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Free vibration analysis of tapered beam-column with pinned ends embedded in Winkler-Pasternak elastic foundation

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Abstract. The current study presents a mathematical model and numerical method for free vibration of tapered piles embedded in two-parameter elastic foundations. The method of Discrete Singular Convolution (DSC) is used for numerical simulation. Bernoulli-Euler beam theory is considered. Various numerical applications demonstrate the validity and applicability of the proposed method for free vibration analysis. The results prove that the proposed method is quite easy to implement, accurate and highly efficient for free vibration analysis of tapered beam-columns embedded in Winkler- Pasternak elastic foundations.

Keywords: free vibration; beam-column; elastic foundation; tapered piles; discrete singular convolution.

1. Introduction

Free vibration or stability analysis of structures is one of the main required tasks for an engineer to accomplish in the engineering design. Free vibration or buckling problems are generally described by a linear partial differential equation associated with a set of related boundary conditions. Analytical solutions of these problems are very limited. Therefore, the analysis of free vibration problem of structures can be treated only by some numerical techniques. Among the various numerical methods, the finite difference, spectral methods, finite elements and differential quadrature methods have been successfully used during the past forty years.

The analysis of structures on elastic foundations is of considerable interest and widely used in several engineering fields, such as foundation, pavement and railroad, pipeline, and some aero-space structures applications. Many problems in the engineering related to soil-structure interaction can be modeled by means of a beam or a beam-column on an elastic foundation. Although few types of foundation models exist, the Winkler foundation model is extensively used by engineers and researchers because of its simplicity. Generally, the foundation is considered to be an array of springs uniformly distributed along the length of the beam. The free vibration of beams or beam-columns has been investigated by many researchers in the past forty years. There are many studies in the literature on theory and analysis of beams. The majority of the available publications are

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based on the analytical and numerical solution of beams. The effect of foundation on the frequencies of beam-columns on elastic foundation was studied in the literature. A few studies concerning the analysis of beam-columns on elastic foundations have been carried out, namely by Zhaohua and Cook (1983), Yankelevsky and Eisenberger (1986), Doyle and Pavlovic (1982), Yokoyama (1991), Valsangkar and Pradhanang (1988), De Rosa and Maurizi (1999), Halabe and Jain (1986), West and Mafi (1984), Matsunaga (1999) and Kameswara et al. (1975).

In this paper, discrete singular convolution method technique is presented for computation of the free vibration analysis of a pile embedded in two-parameter elastic foundations. The accuracy of the solutions is inferred by comparison with analytical and other numerical solutions. Some new results are also provided. To the authors' knowledge, it is the first time the DSC method has been successfully applied to beam-columns embedded in two-parameter elastic foundations for the numerical analysis of vibration.

2. Discrete singular convolution (DSC)

Discrete singular convolution (DSC) method is a relatively new numerical technique in applied mechanics (Wei 2000). The method of discrete singular convolution (DSC) was proposed to solve linear and nonlinear differential equations by Wei (2001, 2001a, 2001b), and later it was introduced to solid mechanics (Wei *et al.* 2002, 2002a). It has been also successfully employed for different vibration problems of structural members such as plates and shells (Zhao *et al.* 2002, 2002a, Civalek 2006, 2006a, 2007, 2007a, 2007b). It is shown from these studies, the method of DSC have high level of accuracy and reliability. It is also emphasized that DSC method yields more efficient and accurate approximation compared to the other numerical methods for higher order frequencies.

The method of discrete singular convolution (DSC) is an effective and simple approach for the numerical verification of singular convolutions, which occur commonly in mathematical physics and engineering. The discrete singular convolution method has been extensively used in scientific computations in past ten years. For more details of the mathematical background and application of the DSC method in solving problems in engineering, the readers may refer to some recently published reference (Wei 2000, Wei *et al.* 2002). In the context of distribution theory, a singular convolution can be defined by (Wei 2001)

$$F(t) = (T*\eta)(t) = \int_{-\infty}^{\infty} T(t-x)\,\eta(x)dx \tag{1}$$

Where T is a kind of singular kernel such as Hilbert, Abel and delta type, and $\eta(t)$ is an element of the space of the given test functions. In the present approach, only singular kernels of delta type are chosen. This type of kernel is defined by (Wei *et al.* 2002)

$$T(x) = \delta^{(r)}(x)\,;\ (r = 0, 1, 2, ...,) \tag{2}$$

where subscript r denotes the rth-order derivative of distribution with respect to parameter x. In order to illustrate the DSC approximation, consider a function $F(x)$. In the method of DSC, numerical approximations of a function and its derivatives can be treated as convolutions with some kernels. According to DSC method, the rth derivative of a function $F(x)$ can be approximated as (Wei 2001a)

$$F^{(r)}(x) \approx \sum_{k=-M}^{M} \delta^{(r)}_{\Delta,\sigma}(x_i - x_k) f(x_k);\ \ (r = 0, 1, 2, ...,) \tag{3}$$

where Δ is the grid spacing, x_k are the set of discrete grid points which are centered around x, and $2M+1$ is the effective kernel, or computational bandwidth. It is also known, the regularized Shannon kernel (RSK) delivers very small truncation errors when it use the above convolution algorithm. The regularized Shannon kernel (RSK) is given by (Zhao *et al.* 2002)

$$\delta_{\Delta,\sigma}(x - x_k) = \frac{\sin[(\pi/\Delta)(x - x_k)]}{(\pi/\Delta)(x - x_k)} \exp\left[-\frac{(x - x_k)^2}{2\sigma^2}\right];\ \sigma > 0 \tag{4}$$

The researchers is generally used the regularized delta Shannon kernel by this time. The required derivatives of the DSC kernels can be easily obtained using the below formulation (Wei 2001b)

$$\delta^{(r)}_{\Delta,\sigma}(x - x_j) = \frac{d^r}{dx^r}[\delta_{\Delta,\sigma}(x - x_j)]\Big|_{x=x_i} \tag{5}$$

In the present study, the governing equation includes second-order derivatives. Thus, the second-order derivative at $x \neq x_k$, can be given as follows (Wei 2000)

$$\begin{aligned}\delta^{2}_{\pi/\Delta,\sigma}(x - x_k) = &-\frac{(\pi/\Delta)\sin(\pi/\Delta)(x - x_k)}{(x - x_k)}\exp[-(x - x_k)^2/2\sigma^2]\\ &-2\frac{\cos(\pi/\Delta)(x - x_k)}{(x - x_k)^2}\exp[-(x - x_k)^2/2\sigma^2]\\ -2\frac{\cos(\pi/\Delta)(x - x_k)}{\sigma^2}&\exp[-(x - x_k)^2/2\sigma^2] + 2\frac{\sin(\pi/\Delta)(x - x_k)}{\pi(x - x_k)^3/\Delta}\exp[-(x - x_k)^2/2\sigma^2]\\ +\frac{\sin(\pi/\Delta)(x - x_k)}{\pi(x - x_k)\sigma^2/\Delta}&\exp[-(x - x_k)^2/2\sigma^2] + \frac{\sin(\pi/\Delta)(x - x_k)}{\pi\sigma^4/\Delta}(x - x_k)\exp[-(x - x_k)^2/2\sigma^2]\end{aligned} \tag{6}$$

For $x=x_k$, this derivative is given by

$$\delta^{(2)}_{\pi/\Delta,\sigma}(0) = -\frac{3 + (\pi^2/\Delta^2)\sigma^2}{3\sigma^2} = -\frac{1}{\sigma^2} - \frac{\pi^2}{3\Delta^2} \tag{7}$$

3. Fundamental equations

Vibration and dynamic analyses of beams, piles or beam-columns on elastic foundations have been treated by researchers in the past (Eisenberger 1995, Lee and Schultz 2004, Şimşek 2009, Şimşek and Kocatürk 2009). Railroad tracks, highway pavements, strip foundations, piles, and many others problems modeled by beam-columns on elastic foundation. The governing equations for free vibration of tapered beam-column embedded in Winkler Pasternak foundation (Fig. 1) using the Euler-Bernoulli beam theory can be written as:

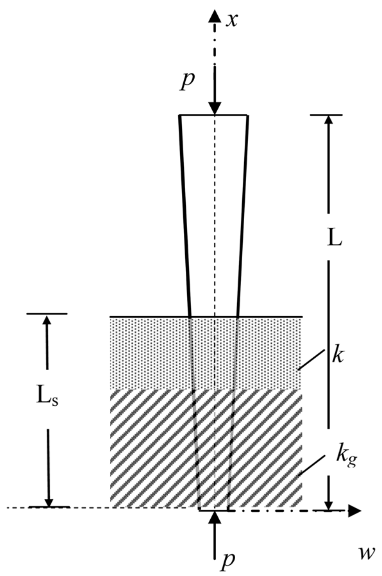

Fig. 1 Tapered piles embedded in elastic foundation

$$EI(x)\frac{\partial^4 w(x,t)}{\partial x^4} + p\frac{\partial^2 w(x,t)}{\partial x^2} + kw(x,t) - k_g\frac{\partial^2 w(x,t)}{\partial x^2} + \rho A(x)\frac{\partial^2 w(x,t)}{\partial t^2} = 0\ , \quad \text{for } 0 < x < L_s \qquad (8a)$$

$$EI(x)\frac{\partial^4 w(x,t)}{\partial x^4} + p\frac{\partial^2 w(x,t)}{\partial x^2} + \rho A(x)\frac{\partial^2 w(x,t)}{\partial t^2} = 0\ . \quad \text{for } L_s < x < L \qquad (8b)$$

in which EI is the flexural rigidity of beam-column, w is the transverse deflection, p is the applied axial load, k_g is the Pasternak parameter, k is the Winkler parameter, ρ is the mass density, A is the cross-sectional area, I the second moment of area of cross-section, E the Young's modulus, L_s is the length of the embedded pile, and ω is the circular frequency. The transverse displacement w is assumed to be

$$w(x,t) = W(x)e^{i\omega t} \qquad (9)$$

Substituting expression (9) into equations (8) and normalizing the equation yields

$$\frac{EI(x)}{L^4}\frac{d^4W}{dX^4} + \frac{p}{L^2}\frac{d^2W}{dX^2} + kW - \frac{k_p}{L^2}\frac{d^2W}{dX^2} - \rho A\omega^2 W = 0\,, \quad \text{for } 0 < x < L_s \qquad (10a)$$

$$\frac{EI(x)}{L^4}\frac{d^4W}{dX^4} + \frac{p}{L^2}\frac{d^2W}{dX^2} - \rho A\omega^2 W = 0\,. \quad \text{for } L_s < x < L \qquad (10b)$$

By using some dimensional quantities, Eqs. (10) can be written as

$$\frac{d^4W}{dX^4} + P\frac{d^2W}{dX^2} + KW - G\frac{d^2W}{dX^2} - \Omega^2 W = 0\,, \quad 0 < X < \gamma \qquad (11a)$$

$$\frac{d^4W}{dX^4} + P\frac{d^2W}{dX^2} - \Omega^2 W = 0\,. \quad \gamma < X < 1 \qquad (11b)$$

Where

$$P = pL^2/EI(x);\ K = kL^4/EI(x);\ G = k_pL^2/EI(x);\ \Omega^2 = \omega L^2\sqrt{\rho A(x)/EI(x)} \tag{12}$$

$$X = x/L,\ W = w/L \text{ and } \gamma = L_s/L$$

The taper ratios (linear) are given as

$$\alpha = h_1/h_0 \quad\text{and}\quad \beta = b_1/b_0 \tag{13}$$

By using DSC discretization the Eqs. (8) take the form

$$\sum_{j=1}^{N} \delta^{(4)}_{\pi/\Delta,\sigma}(\Delta x)W(x_i) + P\sum_{j=1}^{N} \delta^{(2)}_{\pi/\Delta,\sigma}(\Delta x)\,W(x_i)$$

$$-G\sum_{j=1}^{N} \delta^{(2)}_{\pi/\Delta,\sigma}(\Delta x)W(x_i) + KW(x_j) = \Omega^2 W_i \tag{14a}$$

$$\sum_{j=1}^{N} \delta^{(4)}_{\pi/\Delta,\sigma}(\Delta x)W(x_i) + P\sum_{j=1}^{N} \delta^{(2)}_{\pi/\Delta,\sigma}(\Delta x)W(x_i) = \Omega^2 W_i \tag{14b}$$

Pinned boundary conditions are considered for both edges. Related equations for the boundary conditions are given as

$$W = 0 \quad\text{and}\quad EI(x)\frac{\partial^2 W}{\partial x^2} = 0 \tag{15}$$

In this study we consider only pinned ends for beam-columns. After implementation of the given boundary conditions using the standard method proposed by Wei *et al*. (2001, 2001b), Eqs. (14a) and (14b) can be expressed by

$$[\mathbf{R}]\{\mathbf{U}\} = \omega^2\{\mathbf{U}\} \tag{16}$$

where **U** is the displacements vector, **R** is the stiffness matrix. The frequency values for beam-column embedded in elastic foundation are given by the following non-dimensional form

$$\Omega^2 = \omega L^2\sqrt{\frac{\rho A(x)}{EI(x)}} \tag{17}$$

where ω is the circular frequency.

4. Numerical examples

In this section, a number of problems have been solved to demonstrate the performance of the present method. The title problem is analyzed and some of DSC results are compared with results in the open literature (Yokoyama 1991, Valsangkar and Pradhanang 1988) to show the applicability and efficiency of DSC method. Firstly, to check whether or not the purposed formulation and programming are correct, an Euler-Bernoulli beam embedded in a Winkler foundation analyzed. In the present results, P_{cr} is the Euler critical buckling load of a simply-supported beam without elastic foundation. Comparison of first three frequency parameters of Euler-Bernoulli beam column embedded in a Winkler foundation are presented in Table 1-3. The results from finite element method given by Yokoyama (1991) and results obtained by analytical approach (Valsangkar and Pradhanang 1988) have also been provided for comparing the accuracy and for verification. From the tables, it is clear that the results obtained using present method agrees very closely with the other results (Yokoyama 1991, Valsangkar and Pradhanang 1988). It is also shown that the increasing value of K has an important effect on the frequencies and mode shapes. Increase in stiffness parameter, K of beam-column cause increase in the frequencies. Fundamental frequency parameters of Euler-Bernoulli beam column with partially embedded on Winkler-Pasternak foundation are given in Table 4 for different foundation parameters. The general trends of the frequencies are very similar to those of Euler-Bernoulli beam columns embedded in Winkler foundation. However, the effect of Pasternak foundation parameter on frequencies results is less than the Winkler

Table 1 Comparison of frequency parameters of Euler-Bernoulli beam column embedded in Winkler foundation ($G = 0$; $\gamma = 1$)

K	Mode 1				
	Yokoyama (1991)	Valsangkar and Pradhanang (1988)	Present DSC N=9	Present DSC N=11	Present DSC N=15
1	3.15	3.15	3.17	3.15	3.15
100	3.75	3.75	3.76	3.75	3.75
10000	10.02	10.02	10.05	10.03	10.02
1000000	31.62	31.62	31.64	31.63	31.62

Table 2 Comparison of frequency parameters of Euler-Bernoulli beam column embedded in Winkler foundation ($G = 0$; $\gamma = 1$)

K	Mode 2				
	Yokoyama (1991)	Valsangkar and Pradhanang (1988)	Present DSC N=9	Present DSC N=11	Present DSC N=15
1	6.28	6.28	6.30	6.28	6.28
100	6.38	6.38	6.39	6.38	6.38
10000	10.37	10.36	10.38	10.37	10.37
1000000	31.64	31.63	31.66	31.65	31.64

Table 3 Comparison of frequency parameters of Euler-Bernoulli beam column embedded in Winkler foundation ($G = 0$; $\gamma = 1$)

K	Mode 3				
	Yokoyama (1991)	Valsangkar and Pradhanang (1988)	Present DSC N=9	Present DSC N=11	Present DSC N=15
1	9.43	9.42	9.44	9.43	9.43
100	9.45	9.45	9.46	9.45	9.45
10000	11.57	11.57	11.60	11.59	11.58
1000000	31.69	31.72	31.72	31.70	31.70

parameter. In order to examine further the influence of the ratio of supported length to total length of beam-column, an Euler-Bernoulli beam column embedded in Winkler-Pasternak foundation is studied with $N = 15$ grids. The results are presented in Table 5. In general, the value of frequencies gradually increases with increasing of supported length to total length of beam-column for all type of foundation parameters. Fundamental frequency parameters of Euler-Bernoulli beam column embedded in Winkler-Pasternak foundation for different taper ratios are listed in Table 6 and Table 7. The results are presented for different axial load. It is seen that for a beam-column embedded in two-parameter foundation, the frequencies increases slowly as the taper ratio increases. It is also observed from these tables, the frequency parameter for all the four taper ratios decrease as axial load increase. However, it is also shown that, the effect of taper ratios on frequencies is insignificant. Numerical results are presented in Table 8 for first three frequency parameters of Euler-Bernoulli beam column embedded in Winkler-Pasternak foundation. It is shown from the Table; the Pasternak parameter has significant effect on the frequency when the mode number is increase.

Variation of frequency values with the Winkler parameters are depicted in Fig. 2 for a fully supported beam-column. Variations of first three frequencies with the Winkler parameters for different values of γ are also shown in Fig. 3. It is concluded from these figures that, for small values of γ, the effect of K on frequencies is negligible. But in general, the frequency parameters are increased

Table 4 Fundamental frequency parameters of Euler-Bernoulli beam column embedded in Winkler-Pasternak foundation ($\gamma = 0.75$; $P = 0.25P_{cr}$)

Foundation parameters		Present results			
K	G	DSC N=9	DSC N=11	DSC N=13	DSC N=15
100	5	3.790	3.789	3.788	3.788
10000	50	8.636	8.635	8.635	8.635
1000000	500	12.471	12.470	12.469	12.469

Table 5 Fundamental frequency parameters of Euler-Bernoulli beam column embedded in Winkler-Pasternak foundation ($\alpha = \beta = 1$; $N = 15$; $P = 0.25P_{cr}$) for different support ratios

Foundation parameters		Present results			
K	G	$\gamma = 0.25$	$\gamma = 0.5$	$\gamma = 0.75$	$\gamma = 1$
100	5	3.038	3.441	3.788	3.865
10000	50	4.177	5.735	8.635	10.138
1000000	500	4.798	6.971	12.469	31.664

Table 6 Fundamental frequency parameters of Euler-Bernoulli beam column embedded in Winkler-Pasternak foundation ($N = 15$; $\gamma = 0.5$; $P = 0.25P_{cr}$) for different taper ratios

Foundation parameters		Present results		
K	G	$\alpha = \beta = 1$	$\alpha = \beta = 1.25$	$\alpha = \beta = 1.5$
100	5	3.441	3.537	3.649
10000	50	5.735	5.954	6.214
1000000	500	6.971	7.738	7.873

Table 7 Fundamental frequency parameters of Euler-Bernoulli beam column embedded in Winkler-Pasternak foundation (N = 15; $\gamma = 0.5$; P = P_{cr}) for different taper ratios

Foundation parameters		Present results		
K	G	$\alpha = \beta = 1$	$\alpha = \beta = 1.25$	$\alpha = \beta = 1.5$
100	5	2.849	2.857	2.901
10000	50	5.488	5.736	6.079
1000000	500	6.758	7.324	7.723

Table 8 First three frequency parameters of Euler-Bernoulli beam column embedded in Winkler-Pasternak foundation (N = 15; $\gamma = 0.75$; P = 0.25P_{cr}; $\alpha = \beta = 1.5$)

Foundation parameters		Present results		
K	G	Mode 1	Mode 2	Mode 3
10	5	3.605	7.028	10.486
100	5	3.848	7.076	10.509
10000	100	8.821	10.392	12.473
1000000	500	14.135	24.851	30.018

with the increase of *K*. Variation of frequency values with the Winkler parameters for different taper ratios are presented in Fig. 4. Generally, it is shown that the increasing value of taper ratio always increases the frequency parameter. Also, with the increase of *K*, the effect of the taper ratio on the frequency parameter is less significant. Variation of mode shapes with the Winkler and Pasternak parameters are presented in Fig. 5 and Fig. 6, respectively. It is easily shown from these figures, for all mode numbers, the frequencies increase considerably with Winkler parameters. However, the frequency parameter is uniformly increased when the Pasternak ratio increases. Variations of first three frequencies with the taper ratios are presented in Fig. 7. It is shown that the increasing value

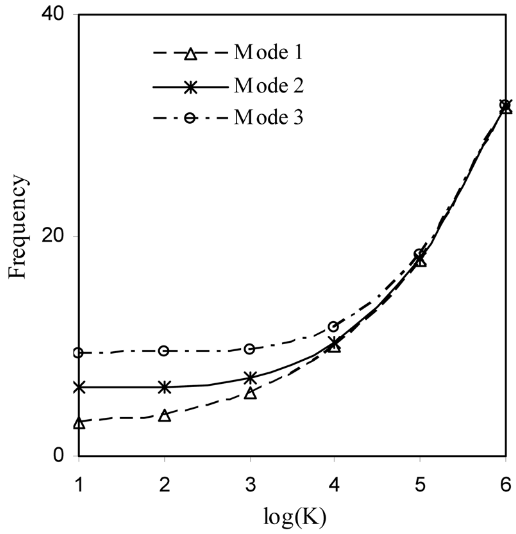

Fig. 2 Variation of frequency values with the Winkler parameters ($\alpha = \beta = 1$; P = 0.25P_{cr}; $\gamma = 1.0$)

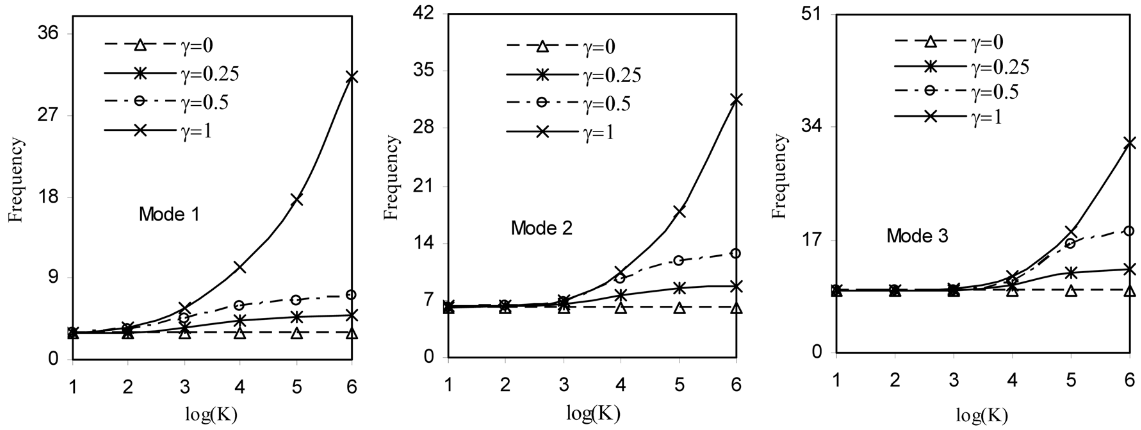

Fig. 3 Variation of first three frequencies with the Winkler parameters for different values of $\gamma = L_s/L$ ($\alpha = \beta = 1$; $P = 0.25P_{cr}$)

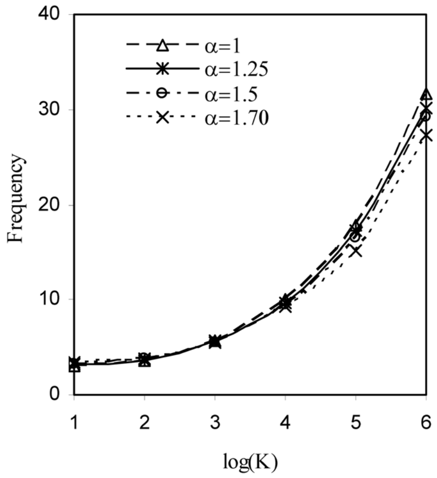

Fig. 4 Variation of frequency values with the Winkler parameters for different taper ratios (r $P = 0.25P_{cr}$; $\gamma = 1.0$)

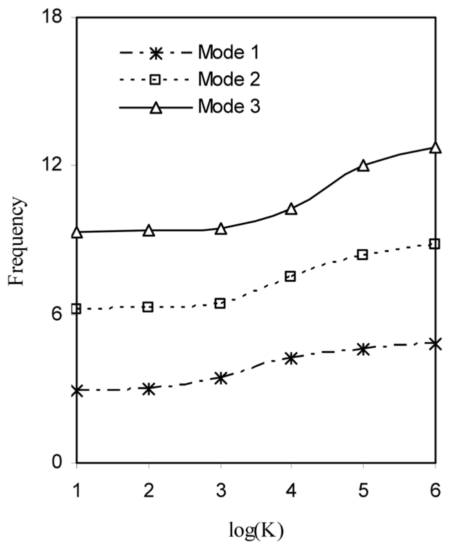

Fig. 5 Variation of frequency values with the Winkler parameters ($\alpha = \beta = 1$; $P = 0.25P_{cr}$; $\gamma = 0.25$)

of taper ratio, always increases the frequency parameter. However, with the increase of mode numbers the effect of the taper ratio on the frequency parameter is less significant. Fig. 8 shows the variation of frequency values with the ratio of supported length to total length of beam-column for first three modes. As can be observed from this figure, the frequency parameters generally increase with increasing the ratio of supported length to total length of beam-column whereas the effect of this ratio on the frequency parameter significant, especially for large values of K.

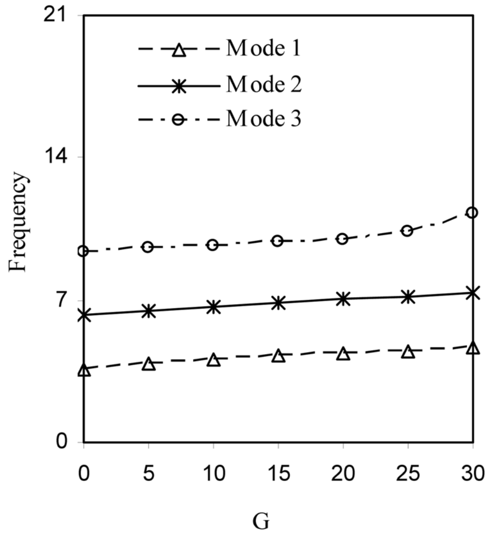

Fig. 6 Variation of frequency values with the Pasternak parameters ($\alpha = \beta = 1$; $P = 0.25P_{cr}$; $\gamma = 1$; $K = 100$)

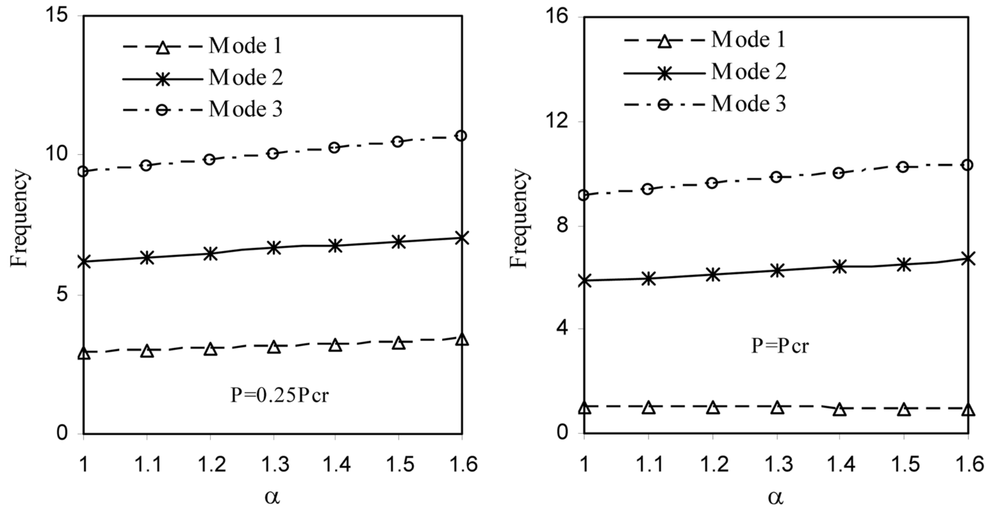

Fig. 7 Variation of frequency values with the taper ratios ($\gamma = 1$; $K = 1$)

5. Conclusions

The free vibration problem of tapered piles with pinned ends embedded in a two-parameter elastic foundation is the focus of the investigation. Several numerical examples are provided to show the effects of taper ratios, axial force, foundation parameters, and partial elastic foundation on frequencies. It is shown that the stiffness parameter of the Winkler foundation and shear parameter of the Pasternak foundation have been found to have a significant influence on frequencies of the beam-column. The effect of the Winkler parameter on the frequencies is greater than the Pasternak parameter. Increase in taper ratios of beam-column cause insufficient increases in the frequencies. The parameter of the Winkler foundation has been found to have a significant influence on the

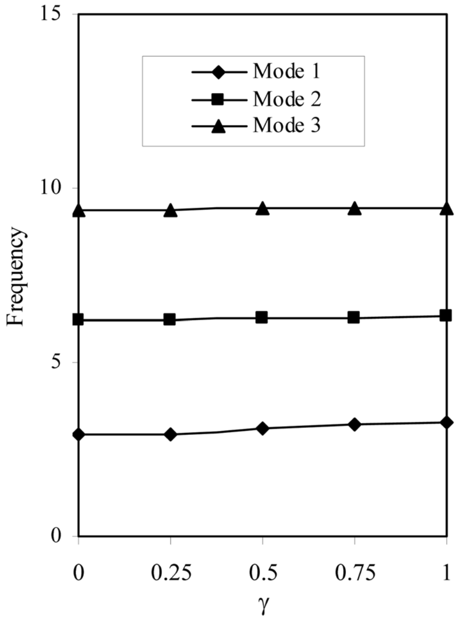

Fig. 8 Variation of frequency values with the ratio of supported length to total length of beam-column ($\alpha = \beta = 1$; $P = 0.25P_{cr}$; $K = 10$; $G = 1$)

frequencies of the beam-column with any taper ratio. Increasing the value of γ has a small effect than the foundation parameters on the frequencies and mode shapes. It may be concluded that increasing the ratio of supported length to total length, γ will always result in increased frequencies. The effect of the value of the axial force equal to or less than the Euler buckling load leads to a significant effect in frequency values. The efficiency and accuracy of the present method have been demonstrated on the basis of presented numerical examples. It is shown that the present method provides an appropriate and sufficient approach for the vibration analysis of beam-columns with pinned ends and embedded in Winkler-Pasternak elastic foundation. Present work also indicates that the method of discrete singular convolution is promising and a potential approach for computational solid mechanics.

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