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Torsional waves in fluid saturated porous layer clamped between two anisotropic media

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Abstract. The paper aims to analyze the behaviour of torsional type surface waves propagating through fluid saturated inhomogeneous porous media clamped between two inhomogeneous anisotropic media. We considered three types of inhomogeneities in upper anisotropic layer which varies exponentially, quadratically and hyperbolically with depth. The anisotropic half space inhomogeneity varies linearly with depth and intermediate layer is taken as inhomogeneous fluid saturated porous media with sinusoidal variation. Following Biot, the dispersion equation has been derived in a closed form which contains Whittaker's function and its derivative, for approximate result that have been expanded asymptotically up to second term. Possible particular cases have been established which are in perfect agreement with standard results and observe that when one of the upper layer vanishes and other layer is homogeneous isotropic over a homogeneous half space, the velocity of torsional type surface waves coincides with that of classical Love type wave. Comparative study has been made to identify the effects of various dimensionless parameters viz. inhomogeneity parameters, anisotropy parameters, porosity parameter, and initial stress parameters on the torsional wave propagation by means of graphs using MATLAB. The study has its own relevance in connection with the propagation of seismic waves in the earth where fluid saturated poroelastic layer is present.

Keywords: torsional surface wave; inhomogeneity; anisotropy; porosity; dispersion equation; group velocity

1. Introduction

Surface waves in elastic medium have been well recognized in the study of seismology, earthquakes, geodynamics and geophysics and also give important information about the layered earth structure and have been used to accurately determine the earthquake epicenter, and also important to seismologists for understanding the causes and estimation of damage due to earthquakes. The wave motion is located at the outside surface itself, and as the depth below this surface increases, wave displacement becomes less and less. When an earthquake or explosion occurs, a part of the energy released through elastic waves is transmitted through the earth. The propagation of seismic waves in layered media is of central interest to the theoretical seismologists, on the grandest scale, with torsional wave as one of the surface waves which propagates horizontally but give a twist to the medium when it propagates observable after several trips around the world, and their systematic study has obvious implications for human safety, as well as for a curiosity concerning the structure and evolution of the earth.

To give a chronological account of the available literature on the propagation of surface waves in layered media, we start with the original work presented by Rayleigh (1885), who studied the propagation of elastic waves on the free surface of a semi-infinite solid, where he showed that the isotropic homogeneous elastic half space does not allow a torsional surface wave to propagate. Later on, Meissner (1921) had inquired that, in an inhomogeneous elastic half-space with a quadratic variation in the depth of the shear modulus and the mass density varying linearly with the depth, torsional surface waves do exist. Birch (1952) investigated that rigidity of the earth layers varies at different rates with respect to depth. The basic literature on the propagation of elastic waves in purely elastic media has been studied extensively in the well-known book of Love (1944). After that the propagation of surface waves in detail is well documented and application of mathematical modeling in elasticity theory and seismology has been given in the text book literature of Bath (1968), Achenbach (1973), Straughan (2008), and Carcione (2015). Biot (1962, 1966) formulated the governing equations and constitutive relations for predicting the frequency-dependent velocities of a fluid saturated porous elastic solid in terms of the dry-rock properties. Since then many of the implications and applications of this medium, a large number of papers by many researchers, over dynamical problems in the propagation of surface waves in a fluid saturated porous media have been published in different journals after the publication of this book. Notable among these are

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Deresiewicz (1964), Yew and Jogi (1976), Konczak (1989), Sharma and Gogna (1991), Dai and Kuang (2006), Ke *et al.* (2006), Ghorai *et al.* (2010), Son and Kang (2012), Wang and Tian (2014), Gupta and Ahmed (2017). Deresiewicz (1964) characterizes Loves waves in a porous layer overlying on elastic half-space and in a porous layer between two elastic half-space and was the first who derived the dispersion equation for Love waves in a porous solid.

Most of the theoretical seismologists and earth scientists are interested in investigating the seismic waves in layered media. In the last few years, continued efforts have been expended upon modeling and dynamical behavior of a surface wave propagation interaction with layered anisotropic fluid saturated porous media, mostly for applications in the fields of earthquake engineering, soil dynamics and fluid dynamics other than seismology. Study of wave motions in fluid-saturated porous rocks has been done by Yew and Jogi (1976). Sharma and Gogna (1991) considered Love waves in an initially stressed medium consisting of a slow elastic layer lying over a liquid-saturated porous half space with small porosity. Wang and Zhang (1998) inquired the propagation of Love waves in a transversely isotropic fluid-saturated porous layered half spaced in detail, and gave dispersion and attenuation curves and an effective iterative method was developed to solve the complex dispersion equation. Dai and Kuang (2006) analyzed the dispersion and attenuation properties of Love wave in double porosity media also they approximated the limit of Love wave speed.

In comparison to the extensive literature on the interaction of plane harmonic waves such as Rayleigh, Love and Stoneley waves with various layered media, much less work is available on the propagation of torsional surface wave in layered media. In the past few years, attention has been given to the problems of generation and propagation of torsional waves in an anisotropic elastic solids or layers of different configurations. By virtue Biot's theory, Samal and Chattaraj (2011), Gupta and Gupta (2011), Gupta *et al.* (2012), Chattopadhyay *et al.* (2013a,b), Shekhar and Parvez (2016a,b) in a series of papers, studied the propagation of torsional surface waves in different medium, different type of irregularities, under different conditions. Recently, Alam *et al.* (2017) presented on propagation of a torsional wave in a doubly-layered half-space structure of an initially stressed heterogeneous viscoelastic layer sandwiched between a dry sandy layer and a half-space of heterogeneous media.

Most natural porous materials, such as rocks and sediments in the Earth, are heterogeneous in the porous continuum so the propagation of torsional wave through rocks is affected by fluid/solid interactions. Over the past few decades, much interest has been focused on the effect of pre-stress or initial stress on the wave propagation. The word 'initial stress' is stress which developed a medium before it is being used for study. The Earth is initially stressed medium, due to presence of differential external forces, gravity variations, slow process of creep, process of quenching, difference of temperature, gravitational field and cold working etc., considerable amount of stresses which are called pre-stresses or initial stresses, remain naturally present in the layers. Remembering the above all facts, the present problem discusses to study the propagation of a torsional type surface wave in an initially stressed fluid saturated porous media sandwiched between inhomogeneous anisotropic layer and inhomogeneous anisotropic half space under influence of initial stress. Bullen (1940) has found that the density inside the earth varies with increase of depth which is possible due to presence of inhomogeneity of the layers. Sharma and Kumar (2016) investigated shear horizontal wave propagation in a layered structure, consisting of granular macromorphic rock substrate underlying a viscoelastic layer of finite thickness. Ke *et al.* (2006) presented Love waves in an inhomogeneous fluid saturated porous layered half-space with linearly varying properties. Gupta and Gupta (2011) studied torsional surface waves in gravitating anisotropic porous half space. Shekhar and Parvez (2016b) investigated the propagation of torsional surface waves in an inhomogeneous anisotropic fluid saturated porous layered half space under initial stress with varying properties also in same year, the propagation of torsional surface waves in a double porous layer lying over a Gibson half space has been studied by the same authors (2016a).

Recently Kumari *et al.* (2016) contributed their thought on the propagation of torsional waves in a viscoelastic layer over a viscoelastic substratum of Voigt types. Chattaraj *et al.* (2015) devised a model on torsional surface waves in a dry sandy layer over an inhomogeneous half space by using WKB approximation method. In this paper, three types of inhomogeneities in the upper anisotropic layer has taken; namely exponential, quadratic and hyperbolic whereas in the fluid saturated porous layer inhomogeneity is taken as sinusoidal under initial stress and for prestressed inhomogeneous anisotropic half space inhomogeneity are taken as linearly and all type of heterogeneity in the anisotropic medium are taken along z-directions. The variable separation method is used for theoretical derivations and analytical solutions are obtained for dispersion relation by means of the Whittaker function and its derivative where asymptotic expansion of Whittaker function and its derivatives has been taken up to second degree term. The dispersion properties of seismic waves in the Earth depend critically on the elastic properties and thicknesses of the layers and also provide important data about the configuration and state of the deep interior of the globe. The study of surface waves in an initially stressed fluid saturated porous medium is of interest not for theoretical taste only but for practical purposes too.

2. Formulation of the problem

In the present paper we consider two layers (M_1) and (M_2) lying over an initially stressed non homogeneous anisotropic half space (M_3) where initial stress, density, and rigidities vary linearly and the cylindrical coordinate system (r, θ, z) is introduced to study torsional surface wave with z-axis directed to downward positive. The origin O is taken at the common interface between inhomogeneous fluid saturated porous layer and the half space which pervades over the region $-\infty < r < \infty, z \geq 0$, see Fig. 1. The sandwiched layer (M_2) of thickness H_1 is inhomogeneous porous media with sinusoidal variation, which is also fluid saturated under presence of initial stress. The density and rigidities of the uppermost inhomogeneous anisotropic layer (M_1) of thickness (H_2-H_1) are varying different types of

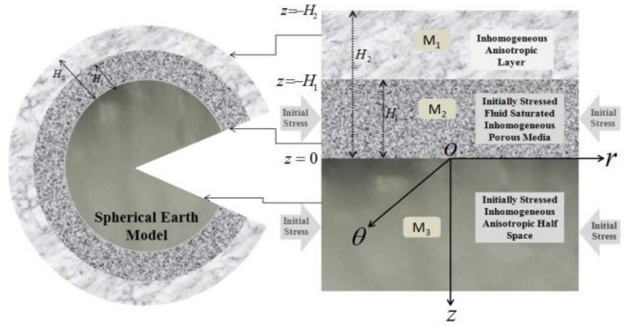


Fig. 1 Geometry of the problem

inhomogeneities, viz. exponential, quadratic and hyperbolic with depth.

3. Solution of the problem

3.1 Solution for the upper inhomogeneous anisotropic layer (M$_1$)

The dynamical equilibrium equations of motion in the absence of external forces for the system in cylindrical coordinate are given by Biot (1966)

$$\left.\begin{aligned}
&\frac{\partial \tau_{rr}}{\partial r}+\frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta}+\frac{\partial \tau_{rz}}{\partial z}+\frac{\tau_{rr}-\tau_{\theta\theta}}{r}=\rho^{(1)}\frac{\partial^2 u}{\partial t^2}\\
&\frac{\partial \tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta}+\frac{\partial \tau_{\theta z}}{\partial z}+\frac{2}{r}\tau_{r\theta}=\rho^{(1)}\frac{\partial^2 v}{\partial t^2}\\
&\frac{\partial \tau_{rz}}{\partial r}+\frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta}+\frac{\partial \tau_{zz}}{\partial z}+\frac{1}{r}\tau_{rz}=\rho^{(1)}\frac{\partial^2 w}{\partial t^2}
\end{aligned}\right\} \tag{1}$$

where $\tau_{rr},\tau_{zz},\tau_{\theta\theta},\tau_{r\theta},\tau_{rz}$ and $\tau_{\theta z}$ are the corresponding stress components in their conventional sense and u, v and w are the displacement components along radial, circumferential and axial direction respectively, and $\rho^{(1)}$ is the density of the medium.

Now, the stress-strain relations for anisotropic medium are given by generalized Hooke's law

$$\tau_{ij}=C_{ijmn}\,e_{mn}\quad (i,j=1,2,..,6) \tag{2}$$

where τ_{ij} are stress components, C_{ijmn} are components of elasticity matrix, and e_{mn} are strain components. Here, linear elasticity, small strains, and the Cauchy stress tensor are considered.

The strain-displacement relations for anisotropic material are

$$\left.\begin{aligned}
&e_{rr}=\frac{1}{2}\left(\frac{\partial u}{\partial r}\right),e_{zz}=\frac{1}{2}\left(\frac{\partial w}{\partial z}\right),e_{\theta\theta}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{u}{r}\right),\\
&e_{r\theta}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial u}{\partial \theta}+\frac{\partial v}{\partial r}-\frac{v}{r}\right),e_{rz}=\frac{1}{2}\left(\frac{\partial w}{\partial r}+\frac{\partial u}{\partial z}\right),e_{\theta z}=\frac{1}{2}\left(\frac{\partial v}{\partial z}+\frac{1}{r}\frac{\partial w}{\partial \theta}\right)
\end{aligned}\right\} \tag{3}$$

Assuming that the torsional surface wave travels in radial direction and that of all the mechanical properties associated with it are independent of θ, so for torsional surface wave, u=0, v=$v_1(r, z, t)$ and w=0. So the only non-vanishing equation of motion for the propagation of a torsional surface wave without body force may be written as

$$\frac{\partial \tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta}+\frac{\partial \tau_{\theta z}}{\partial z}+\frac{2}{r}\tau_{r\theta}=\rho^{(1)}(z)\frac{\partial^2 v_1}{\partial t^2} \tag{4}$$

with $v_1(r, z, t)$ being the displacement along the θ (azimuthal) direction.

For an inhomogeneous anisotropic elastic medium, the non-vanishing stresses are related to the strain component by

$$\tau_{r\theta}=2N^{(1)}e_{r\theta},\tau_{z\theta}=2L^{(1)}e_{z\theta} \tag{5}$$

where $N^{(1)}$ and $L^{(1)}$ are the directional rigidities of the medium along the r- and z- directions, respectively.

Using the above relations, Eq. (4) takes the form

$$N^{(1)}\left(\frac{\partial^2 v_1}{\partial r^2}+\frac{1}{r}\frac{\partial v_1}{\partial r}-\frac{v_1}{r^2}\right)+\frac{\partial}{\partial z}\left(L^{(1)}\frac{\partial v_1}{\partial z}\right)=\rho^{(1)}(z)\frac{\partial^2 v_1}{\partial t^2} \tag{6}$$

The solution of Eq. (6) when a wave propagates along the radial direction with an amplitude of displacement as a function of depth may be taken as

$$v_1=V_{10}(z)J_1(kr)e^{i\omega t} \tag{7}$$

where $V_{10}(z)$ is the solution of

$$\frac{d^2V_{10}}{dz^2}+\frac{1}{L^{(1)}}\frac{dL^{(1)}}{dz}\frac{dV_{10}}{dz}-\frac{k^2N^{(1)}}{L^{(1)}}\left(1-\frac{c^2\rho^{(1)}}{N^{(1)}}\right)V_{10}(z)=0 \tag{8}$$

where k is the wave number, c is the common wave velocity or phase velocity of torsional wave, $\omega(=kc)$ is the circular frequency and $J_1(kr)$ is the Bessel's function of first kind and of order one.

Putting $V_{10}(z)=V_{11}(z)\big/\sqrt{L^{(1)}}$ in Eq. (8) we get

$$\frac{d^2V_{11}}{dz^2}-\frac{1}{2L^{(1)}}\left\{\frac{d^2L^{(1)}}{dz^2}-\frac{1}{2L^{(1)}}\left(\frac{dL^{(1)}}{dz}\right)^2\right\}V_{11}=\frac{k^2N^{(1)}}{L^{(1)}}\left(1-\frac{\rho^{(1)}c^2}{N^{(1)}}\right)V_{11} \tag{9}$$

The following variations in directional rigidities and density of the layer are taken as follows

$$\left.\begin{aligned}
&\text{Case – A:}\quad N^{(1)}=N_1e^{z/a},L^{(1)}=L_1e^{z/a},\rho^{(1)}=\rho_1e^{z/a}\\
&\text{Case – B:}\quad N^{(1)}=N_1(1+z/a)^2,L^{(1)}=L_1(1+z/a)^2,\rho^{(1)}=\rho_1(1+z/a)^2\\
&\text{Case – C:}\quad N^{(1)}=N_1\cosh^2(z/a),L^{(1)}=L_1\cosh^2(z/a),\rho^{(1)}=\rho_1\cosh^2(z/a)
\end{aligned}\right\} \tag{10}$$

Using Eq. (10), Eq. (9) becomes

$$\frac{d^2V_{11}}{dz^2}+\lambda_l^2V_{11}=0 \tag{11}$$

where $\lambda_1(l$=1,2,3) are given in Appendix-A and indices l=1,2,3 denotes the cases for A, B, and C respectively and $c_1=\sqrt{N_1/\rho_1}$ is the shear wave velocity in the layer.

The solution of Eq. (11) is given by

$$V_{11}(z)=D_1\sin\lambda_l z+D_2\cos\lambda_l z \tag{12}$$

where D_1 and D_2 are arbitrary constants. Therefore the displacement component for torsional wave in the upper inhomogeneous anisotropic layer is given by

$$v_1=\left(D_1\sin\lambda_l z+D_2\cos\lambda_l z\right)J_1(kr)\left(L^{(1)}\right)^{-1/2}e^{i\omega t} \tag{13}$$

3.2 Solution for the intermediate fluid saturated porous layer under initial stress (M$_2$)

Neglecting the viscosity of the liquid and body force, dynamical equations for poroelastic medium under initial stress $P^{(2)}$, are given by Biot 1962. Those are

$$\left.\begin{aligned}&\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta}+\frac{\partial\sigma_{rz}}{\partial z}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}-P^{(2)}\left(\frac{\partial\omega_z}{\partial\theta}-\frac{\partial\omega_\theta}{\partial z}\right)=\frac{\partial^2}{\partial t^2}(\rho_{rr}u_r+\rho_{r\theta}U_r)\\&\frac{\partial\sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+\frac{\partial\sigma_{\theta z}}{\partial z}+\frac{2}{r}\sigma_{r\theta}-P^{(2)}\frac{\partial\omega_z}{\partial r}=\frac{\partial^2}{\partial t^2}(\rho_{rr}v_\theta+\rho_{r\theta}V_\theta)\\&\frac{\partial\sigma_{rz}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta z}}{\partial\theta}+\frac{\partial\sigma_{zz}}{\partial z}+\frac{1}{r}\sigma_{rz}+P^{(2)}\frac{\partial\omega_\theta}{\partial r}=\frac{\partial^2}{\partial t^2}(\rho_{rr}w_z+\rho_{r\theta}W_z)\end{aligned}\right\}\tag{14}$$

and

$$\frac{\partial\sigma}{\partial r}=\frac{\partial^2}{\partial t^2}(\rho_{r\theta}u_r+\rho_{\theta\theta}U_r),\frac{\partial\sigma}{\partial\theta}=\frac{\partial^2}{\partial t^2}(\rho_{r\theta}v_\theta+\rho_{\theta\theta}V_\theta),\frac{\partial\sigma}{\partial z}=\frac{\partial^2}{\partial t^2}(\rho_{r\theta}w_z+\rho_{\theta\theta}W_z)\tag{15}$$

where $\sigma_{ij}(i, j,=r, \theta, z)$ are the incremental stress components and (u_r, v_θ, w_z) are the components of the displacement vector of the solid, (U_r, V_θ, W_z) are the component of the displacement vector of the liquid and σ is the stress vector due to liquid and

$$\omega_r=\frac{1}{2r}\left(\frac{\partial w_z}{\partial\theta}-r\frac{\partial v_\theta}{\partial z}\right),\omega_\theta=\frac{1}{2}\left(\frac{\partial u_r}{\partial z}-\frac{\partial w_z}{\partial r}\right),\omega_z=\frac{1}{2r}\left(\frac{\partial(rv_\theta)}{\partial r}-\frac{\partial v_\theta}{\partial\theta}\right)\tag{16}$$

are the components of the rotational vector ω.

The porous medium represents volumetrically interacting solid-fluid aggregates, which can be modeled using continuum porous media theories by allowing for both solid-matrix deformation and fluid flow so the stress-strain relations for the liquid saturated anisotropic porous layer under normal stress $P^{(2)}$ are

$$\left.\begin{aligned}&\sigma_{rr}=(A^{(2)}+P^{(2)})\delta_{rr}+(A^{(2)}-2N^{(2)}+P^{(2)})\delta_{\theta\theta}+(F^{(2)}+P^{(2)})\delta_{zz}+S_\varepsilon,\\&\sigma_{r\theta}=2N^{(2)}\delta_{r\theta},\sigma_{\theta\theta}=(A^{(2)}-2N^{(2)})\delta_{rr}+A^{(2)}\delta_{\theta\theta}+F^{(2)}\delta_{zz}+S_\varepsilon,\\&\sigma_{\theta z}=2L^{(2)}\delta_{\theta z},\sigma_{zz}=F^{(2)}\delta_{rr}+F^{(2)}\delta_{\theta\theta}+C^{(2)}\delta_{zz}+S_\varepsilon,\sigma_{zr}=2L^{(2)}\delta_{zr}\end{aligned}\right\}\tag{17}$$

where $A^{(2)}$, $F^{(2)}$, $C^{(2)}$, $N^{(2)}$ and $L^{(2)}$ are elastic constants of the medium. $N^{(2)}$ and $L^{(2)}$ are in particular, shear moduli of the anisotropic layer in the radial and the z-direction respectively, and

$$\left.\begin{aligned}&\delta_{rr}=\frac{\partial u_r}{\partial r},\delta_{\theta\theta}=\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{u_r}{r},\delta_{zz}=\frac{\partial w_z}{\partial z},\delta_{zr}=\frac{1}{2}\left(\frac{\partial w_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)\\&\delta_{r\theta}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}+\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}\right),\delta_{\theta z}=\frac{1}{2}\left(\frac{\partial v_\theta}{\partial z}+\frac{1}{r}\frac{\partial w_z}{\partial\theta}\right)\end{aligned}\right\}\tag{18}$$

Further, S_ε being the measure of coupling between the volume change of the solid and the liquid is a positive quantity. The relation between stress vector σ due to presence of liquid in the porous solid and the fluid pressure P_* can be represented as

$$\sigma=-\phi P_*\tag{19}$$

where ϕ is the porosity of the medium. The mass coefficients ρ_{rr}, $\rho_{r\theta}$ and $\rho_{\theta\theta}$ are related to the densities ρ, ρ_s and ρ_f of the layer, the solid and the liquid respectively by

$$\rho_{rr}+\rho_{r\theta}=(1-\phi)\rho_s\,,\rho_{r\theta}+\rho_{\theta\theta}=\phi\rho_f\tag{20}$$

So the mass density of the of the aggregate is

$$\rho'=\rho_{rr}+2\rho_{r\theta}+\rho_{\theta\theta}=\rho_s+\phi(\rho_f-\rho_s)\tag{21}$$

Also the mass coefficients obey the following inequalities

$$\rho_{rr}>0,\rho_{\theta\theta}>0,\rho_{r\theta}<0,\rho_{rr}\rho_{\theta\theta}-\rho_{r\theta}^2>0\tag{22}$$

For the propagation of torsional surface waves along the radial direction and having displacement of particles along the θ direction we have

$$u_r=0,v_\theta=v_2(r,z,t),w_z=0;\;U_r=0,V_\theta=V_2(r,z,t),W_z=0\tag{23}$$

The above displacements will produce $\delta_{\theta z}$ and $\delta_{r\theta}$ strain components and the other strain components will be zero. Hence the final stress-strain relations are

$$\sigma_{\theta z}=2L^{(2)}\delta_{\theta z},\sigma_{r\theta}=2N^{(2)}\delta_{r\theta}\tag{24}$$

Now Eq. (23) makes, Eqs. (14) and (15) as follows

$$\frac{\partial\sigma_{r\theta}}{\partial r}+\frac{\partial\sigma_{\theta z}}{\partial z}+\frac{2}{r}\sigma_{r\theta}-P^{(2)}\frac{\partial\omega_z}{\partial r}=\frac{\partial^2}{\partial t^2}(\rho_{rr}v_2+\rho_{r\theta}V_2)\tag{25}$$

$$\frac{\partial^2}{\partial t^2}(\rho_{r\theta}v_2+\rho_{\theta\theta}V_2)=0\tag{26}$$

Using stress-strain relations of Eq. (17), Eq. (25) can be written as

$$\left(N^{(2)}-\frac{P^{(2)}}{2}\right)\left(\frac{\partial^2v_2}{\partial r^2}-\frac{v_2}{r^2}+\frac{1}{r}\frac{\partial v_2}{\partial r}\right)+\frac{\partial}{\partial z}\left(L^{(2)}\frac{\partial v_2}{\partial z}\right)=\frac{\partial^2}{\partial t^2}(\rho_{rr}v_2+\rho_{r\theta}V_2)\tag{27}$$

From Eq. (26), let us assume $d''=\rho_{r\theta}v_2+\rho_{\theta\theta}V_2$ i.e., $V_2=(d''-\rho_{r\theta}v_2)/\rho_{\theta\theta}$ and follows

$$\frac{\partial^2}{\partial t^2}(\rho_{rr}v_2+\rho_{r\theta}V_2)=\mathrm{d}'\frac{\partial^2v_2}{\partial t^2}\tag{28}$$

where $d'=\rho_{rr}-\rho_{r\theta}^2/\rho_{\theta\theta}$.

Using Eq. (28) in Eq. (27), we get

$$\left(N^{(2)}-\frac{P^{(2)}}{2}\right)\left(\frac{\partial^2v_2}{\partial r^2}-\frac{v_2}{r^2}+\frac{1}{r}\frac{\partial v_2}{\partial r}\right)+\frac{\partial}{\partial z}\left(L^{(2)}\frac{\partial v_2}{\partial z}\right)=d'\frac{\partial^2v_2}{\partial t^2}\tag{29}$$

For the wave propagating alone r- direction, the solution of Eq. (29) can be assume as

$$v_2=V_{20}(z)J_1(kr)e^{i\omega t}\tag{30}$$

where $\omega=ck$ is the angular velocity; c is the phase velocity; k is the angular wave number of the torsional surface waves; and $J_1(kr)$ is the Bessel's function of first kind with order one.

Thus, Eq. (29) can be expressed as

$$\frac{d^2V_{20}(z)}{dz^2}+\frac{1}{L^{(2)}}\frac{dL^{(2)}}{dz}\frac{dV_{20}(z)}{dz}-\frac{k^2}{L^{(2)}}\left\{\left(N^{(2)}-\frac{P^{(2)}}{2}\right)-d\rho'c^2\right\}V_{20}(z)=0\tag{31}$$

where $d=d'/\rho'=\left(\rho_{rr}-\rho_{r\theta}^2/\rho_{\theta\theta}\right)/\rho'=\gamma_{11}-\gamma_{12}^2/\gamma_{22}$ is a non-dimensional constant, called poro- elastic parameter and $\gamma_{11}=\rho_{rr}/\rho',\gamma_{12}=\rho_{r\theta}/\rho',\gamma_{22}=\rho_{\theta\theta}/\rho'$ are also non-dimensional parameters.

Here the following cases may be discussed,

(i) For porous layer: If $\phi\to1$ then $\rho_f\to\rho'$, thus the bulk material becomes fluid i.e., $\gamma_{11}-\gamma_{12}^2/\gamma_{22}\to0$ or, $d\to0$. It means shear wave do not exist.

(ii) For non-porous layer: If the layer is non porous then $d\to0$and $\rho_s\to\rho'$ which leads to $\gamma_{11}+\gamma_{12}\to1$ and $\gamma_{11}+\gamma_{22}\to0$, which gives to $\gamma_{11}-\gamma_{12}^2/\gamma_{22}\to1$ or $d\to1$.

Thus for porous layer 0<d<1.

Putting the substitution $V_{20}(z)=V_{21}(z)\left(L^{(2)}(z)\right)^{-1/2}$ in Eq. (31), there follows

$$\frac{d^2V_{21}(z)}{dz^2}-\frac{1}{2L^{(2)}}\left\{\frac{d^2L^{(2)}}{dz^2}-\frac{1}{2L^{(2)}}\left(\frac{dL^{(2)}}{dz}\right)^2\right\}V_{21}(z)-\frac{k^2}{L^{(2)}}\left\{\left(N^{(2)}-\frac{P^{(2)}}{2}\right)-d\rho'c^2\right\}V_{21}(z)=0 \quad (32)$$

Now, the variation of directional rigidity, initial stress and density with depth z, in the fluid saturated inhomogeneous porous layer has been taken as

$$N^{(2)}=N_2\left(1+\sin bz\right), L^{(2)}=L_2\left(1+\sin bz\right), P^{(2)}=P_2\left(1+\sin bz\right), \rho'=\rho_2\left(1+\sin bz\right) \quad (33)$$

where b is the constant having dimension that is inverse of length.

Using relation (33), Eq. (32) takes the form

$$\frac{d^2V_{21}(z)}{dz^2}+q^2V_{21}(z)=0 \quad (34)$$

The solution of Eq. (34) may be taken as $V_{21}(z)=\left(D_3\sin qz+D_4\cos qz\right)$ giving

$$v_2=\left(D_3\sin qz+D_4\cos qz\right)J_1(kr)\left(L^{(2)}\right)^{-1/2}e^{i\omega t} \quad (35)$$

where $q^2=k^2\left[\frac{N_2}{L_2}\left\{d\left(\frac{c}{c_2}\right)^2-1\right\}+\xi+\frac{b^2}{4k^2}\right]$, $\xi=\frac{P_2}{2L_2}$ and $c_2=\sqrt{\frac{N_2}{\rho_2}}$ is the velocity of the shear wave in the corresponding initial stress free poroelastic layer along the radial direction.

3.3 *Solution for initially stressed inhomogeneous anisotropic half space* (M_3)

Assuming r and θ are the radial and circumferential coordinates, respectively, and if the wave travels along the radial direction only, then the dynamical equation of motion for the initially stressed inhomogeneous anisotropic half space is given by Biot (1966)

$$\frac{\partial s_{r\theta}}{\partial r}+\frac{\partial s_{z\theta}}{\partial z}+\frac{2}{r}s_{r\theta}-\frac{\partial}{\partial z}\left(P^{(3)}\varepsilon_{z\theta}\right)=\rho^{(3)}\frac{\partial^2 v_3}{\partial t^2} \quad (36)$$

where $v_3(r, z, t)$ is the displacement along the θ directions; $\rho^{(3)}$ is the initial stress in the medium along the r-direction and $\rho^{(3)}$ is the density of the medium.

For the anisotropic elastic medium under initial stress, the stresses are related to strain by

$$s_{r\theta}=2N^{(3)}\varepsilon_{r\theta}, s_{z\theta}=2L^{(3)}\varepsilon_{z\theta} \quad (37)$$

where $\varepsilon_{r\theta}=\frac{1}{2}\left(\frac{\partial v_3}{\partial r}-\frac{v_3}{r}\right), \varepsilon_{z\theta}=\frac{1}{2}\left(\frac{\partial v_3}{\partial z}\right)$ and $N^{(3)}, L^{(3)}$ are rigidities of the medium along the r- and z- directions respectively.

Using the above relations, Eq. (37) becomes

$$N^{(3)}\left(\frac{\partial^2 v_3}{\partial r^2}+\frac{1}{r}\frac{\partial v_3}{\partial r}-\frac{v_3}{r^2}\right)+\frac{\partial}{\partial z}\left(\tilde{G}\frac{\partial v_3}{\partial z}\right)=\rho^{(3)}(z)\frac{\partial^2 v_3}{\partial t^2} \quad (38)$$

where $\tilde{G}(z)=L^{(3)}(z)-P^{(3)}(z)/2$.

The solution of Eq. (38) when a wave propagates along the radial direction with amplitude of displacement as a function of depth may be taken as

$$v_3(r,z,t)=V_{30}(z)J_1(kr)e^{i\omega t} \quad (39)$$

where $V_{30}(z)$ is the solution of

$$\frac{d^2V_{30}}{dz^2}+\frac{1}{\tilde{G}}\frac{d\tilde{G}}{dz}\frac{dV_{30}}{dz}+\frac{1}{\tilde{G}}\left(\rho^{(3)}\omega^2-N^{(3)}k^2\right)V_{30}(z)=0 \quad (40)$$

where k is the wave number, c is the common wave velocity, $\omega(=kc)$ is the circular frequency and $J_1(kr)$ is the Bessel's function of first kind and of order one.

On substituting $V_{30}(z)=V_{31}(z)/\sqrt{\tilde{G}}$ in Eq. (40) we get

$$\frac{d^2V_{31}}{dz^2}+\frac{1}{4}\frac{V_{31}}{\tilde{G}^2}\left(\frac{d\tilde{G}}{dz}\right)^2-\frac{V_{31}}{2\tilde{G}}\frac{d^2\tilde{G}}{dz^2}+\frac{k^2N^{(3)}}{\tilde{G}}\left(\frac{\rho^{(3)}c^2}{N^{(3)}}-1\right)V_{31}=0 \quad (41)$$

Now, the variations in directional rigidities (elastic moduli), density and initial stress in the substratum are taken as

$$N^{(3)}=N_3(1+\alpha_3 z), L^{(3)}=L_3(1+\beta_3 z), P^{(3)}=P_3(1+\gamma_3 z), \rho^{(3)}=\rho_3(1+\delta_3 z) \quad (42)$$

where α_3, β_3, γ_3 and δ_3 are the constants having dimension that are inverse of length.

Using Eq. (42) in Eq. (41), there follows

$$\frac{d^2V_{31}}{dz^2}+\left[\kappa^2\left\{\left(\frac{c}{c_3}\right)^2\frac{1+\delta_3 z}{1+\zeta z}-\frac{1+\alpha_3 z}{1+\beta_3 z}\right\}+\frac{1}{4}\frac{\zeta^2}{\left(1+\zeta z\right)^2}\right]V_{31}=0 \quad (43)$$

where $\kappa^2=\frac{k^2N_3}{\mathrm{h}}, \mathrm{h}=L_3(1-\mu), \zeta=\frac{\beta_3-\mu\gamma_3}{1-\mu}, \mu=\frac{P_3}{2L_3}$ and $c_3=\sqrt{\frac{N_3}{\rho_3}}$ is the shear wave velocity.

On substituting $V_{31}(\mathrm{z})=\psi(\chi)$ in Eq. (43), where $\chi=\left(2F\kappa(1+\zeta z)\right)/\zeta$ we get

$$\psi''(\chi)+\left\{\frac{1}{4\chi^2}+\frac{M}{\chi}-\frac{1}{4}\right\}\psi(\chi)=0 \quad (44)$$

where $F=\sqrt{\frac{\alpha_3}{\zeta}-\frac{\delta_3}{\zeta}\left(\frac{c}{c_3}\right)^2}$ and $M=\frac{\kappa}{2F\zeta}\left\{F^2+\left(\frac{c}{c_3}\right)^2-1\right\}$ are dimensionless quantities and Eq. (44) is the Whittaker equation (Whittaker and Watson (1991)) and solution is obtained as

$$\psi(\chi)=D_5W_{M,0}(\chi)+D_6W_{-M,0}(-\chi) \quad (45)$$

where D_5 and D_6 are arbitrary constants and $W_{M,0}(\chi)$ is the Whittaker function.

As the solution of Eq. (44) must be bounded and vanishes as $z\to\infty$ for the surface wave, i.e., $\chi\to\infty$, we may take the solution as $\psi(\chi)=D_5W_{M,0}(\chi)$.

Therefore, the displacement in a pre-stressed non-homogeneous anisotropic half-space is given by

$$v_3(r,z,t)=D_5\left\{W_{M,0}\left(2F\kappa(1+\zeta z)/\zeta\right)\right\}\Big/\sqrt{\mathrm{h}(1+\zeta z)}\;J_1(kr)e^{i\omega t} \quad (46)$$

Considering Whittaker function up to linear term, Eq. (46) takes the form

$$v_3=\frac{D_5}{\sqrt{\mathrm{h}(1+\zeta z)}}e^{-\frac{F\kappa(1+\zeta z)}{\zeta}}\sqrt{\frac{2F\kappa(1+\zeta z)}{\zeta}}\left\{1-\left(M-\frac{1}{2}\right)\frac{2F\kappa(1+\zeta z)}{\zeta}\right\}J_1(kr)e^{i\omega t} \quad (47)$$

4. Boundary conditions and frequency equation

Assuming that the interface of the layers is of welded contact, geometry of the problem leads to the following boundary conditions;

1. At the free surface $z=-H_2$, stress of the upper layer vanishes, i.e.,

$$\left(\tau_{z\theta}\right)_{M_1}=0 \tag{48}$$

2. At the interface of the upper layer and the intermediate layer, the displacement component and stress component are continuous, i.e.,

$$\left(v_1\right)_{M_1}=\left(v_2\right)_{M_2} \text{ at } z=-H_1, \tag{49}$$

$$\left(\tau_{z\theta}\right)_{M_1}=\left(\sigma_{z\theta}\right)_{M_2} \text{ at } z=-H_1, \tag{50}$$

3. At the interface of the intermediate layer and the lower half space, the displacement component and stress component are continuous, i.e.,

$$\left(v_2\right)_{M_2}=\left(v_3\right)_{M_3} \text{ at } z=0, \tag{51}$$

$$\left(\sigma_{z\theta}\right)_{M_2}=\left(s_{z\theta}\right)_{M_3} \text{ at } z=0. \tag{52}$$

Using the above boundary conditions, Eqs. (48)-(52), with the help of Eq. (13), Eq. (35), and Eq. (47) we get five homogeneous equations with five unknowns, D_1, D_2, D_3, D_4 and D_5 and eliminating the arbitrary constants we get

$$\tan\left(qH_1\right)=\frac{\Upsilon_1^{(l)}+\Upsilon_2^{(l)}\tan\left(\lambda_l\left(H_2-H_1\right)\right)}{\Upsilon_3^{(l)}+\Upsilon_4^{(l)}\tan\left(\lambda_l\left(H_2-H_1\right)\right)} \tag{53}$$

where $\Upsilon_j^{(l)}$ ($j=1,..,4$; $l=1,2,3$) are given in Appendix A.

Eq. (53) gives the dispersion equation of torsional surface wave in an initially stressed inhomogeneous fluid saturated porous media sandwiched between inhomogeneous anisotropic layer and initially stressed inhomogeneous anisotropic half space.

The expression for group velocity regarding the studied three layered model is given explicitly as

$$\begin{aligned}&\left\{\Theta_1^{(l)}+\Theta_2^{(l)}\tan\left(\lambda_l\left(H_2-H_1\right)\right)+\Theta_3^{(l)}\sec^2\left(\lambda_l\left(H_2-H_1\right)\right)\frac{d\lambda_l}{dk}\left(H_2-H_1\right)\right\}\tan\left(qH_1\right)\\&+\left\{\Theta_4^{(l)}+\Theta_3^{(l)}\tan\left(\lambda_l\left(H_2-H_1\right)\right)\right\}\sec^2\left(qH_1\right)\frac{dq}{dk}H_1+\Theta_5^{(l)}+\Theta_6^{(l)}\tan\left(\lambda_l\left(H_2-H_1\right)\right)\\&+\Theta_7^{(l)}\sec^2\left(\lambda_l\left(H_2-H_1\right)\right)\frac{d\lambda_l}{dk}\left(H_2-H_1\right)=0\end{aligned} \tag{54}$$

where $\Theta_j^{(l)}$ ($j=1,..,7;l=1,2,3$) are given in Appendix A.

For l=1, 2 and 3 in both Eqs. (53) and (54) correspond to the case when upper anisotropic layer is of exponential variation (case A), quadratic variation (case B) and hyperbolic variation (case C) respectively.

5. Particular cases

Case I: When uppermost layer is homogeneous (i.e., $1/a\to 0, N_1=L_1$) i.e., the directional rigidities and density becomes constant then the dispersion Eq. (53) takes the form (all three cases)

$$\tan\left(\tilde{q}H_1\right)=\left\{\eta\tilde{q}\tilde{\lambda}+\left(\tilde{q}L_1\tilde{\lambda}^2/(2L_2)\right)\tan\left(\tilde{\lambda}\left(H_2-H_1\right)\right)\right\}\Big/\left\{\tilde{q}^2\tilde{\lambda}-\left(\eta L_1\tilde{\lambda}^2/2L_2\right)\tan\left(\tilde{\lambda}\left(H_2-H_1\right)\right)\right\}$$

which is the dispersion equation of torsional surface wave in an initially stressed inhomogeneous anisotropic fluid saturated porous media constrained between homogeneous layer and initially stressed inhomogeneous anisotropic half space and $\tilde{\lambda},\tilde{q}$ are given in Appendix-B.

Case II*:* When uppermost layer is homogeneous (i.e., $1/a\to 0, N_1=L_1$) i.e., the directional rigidities and density becomes constant and also sandwiched layer is non porous homogeneous isotropic elastic without initial stress (i.e., $N_2=L_2, d\to 1, P_2\to 0, b\to 0$) then the dispersion Eq. (53) takes the form (all three cases)

$$\tan\left(\tilde{q}H_1\right)=\left\{\left(\tilde{q}L_1\tilde{\lambda}^2/(2L_2)\right)\tan\left(\tilde{\lambda}\left(H_2-H_1\right)\right)\right\}\Big/\left\{\tilde{q}^2\tilde{\lambda}\right\}$$

which is the dispersion equation of torsional surface wave in an isotropic homogeneous elastic media constrained between homogeneous layer and initially stressed inhomogeneous anisotropic half space and $\tilde{\lambda},\tilde{q}$ are given in Appendix B.

Case III: When uppermost layer is homogeneous (i.e., $1/a\to 0, N_1=L_1$) i.e., the directional rigidities and density becomes constant and sandwiched layer is non porous homogeneous isotropic elastic without initial stress (i.e., $N_2=L_2, d\to 1, P_2\to 0, b\to 0$) and also lower half space is without initially stressed and isotropic homogeneous (i.e., $N_3=L_3, \alpha_3\to 0, \beta_3\to 0, \quad \delta_3\to 0, P_3\to 0$) then the dispersion Eq. (53) takes the form (all three cases)

$$\tan\left(\tilde{q}H_1\right)=\left\{\Upsilon_1^{(l)}+\left(\Upsilon_2^{(l)}\right)_{\mathrm{IV}}\tan\left(\lambda_l\left(H_2-H_1\right)\right)\right\}\Big/\left\{\Upsilon_3^{(l)}+\left(\Upsilon_4^{(l)}\right)_{\mathrm{IV}}\tan\left(\lambda_l\left(H_2-H_1\right)\right)\right\}$$

which is the dispersion equation of torsional surface wave in an isotropic homogeneous elastic media constrained between two homogeneous media and $\tilde{q}$, $\left(\Upsilon_2^{(l)}\right)_{\mathrm{IV}}$ and $\left(\Upsilon_4^{(l)}\right)_{\mathrm{IV}}$ are given in Appendix B.

Case IV: When uppermost layer is inhomogeneous anisotropic and sandwiched layer is non porous isotropic homogeneous elastic without initial stress (i.e., $N_2=L_2, d\to 1, P_2\to 0, b\to 0$) and lower half space is inhomogeneous anisotropic without initial stress (i.e., $P_3\to 0$) then the dispersion Eq. (53) takes the form

$$\tan\left(qH_1\right)=\left\{\left(\Upsilon_1^{(l)}\right)_{\mathrm{V}}+\left(\Upsilon_2^{(l)}\right)_{\mathrm{V}}\tan\left(\lambda_l\left(H_2-H_1\right)\right)\right\}\Big/\left\{\left(\Upsilon_3^{(l)}\right)_{\mathrm{V}}+\left(\Upsilon_4^{(l)}\right)_{\mathrm{V}}\tan\left(\lambda_l\left(H_2-H_1\right)\right)\right\}$$

which is the dispersion equation of torsional surface wave in an isotropic homogeneous elastic media constrained between two non-homogeneous anisotropic media and $\left(\Upsilon_1^{(l)}\right)_{\mathrm{V}}$, $\left(\Upsilon_2^{(l)}\right)_{\mathrm{V}}$, $\left(\Upsilon_3^{(l)}\right)_{\mathrm{V}}$, $\left(\Upsilon_4^{(l)}\right)_{\mathrm{V}}$; ($l$=1, 2, 3) are given in Appendix-B.

Case V: When uppermost layer is inhomogeneous anisotropic and sandwiched layer is initially stressed fluid saturated porous inhomogeneous layer and lowermost half-space is isotropic homogeneous elastic, without initial stress (i.e., $N_3=L_3, \alpha_3\to 0, \beta_3\to 0, \delta_3\to 0, P_3\to 0$) then the dispersion Eq. (53) takes the form

$$\tan\left(qH_1\right)=\left\{\left(\Upsilon_1^{(l)}\right)_{\mathrm{V}}+\left(\Upsilon_2^{(l)}\right)_{\mathrm{V}}\tan\left(\lambda_l\left(H_2-H_1\right)\right)\right\}\Big/\left\{\left(\Upsilon_3^{(l)}\right)_{\mathrm{V}}+\left(\Upsilon_4^{(l)}\right)_{\mathrm{V}}\tan\left(\lambda_l\left(H_2-H_1\right)\right)\right\}$$

which is the dispersion equation of torsional surface wave in an inhomogeneous anisotropic fluid saturated porous media constrained between inhomogeneous anisotropic layer and homogeneous isotropic half space and $\left(\Upsilon_1^{(l)}\right)_{\mathrm{V}}$, $\left(\Upsilon_2^{(l)}\right)_{\mathrm{V}}$, $\left(\Upsilon_3^{(l)}\right)_{\mathrm{V}}$, $\left(\Upsilon_4^{(l)}\right)_{\mathrm{V}}$; ($l$=1, 2, 3) are given in Appendix B.

Case VI: When uppermost layer is inhomogeneous anisotropic and sandwiched layer is non porous isotropic homogeneous elastic without initial stress (i.e.,

$N_2=L_2, d\to 1, P_2\to 0, b\to 0$) and also lowermost half-space is isotropic homogeneous elastic, without initial stress (i.e., $N_3=L_3, \alpha_3\to 0, \beta_3\to 0, \delta_3\to 0, P_3\to 0$) then the dispersion Eq. (53) takes the form

$$\tan(\tilde{q}H_1)=\left\{\left(\Upsilon_1^{(l)}\right)_{\mathrm{VI}}+\left(\Upsilon_2^{(l)}\right)_{\mathrm{VI}}\tan\left(\lambda_l(H_2-H_1)\right)\right\}\Big/\left\{\left(\Upsilon_3^{(l)}\right)_{\mathrm{VI}}+\left(\Upsilon_4^{(l)}\right)_{\mathrm{VI}}\tan\left(\lambda_l(H_2-H_1)\right)\right\}$$

which is the dispersion equation of torsional surface wave in an isotropic homogeneous elastic media constrained between inhomogeneous anisotropic layer and homogeneous isotropic half space and $\tilde{q}$, $\left(\Upsilon_1^{(l)}\right)_{\mathrm{V}}$, $\left(\Upsilon_2^{(l)}\right)_{\mathrm{V}}$, $\left(\Upsilon_3^{(l)}\right)_{\mathrm{V}}$, $\left(\Upsilon_4^{(l)}\right)_{\mathrm{V}}$; ($l$=1, 2, 3) are given in Appendix-B.

Case VII: When sandwiched layer is absent {i.e., $H_1\to 0$} and uppermost layer is homogeneous and perfectly elastic (i.e., $1/a\to 0, N_1=L_1=\mu_1$) and lowermost half-space is isotropic homogeneous elastic, without initial stress (i.e., $N_3=L_3=\mu_3, P_3\to 0, \alpha_3\to 0, \beta_3\to 0, \delta_3\to 0$) then the dispersion Eq. (53) takes the form (all three cases)

$$\tan\left(kH_2\sqrt{c^2/c_1^2-1}\right)=\left(\mu_3\sqrt{1-c^2/c_3^2}\right)\Big/\left(\mu_1\sqrt{c^2/c_1^2-1}\right)$$

Case VIII: When uppermost layer is absent {i.e., $(H_2-H_1)\to 0$ }, sandwiched layer is non porous homogeneous isotropic elastic without initial stress (i.e., $N_2=L_2=\mu_2, d\to 1, P_2\to 0,\ b\to 0$) and lowermost half-space is isotropic homogeneous elastic, without initial stress (i.e., $N_3=L_3=\mu_3, \alpha_3\to 0, \beta_3\to 0, \delta_3\to 0, P_3\to 0$) then the dispersion Eq. (53) takes the form

$$\tan\left(kH_1\sqrt{c^2/c_2^2-1}\right)=\left(\mu_3\sqrt{1-c^2/c_3^2}\right)\Big/\left(\mu_2\sqrt{c^2/c_2^2-1}\right)$$

Dispersion equation for the cases VII and VIII are the well known classical Love wave equation.

6. Numerical results and discussion

In order to emerge with the effect of various parameters such as $\{1/(ak), b/k, \alpha_3/k, \beta_3/k, \gamma_3/k, \delta_3/k\}$, porosity $\{d\}$, initial stresses $\{\xi, \mu\}$, ratio of thickness of the layers $\{H=(H_2/H_1)\}$ on the propagation of torsional wave, the curves have been plotted from the dispersion equations (Eq. (53)), and from group velocity (Eq. (54)) for different values of elastic constants. For the purpose of numerical computation, following data have been considered:

For uppermost anisotropic non-homogeneous layer: (Tierstein 1969)

$$N_1=3.99\times 10^{10}\,N/m^2,\ L_1=5.79\times 10^{10}\,N/m^2,\ \rho_1=2649\,kg/m^3$$

(Anisotropic quartz material)

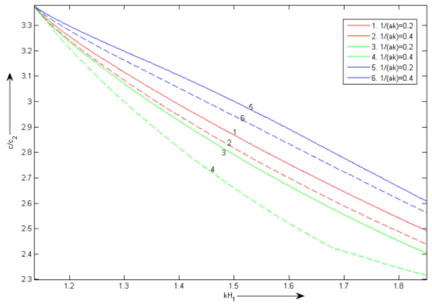
For initially stressed fluid saturated homogeneous intermediate layer: (Samal and Chattaraj 2011)

$$N_2=0.2774\times 10^{10}\,N/m^2, L_2=0.1387\times 10^{10}\,N/m^2, \phi=0.26, \rho_{rr}=1.926137\times 10^3\,kg/m^3,$$
$$\rho_{r\theta}=-0.002137\times 10^3\,kg/m^3, \rho_{rr}=0.215337\times 10^3\,kg/m^3 \text{ (Water saturated lime stone)}$$

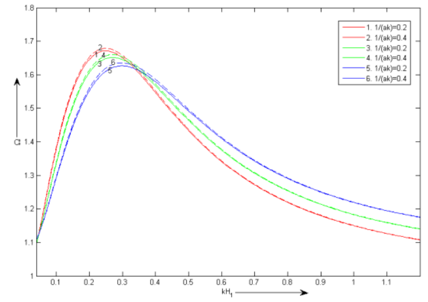
For lowermost initially stressed anisotropic non-homogeneous half space: (Gubbins 1990)

$$N_3=6.34\times 10^{10}\,N/m^2,\ L_3=7.5\times 10^{10}\,N/m^2,\ \rho_3=3338\,kg/m^3$$

(Anisotropic sandstone material)

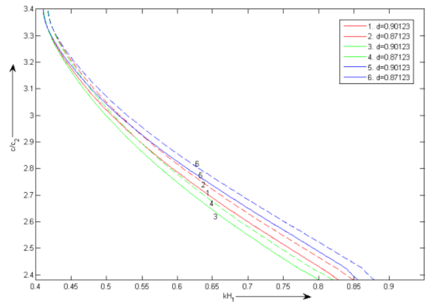


(a)

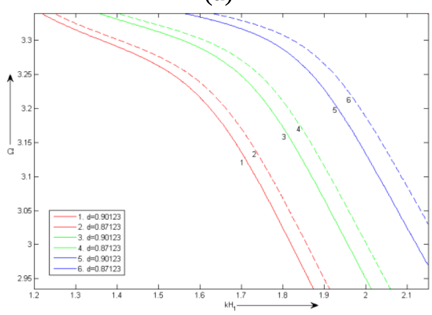


(b)

Fig. 2 Variation of the non-dimensional wave number kH_1 against the (a) dimensionless phase velocity c/c_2 and (b) dimensionless group velocity (Ω) for different values of (1/(ak))



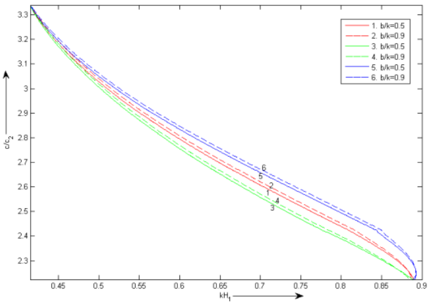
(a)



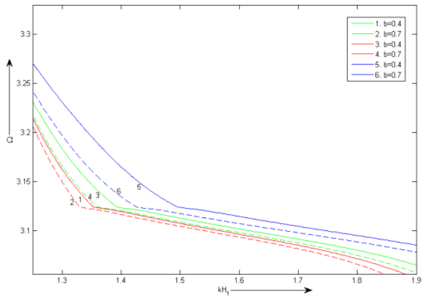
(b)

Fig. 3 Variation of the non-dimensional wave number kH_1 against the (a) dimensionless phase velocity (c/c_2) and (b) dimensionless group velocity (Ω) for different values of (d).

For graphical representation, the numerical values of all non-dimensional parameter in the figures have been consider as 1/(ak)=0.3, b/k=0.25, d=0.90123, ξ=0.2, α_3/k=0.1, β_3/k=0.2, γ_3/k=0.2, δ_3/k=0.2, μ=0.3, H=1.7 and ω/k=0.5, unless otherwise stated.



(a)



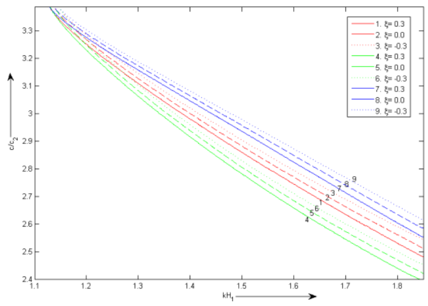
(b)

Fig. 4 Variation of the non-dimensional wave number kH_1 against the (a) dimensionless phase velocity (c/c_2) and (b) dimensionless group velocity (Ω) for different values of (b/k)

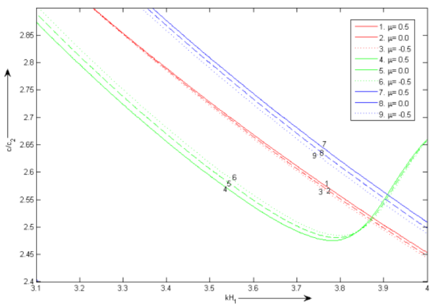
In all of the figures, curves have been plotted with horizontal axis as dimensionless wave number kH_1 against vertical axis as (i) dimensionless phase velocity (c/c_2) or (ii) dimensionless group velocity ($\Omega=d(kc)/dk$) and also each of the figures, curves are in red, green and blue colours correspond to the case when upper layer is of exponential variation (Case-A), quadratic variation (Case-B) and hyperbolic variation (Case-C) respectively.

Fig. 2 reflects the effect of inhomogeneity factor ($1/(ak)$) associated with directional rigidities and density of the upper non-homogeneous layer. Under the above-considered values in this figure we observed that, in presence of inhomogeneity, for a fixed value of wave number kH_1, phase velocity (c/c_2) of torsional surface wave decreases with the gradual increase of dimensionless inhomogeneity parameter ($1/(ak)$) but group velocity (Ω) increases with the inhomogeneity parameter ($1/(ak)$). Fig. 3 represents the variation of dimensionless (a) phase velocity (b) group velocity against the dimensionless wave number for different values of porosity parameter (d) in the sandwiched layer. From these figures it have been observed that as the porosity of the intermediate layer decreases, the phase velocity (c/c_2) of the torsional wave number increases for a particular wave number kH_1. More interestingly, it has been observed that for a slight change in porosity parameter associated with the intermediate layer, the velocity is affected considerably. In addition to this, it can be quoted that the porosity existing in the layer resists the velocities of torsional wave propagating through it. Fig. 4 delineates the influence of inhomogeneity parameter (b/k) present in the poroelastic layer on the phase and group velocities of torsional wave. It is evident from this figure that as long as inhomogeneity prevails in the medium, the phase velocity gets increased for a particular wave number in all considered cases. It is adduced from this figure that group velocity of torsional wave increases with increase in inhomogeneity associated with poroelastic layer in all considered cases. Fig. 5(a) has been plotted to depict the effect of non-dimensional initial stress parameter (ξ) associated with the intermediate porous layer on the phase velocity (c/c_2) of the torsional surface wave in anisotropic inhomogeneous porous layer. When $\xi>0$, it is termed as compressive initial stress, whereas for $\xi<0$, is called tensile initial stress. Under these considered values and for a fixed wave number this leads to the facts that initial stress (ξ) of the sandwiched medium is inversely proportional to the phase velocity (c/c_2) of the torsional surface wave. Fig. 5 (b) describes the effect of non-dimensional initial stress parameter (μ) associated with the non-homogeneous anisotropic half space. Here we show the effect of two types of initial stress, namely compressive initial stress and tensile initial stress. Thus one can conclude that under these mentioned values, as the value of initial stress (μ) decreases, the dimensionless phase velocity (c/c_2) increases at the same frequency except the case of quadratic variation but after certain wave number it shows resemblance. Fig. 6(a) demonstrates the effect of dispersion curves and group velocity in the presence of the anisotropy parameter (α_3/k) incorporated in the directional rigidity associated with the half space on the torsional wave velocity. The effect found here in this figure is that the value of phase velocity decreases as the anisotropy factor increases for a fixed value of the wave number.

In Fig. 6(b), an attempt has been made to come out with effect of anisotropy factor (β_3/k). It has been observed that as the rigidity increases the phase velocity increases for a particular wave number. This figure manifests that the phase velocity of torsional wave is directly proportional to

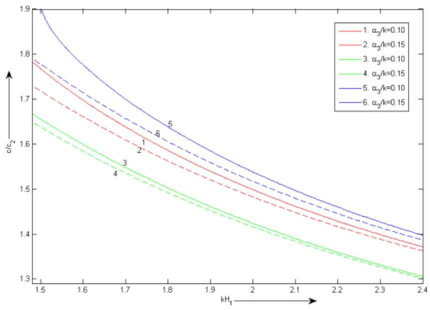


(a)

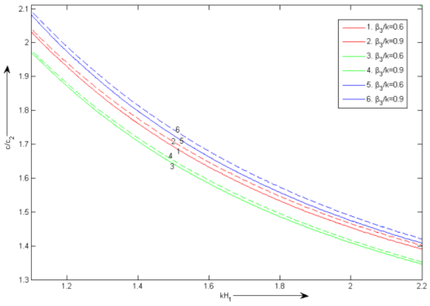


(b)

Fig. 5 Variation of the dimensionless phase velocity (c/c_2) against the non-dimensional wave number kH_1 for different values of initial stress parameter (a) ξ and (b) μ

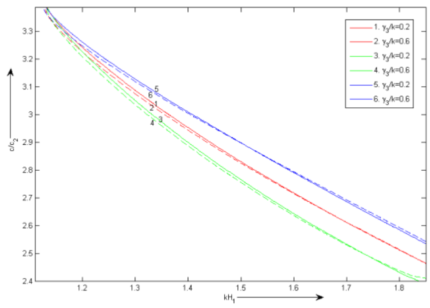


(a)

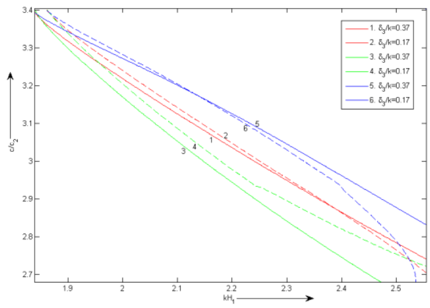


(b)

Fig. 6 Variation of the dimensionless phase velocity (c/c_2) against the non-dimensional wave number kH_1 for different values of non-dimensional anisotropy parameter (a) (α_3/k) and (b) (β_3/k)



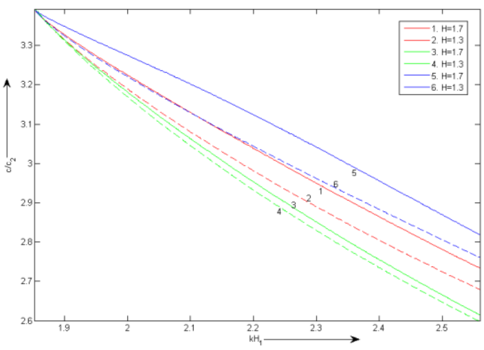
(a)



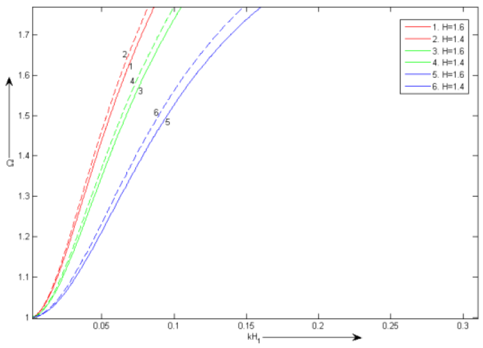
(b)

Fig. 7 Variation of the dimensionless phase velocity (c/c_2) against the non-dimensional wave number kH_1 for different values of non-dimensional parameter (a) (γ_3/k) and (b) (δ_3/k)

rigidity of the half-space. Fig. 7(a) signifies the effect of anisotropy factor (γ_3/k) present in the initial stress in the half space. These curves elucidate that the phase velocity (c/c_2) of the torsional surface wave decreases remarkably as the value of anisotropy factor increases but it gets reversed as



(a)



(b)

Fig. 8 Variation of the non-dimensional wave number kH_1 against the (a) dimensionless phase velocity (c/c_2) and (b) dimensionless group velocity (Ω) for different values of H

the wave number increases. A minute observation of Fig. 7(a) illustrates that difference in phase velocity becomes little considerable with growing magnitude of wave number for all three cases. Fig. 7(b) renders the dispersion curves of a torsional surface wave under the effect of the inhomogeneity factor (δ_3/k) associated with the density in the half space. It has also been observed that initially as the inhomogeneity parameter of the half space decreases, velocity of the torsional surface wave also decreases at the same frequency but after particular point it gets reversed (except in case of quadratic variation). This figure shows that density inhomogeneity parameter (δ_3/k) has perfect influence over the velocity of torsional surface wave. Fig. 8 discusses the dispersion curves at different values of ratio of thickness (H) of the upper and intermediate layer. Also, it has been noted that as the ratio of thickness of the layers (H) decreases, (a) the phase velocity (c/c_2) of torsional surface decreases but (b) group velocity (Ω) increases at a particular frequency of wave number kH_1, which justifies the fact that phase velocity of torsional surface wave is directly proportional to the ratio of thickness of the layers. An overview of all the curves in Figs. 2-8, that the influence of inhomogeneity associated with upper layer on phase velocity for the case when layer is with hyperbolic variation is much more pronounced as compared to the case when the layer is of quadratic variation.

7. Conclusions

In this present study we investigated the propagation of torsional surface waves in an initially stressed inhomogeneous fluid saturated porous media sandwiched between inhomogeneous anisotropic layer and initially stressed inhomogeneous anisotropic substratum. The analytical solution for displacement for each medium have been derived separately and also the final dispersion equation thus obtained in a closed form reduces into classical equation of Love wave, thereby validating the proposed solution of the problem when one of the layer vanishes and also initial stress, other technical parameters are neglected. Regarding this model we have also elucidated expression for group velocity. The numerical computations for the dispersion relation and group velocity are performed and the effects of relevant parameters are studied and shown graphically by using MATLAB. From the overall study, we arrive at the following conclusions:

•As the inhomogeneity parameter ($1/(ak)$) associated with upper layer increases, group velocity of a torsional surface wave increases, but it gets reversed in case of phase velocity.

•The porosity of the medium plays an important role in the wave propagation. As the porosity parameters (d) decreases, the intermediate layer will become an elastic solid with less pores and the phase velocity of torsional surface waves decreases and ultimately vanishes when the medium is elastic solid, also observed that in the limiting case if the porous medium changes to a liquid layer then a torsional surface wave does not exist. Similar behavior of the porosity parameter was found by Ghorai *et al.* (2010) on Love wave propagation.

•The anisotropy in the medium also reduces the velocity of torsional wave propagation.

•Form the present study it is clear that width of the intermediate porous layer and width of the topmost layer plays an important role in the study of torsional surface waves.

•The initial stresses present in the half space and fluid filled porous layer also have effect in the phase velocity of propagation. In porous medium as decreases of initial stress, phase velocity increases. This result is similar to that observed by Shekhar and Parvez (2016a).

From this study, it can be concluded that the presence of heterogeneity, porosity, layer width, and initial stress in the layer and substratum affect the torsional wave energy. Torsional wave in an inhomogeneous prestressed elastic layer overlying an inhomogeneous elastic half-space under the effect of rigid boundary was illustrated by Kakar (2015). Singh *et al.* (2017) studied the propagation of torsional surface wave in an initially stressed visco-elastic layer sandwiched between upper and lower initially stressed dry-sandy Gibson half-spaces. In this present study, torsional surface wave in layered media has possible application in geophysics and for better understanding the cause and effect due to earthquake and artificial explosions and also important to seismologists and geophysicists to find the location of the earthquakes as well as their energy, mechanism etc. So, these types of studies enable the seismologists and geophysicists to sketch the real Earth model more profoundly, imparting a detailed notion about the interior of the Earth at all scales and also helpful for civil engineers in estimating the damages during an earthquake, empowering them to better deal with the practical situations.

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Appendix A

$$\lambda_1^2 = k^2\left\{\frac{N_1}{L_1}\left(\frac{c^2}{c_1^2}-1\right)-\frac{1}{4a^2k^2}\right\},\quad \lambda_2^2 = k^2\,\frac{N_1}{L_1}\left(\frac{c^2}{c_1^2}-1\right),\quad \lambda_3^2 = k^2\left\{\frac{N_1}{L_1}\left(\frac{c^2}{c_1^2}-1\right)-\frac{1}{a^2k^2}\right\},\quad \lambda_a = \lambda_1^2+\frac{1}{4a^2},$$

$$\Upsilon_1^{(1)} = -A_2\lambda_1,\quad \Upsilon_2^{(1)} = \left(qL_1\lambda_a/(2L_2)\right)e^{-H_1/a} - A_2/(2a),\quad \Upsilon_3^{(1)} = A_1\lambda_1,\quad \Upsilon_4^{(1)} = A_1/(2a)-\left(\eta L_1\lambda_a/(2L_2)\right)e^{-H_1/a}$$

$$\Upsilon_1^{(2)} = -A_2\Lambda_1 - q\lambda_2\Lambda_2,\quad \Upsilon_2^{(2)} = q\Lambda_4 - A_2\Lambda_3,\quad \Upsilon_3^{(2)} = A_1\Lambda_1+\eta\lambda_2\Lambda_2,\quad \Upsilon_4^{(2)} = A_1\Lambda_3-\eta\Lambda_4$$

$$\Lambda_1 = \lambda_2/\left(1-H_2/a\right),\quad \Lambda_2 = \frac{L_1}{2aL_2}\left\{\left(\frac{1-H_1/a}{1-H_2/a}\right)^2-\frac{1-H_1/a}{1-H_2/a}\right\},\quad \Lambda_3 = \frac{1}{a\left(1-H_2/a\right)^2},\quad \Lambda_4 = \frac{L_1}{2L_2}\left\{\frac{\lambda_2^2\left(1-H_1/a\right)^2}{\left(1-H_2/a\right)}+\frac{1-H_1/a}{a^2\left(1-H_2/a\right)^2}\right\}$$

$$\Upsilon_1^{(3)} = -A_2\Gamma_1 - q\lambda_3\Gamma_2,\quad \Upsilon_2^{(3)} = A_2\Gamma_3+q\Gamma_4,\quad \Upsilon_3^{(3)} = A_1\Gamma_1+\eta\lambda_3\Gamma_2,\quad \Upsilon_4^{(3)} = -A_1\Gamma_3-\eta\Gamma_4$$

$$\Gamma_1 = \lambda_3/\cosh(H_2/a),\quad \Gamma_2 = \frac{L_1\cosh(H_1/a)}{2aL_2}\left\{\frac{\sinh(H_1/a)}{\cosh(H_2/a)}-\frac{\sinh(H_2/a)\cosh(H_1/a)}{\cosh^2(H_2/a)}\right\},\quad \Gamma_3 = \frac{\sinh(H_2/a)}{a\cosh^2(H_2/a)},\quad \Gamma_4 = \frac{L_1\cosh(H_1/a)}{2L_2}\left\{\frac{\lambda_3^2\cosh(H_1/a)}{\cosh(H_2/a)}+\frac{\sinh(H_2/a)\sinh(H_1/a)}{a^2\cosh^2(H_2/a)}\right\}$$

$$A_1 = q^2\cos^2\frac{bH_1}{2}+\frac{b\eta}{4}\cos bH_1,\quad A_2 = \eta q\cos^2\frac{bH_1}{2}-\frac{bq}{4}\cos bH_1,\quad \eta = \frac{S_2}{L_2S_1}+\frac{b}{2},\quad S_1 = \left[v_3\right]_{z=0},\quad S_2 = \left[L^{(3)}\frac{\partial v_3}{\partial z}\right]_{z=0},\quad \frac{S_2}{S_1}\approx\frac{Q_2}{Q_1}$$

$$Q_1 = 1+\frac{F\kappa}{\zeta}-\frac{2F\kappa\mathcal{M}}{\zeta},\quad Q_2 = L_3F\kappa\left(\frac{2\mathcal{M}F\kappa}{\zeta}-\frac{F\kappa}{\zeta}-2\mathcal{M}\right),\quad \cancel{T} = \frac{\sqrt{L_2}}{\cos(bH_1/2)},\quad \cancel{L} = \frac{L_1}{2\sqrt{L_2}\cos(bH_1/2)}$$

$$\eta^* = \frac{Q_2}{L_2Q_1}+\frac{b}{2},\quad \cancel{\varkappa} = \frac{1}{L_2}\frac{dQ_2}{dk}+\frac{b}{2}\frac{dQ_1}{dk},\quad \Im = \frac{dq}{dk}Q_1+q\frac{dQ_1}{dk},\quad \Delta_j = \left(\frac{dq}{dk}\lambda_jQ_1+q\frac{d\lambda_j}{dk}Q_1+q\lambda_j\frac{dQ_1}{dk}\right)(j=2,3)$$

$$A_1^* = \cancel{T}Q_1\left(q^2\cos^2\frac{bH_1}{2}+\frac{b\eta^*}{4}\cos bH_1\right),\quad A_2^* = \cancel{T}Q_1\left(\eta^*q\cos^2\frac{bH_1}{2}-\frac{bq}{4}\cos bH_1\right),\quad \mathrm{X}_j^{(l)} = \frac{d\lambda_l}{dk}A_j^*+\lambda_l\frac{dA_j^*}{dk}\ (j=1,2;l=1,2,3),$$

$$\Theta_1^{(1)} = \mathrm{X}_1^{(1)},\quad \Theta_2^{(1)} = \frac{1}{2a}\frac{dA_1^*}{dk}-\cancel{L}e^{-H_1/a}\left\{2\lambda_1\frac{d\lambda_1}{dk}Q_1\eta^*+\cancel{\varkappa}\lambda_a\right\},\quad \Theta_3^{(1)} = \frac{A_1^*}{2a}-\cancel{L}Q_1\eta^*\lambda_ae^{-H_1/a},\quad \Theta_4^{(1)} = \lambda_1A_1^*,\quad \Theta_5^{(1)} = \mathrm{X}_2^{(1)},$$

$$\Theta_6^{(1)} = \frac{1}{2a}\frac{dA_2^*}{dk}-\cancel{L}e^{-H_1/a}\left\{2qQ_1\lambda_1\frac{d\lambda_1}{dk}+\Im\lambda_a\right\},\quad \Theta_7^{(1)} = \frac{A_2^*}{2a}-qQ_1\lambda_a\cancel{L}e^{-H_1/a}$$

$$\Theta_1^{(2)} = \frac{\mathrm{X}_1^{(2)}}{1-H_2/a}+\Lambda_2\lambda_2\cancel{T}\left\{\cancel{\varkappa}+\frac{Q_1\eta^*}{\lambda_2}\frac{d\lambda_2}{dk}\right\},\quad \Theta_2^{(2)} = \frac{dA_1^*}{dk}\Lambda_3-\cancel{T}\left\{\cancel{\varkappa}\Lambda_4+Q_1\eta^*\frac{d\Lambda_4}{dk}\right\},\quad \Theta_3^{(2)} = A_1^*\Lambda_3-\eta^*Q_1\cancel{T}\Lambda_4,$$

$$\Theta_4^{(2)} = A_1^*\Lambda_1+\cancel{T}\Lambda_2\lambda_2\eta^*Q_1,\quad \Theta_5^{(2)} = \mathrm{X}_2^{(2)}/\left(1-H_2/a\right)+\cancel{T}\Lambda_2\Delta_2,\quad \Theta_6^{(2)} = \frac{dA_2^*}{dk}\Lambda_3-\cancel{T}\left\{\Lambda_4\Im+\frac{d\Lambda_4}{dk}Q_1q\right\},\quad \Theta_7^{(2)} = A_2^*\Lambda_3-\cancel{T}qQ_1\Lambda_4$$

$$\Theta_1^{(3)} = \frac{\mathrm{X}_1^{(3)}}{\cosh(H_2/a)}+\cancel{T}\Gamma_2\lambda_3\left\{\cancel{\varkappa}+\frac{Q_1\eta^*}{\lambda_3}\frac{d\lambda_3}{dk}\right\},\quad \Theta_2^{(3)} = -\Gamma_3\frac{dA_1^*}{dk}-\cancel{T}\left\{\cancel{\varkappa}\Gamma_4+\eta^*Q_1\frac{d\Gamma_4}{dk}\right\},\quad \Theta_3^{(3)} = -A_1^*\Gamma_3-\eta^*Q_1\cancel{T}\Gamma_4,$$

$$\Theta_4^{(3)} = A_1^*\Gamma_1+\cancel{T}\Gamma_2\lambda_3\eta^*Q_1,\quad \Theta_5^{(3)} = \frac{\mathrm{X}_2^{(3)}}{\cosh(H_2/a)}+\cancel{T}\Gamma_2\Delta_3,\quad \Theta_6^{(3)} = -\frac{dA_2^*}{dk}\Gamma_3-\cancel{T}\left\{\Gamma_4\Im+\frac{d\Gamma_4}{dk}Q_1q\right\},\quad \Theta_7^{(3)} = -A_2^*\Gamma_3-\cancel{T}qQ_1\Gamma_4$$

Appendix B

$$\left(\Upsilon_2^{(1)}\right)_{\mathrm{IV}} = \left(\tilde{q}L_1\lambda_a/(2L_2)\right)e^{-H_1/a} - \left(\eta_0\tilde{q}/(2a)\right), \left(\Upsilon_2^{(2)}\right)_{\mathrm{IV}} = \tilde{q}\Lambda_4 - \eta_0\tilde{q}\Lambda_3, \left(\Upsilon_2^{(3)}\right)_{\mathrm{IV}} = -\eta_0\tilde{q}\Gamma_3 + \tilde{q}\Gamma_4,$$

$$\left(\Upsilon_4^{(1)}\right)_{\mathrm{IV}} = \tilde{q}^2/(2a) - \left(\eta_0 L_1\lambda_a/(2L_2)\right)e^{-H_1/a}, \left(\Upsilon_4^{(2)}\right)_{\mathrm{IV}} = \tilde{q}^2\Lambda_3 - \eta_0\Lambda_4, \left(\Upsilon_4^{(3)}\right)_{\mathrm{IV}} = -\tilde{q}^2\Gamma_3 - \eta_0\Gamma_4,$$

$$\left(\frac{S_2}{S_1}\right)_0 = Fk\left\{\left(F^2 + \left(\frac{c}{c_3}\right)^2 - 1\right)\left(\frac{k^2(N_3)^{3/2}}{\zeta\sqrt{L_3}} - \frac{kN_3}{F\zeta}\right) - \frac{FkN_3}{\zeta}\right\} \Big/ \left\{1 + \frac{Fk}{\zeta}\sqrt{\frac{N_3}{L_3}} - \frac{k^2N_3}{\zeta L_3}\left(F^2 + \left(\frac{c}{c_3}\right)^2 - 1\right)\right\},$$

$$(A_1)_0 = q^2\cos^2\frac{bH_1}{2} + \frac{b^2}{8}\cos bH_1, (A_2)_0 = \frac{bq}{2}\left(\cos^2\frac{bH_1}{2} - \frac{\cos bH_1}{2}\right), \tilde{q} = k\sqrt{\frac{c^2}{c_2^2} - 1}, \tilde{\lambda} = k\sqrt{\frac{c^2}{c_1^2} - 1},$$

$$\left(\Upsilon_1^{(1)}\right)_{\mathrm{V}} = -(A_2)_0\lambda_1, \left(\Upsilon_2^{(1)}\right)_{\mathrm{V}} = \frac{qL_1}{2L_2}\lambda_a e^{-H_1/a} - \frac{(A_2)_0}{2a}, \left(\Upsilon_3^{(1)}\right)_{\mathrm{V}} = (A_1)_0\lambda_1, \left(\Upsilon_4^{(1)}\right)_{\mathrm{V}} = \frac{(A_1)_0}{2a} - \frac{bL_1}{4L_2}\lambda_a e^{-H_1/a},$$

$$\left(\Upsilon_1^{(2)}\right)_{\mathrm{V}} = -(A_2)_0\Lambda_1 - q\lambda_2\Lambda_2, \left(\Upsilon_2^{(2)}\right)_{\mathrm{V}} = q\Lambda_4 - (A_2)_0\Lambda_3, \left(\Upsilon_3^{(2)}\right)_{\mathrm{V}} = (A_1)_0\Lambda_1 + \lambda_2\Lambda_2 b/2,$$

$$\left(\Upsilon_4^{(2)}\right)_{\mathrm{V}} = (A_1)_0\Lambda_3 - b\Lambda_4/2, \left(\Upsilon_1^{(3)}\right)_{\mathrm{V}} = -(A_2)_0\Gamma_1 - q\lambda_3\Gamma_2, \left(\Upsilon_2^{(3)}\right)_{\mathrm{V}} = (A_2)_0\Gamma_3 + q\Gamma_4, \eta_0 = (S_2/S_1)_0/L_2 + b/2$$

$$\left(\Upsilon_3^{(3)}\right)_{\mathrm{V}} = (A_1)_0\Gamma_1 + \lambda_3\Gamma_2 b/2, \left(\Upsilon_4^{(3)}\right)_{\mathrm{V}} = -(A_1)_0\Gamma_3 - \Gamma_4 b/2, \left(\Upsilon_1^{(1)}\right)_{\mathrm{VI}} = 0, \left(\Upsilon_2^{(1)}\right)_{\mathrm{VI}} = \left(\tilde{q}L_1\lambda_a/(2L_2)\right)e^{-H_1/a},$$

$$\left(\Upsilon_3^{(1)}\right)_{\mathrm{VI}} = \tilde{q}^2\lambda_1, \left(\Upsilon_4^{(1)}\right)_{\mathrm{VI}} = \tilde{q}^2/(2a), \left(\Upsilon_1^{(2)}\right)_{\mathrm{VI}} = -\tilde{q}\lambda_2\Lambda_2, \left(\Upsilon_2^{(2)}\right)_{\mathrm{VI}} = \tilde{q}\Lambda_4, \left(\Upsilon_3^{(2)}\right)_{\mathrm{VI}} = \tilde{q}^2\Lambda_1, \left(\Upsilon_4^{(2)}\right)_{\mathrm{VI}} = \tilde{q}^2\Lambda_3,$$

$$\left(\Upsilon_1^{(3)}\right)_{\mathrm{VI}} = -q\lambda_3\Gamma_2, \left(\Upsilon_2^{(3)}\right)_{\mathrm{VI}} = \tilde{q}\Gamma_4, \left(\Upsilon_3^{(3)}\right)_{\mathrm{VI}} = \tilde{q}^2\Gamma_1, \left(\Upsilon_4^{(3)}\right)_{\mathrm{VI}} = -\tilde{q}^2\Gamma_3.$$