other inhomogeneities. The phase velocity of Stoneley wave not only depends on gravity field but also on the non-homogeneity, magnetic field, electric field, temperature, initial stress and granular notations of the material medium. Fig. 3 shows the effect of the density on Stoneley wave determinant with respect to initial stress. It is obvious that Stoneley wave velocity decreases with an increasing of the various values of the initial stress P also with the wave number. Fig. 4 is plotted to observe the effect of magnetic and electric field on Stoneley waves velocity with respect depth. The velocity of Stoneley waves is slowed down in the presence of magnetic

Table 1 Material properties

C_{11}	C_{13}	C_{44}	C_{33}	ρ_0
135 GPa	67.9 GPa	22.2 GPa	113 GPa	7500 Kg/m^3

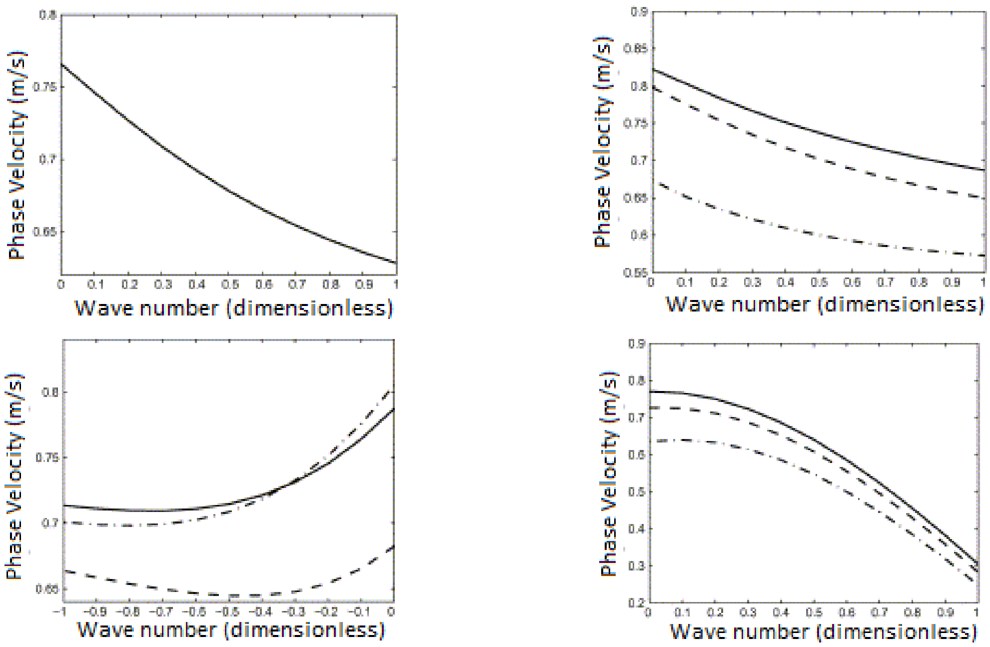

Fig. 2 Effect of coupled non-homogeneities (%) on Stoneley waves velocity with respect wave number keeping initial stress at P = 1, P = 0.5 and P = 0.1

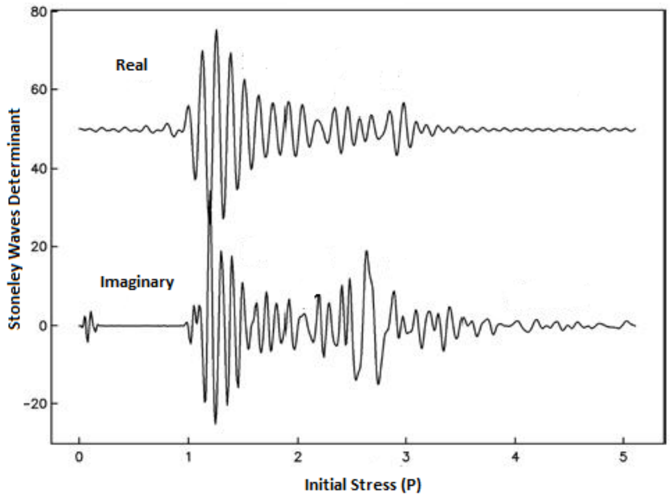

Fig. 3 Effect of the density on Stoneley waves determinant with respect to initial stress

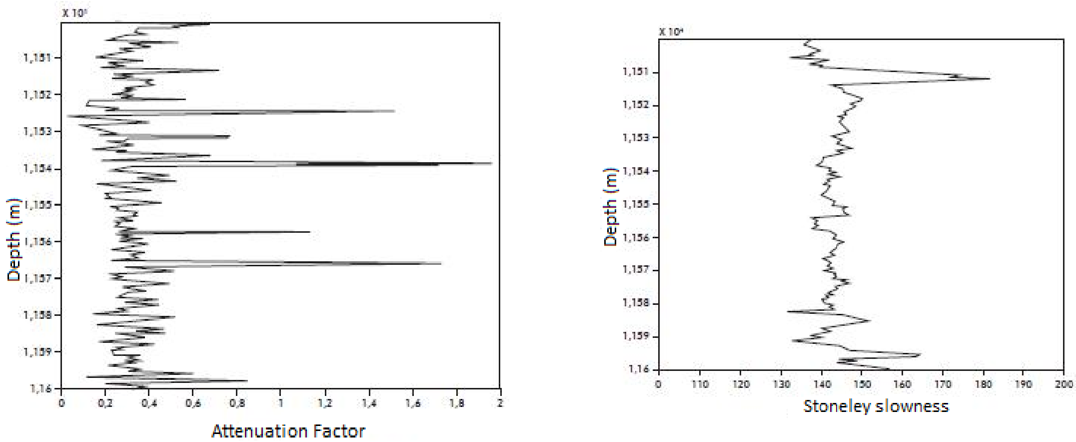

Fig. 4 Effect of the magnetic field and electric field on Stoneley waves with depth

and electric field of the medium. Fig. 5 shows the effect of initial compression on the Rayleigh Waves, it is obvious that Rayleigh wave velocity decreases with an increasing of the various values of the initial stress P also with the wave number. Fig. 6 represents the variation of phase velocity with dimensionless less wave number at different values of initial stress. The three modes of Rayleigh waves have been plotted at two different values of initial stress i.e., at P = 1 and P = 0.1. The value of magnetic field, electric field and temperature is fixed at 0.4 Tesla, 50 V/m and 293 K. It is clear from Fig. 6 as the value of initial compression increases the phase velocity decreases sharply with dimension less wave number. Fig. 7 is plotted to observe the effect of various non-homogeneities factor W'in (%) on Rayleigh waves velocity with respect wave number at P = 1and P = 0.1. In graph W represents the zeroth level of non-homogeneities. Fig. 8 represents the effect of depth (dimensionless) on frequency (dimensionless) on Rayleigh wave velocity.

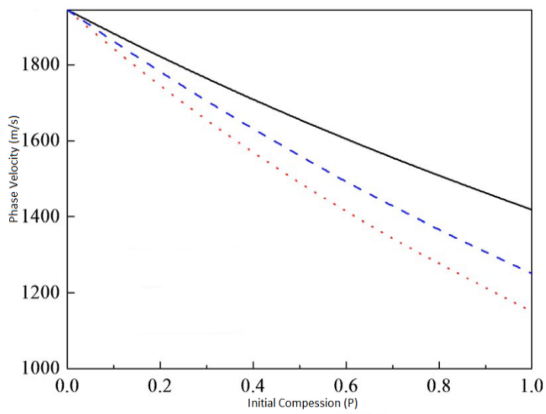

Fig. 5 Variation of Rayleigh waves velocity respect to initial stress with the various values of the wave number, $H = 0.4$ Tesla, $g = 9.8$ m/s^2, $T = 293$ K, $E = 50$ V/m, granular rotations = 0

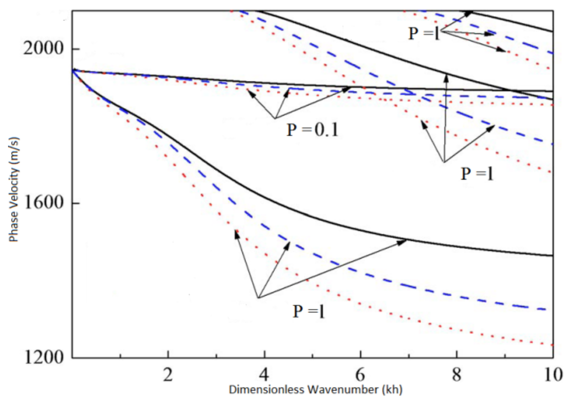

Fig. 6 Variation of Rayleigh waves velocity respect wave number, H = 0.4 Tesla, E = 50 V/m, T = 293 K, g = 9.8 m/s^2, P = 1, P = 0.1, granular rotations = 0

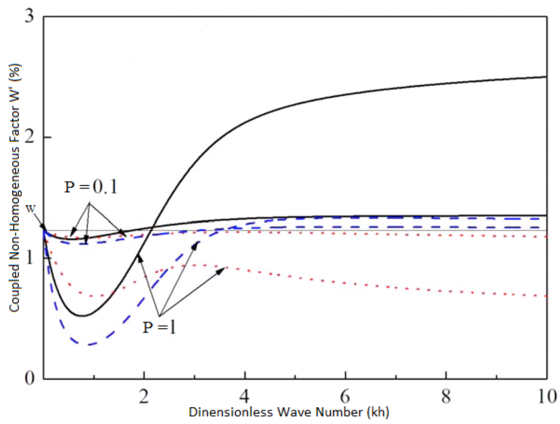

Fig. 7 Effect of coupled non-homogeneities (%) on Rayleigh waves velocity with respect wave number keeping initial stress at P = 1and P = 0.1

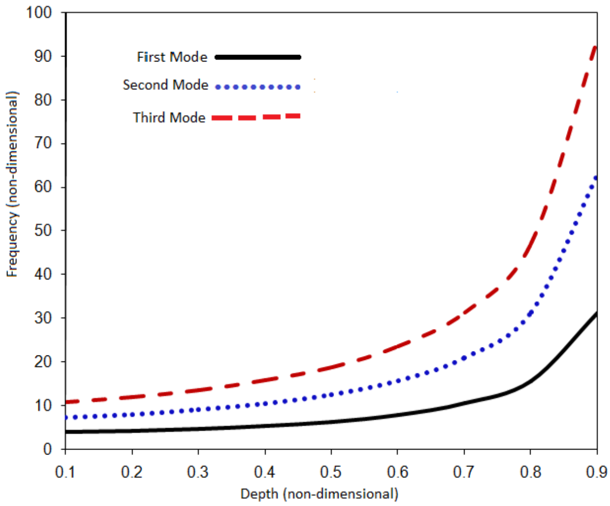

Fig. 8 Variation of Rayleigh waves frequency with respect depth, $H = 0.4$ Tesla, $E = 50$ V/m, $T = 293$ K, $g = 9.8$ m/s^2, $P = 1$, $P = 0.1$, granular rotations = 0

9. Conclusions

The frequency equation for surface wave contains terms involving gravity and non-homogeneity, so the phase velocity of Stoneley and Rayleigh waves not only depend on gravity field but also on the non-homogeneity, magnetic field, electric field, temperature, initial stress and granular notations of the material medium.

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