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Theoretical and numerical analysis of the influence of initial stress gradient on wave propagations

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Abstract. The investigation of stress wave propagation in a medium with initial stress has very important application in the field of engineering. However, the previous research less consider the influence of initial stress gradient on wave propagation. In the present paper, the governing equation of wave propagation in elastic continuum material with inhomogeneous initial stress is derived, which indicated that the inhomogeneous initial stress changed the governing equation of wave propagation. Additionally, the definite problem of wave propagation in material with initial stress gradient is verified by using mathematical physics method. Based on the definite problem, the elastic displacement-time relationship of wave propagation is explored, which indicated that the inhomogeneous initial stress changed waveform and relationship of displacement-time histories. Furthermore, the spall process of blasting wave propagation from underground to earth surface is simulated by using LS-DYNA.

Keywords: inhomogeneous initial stress; mathematical physics; wave propagation; governing equations

1. Introduction

In civil and mining engineering, the problems involve tectonics and earth masses that are initially under stress state, the dynamic process is stress wave propagation in a material with initial stress. Therefore, the theory of elastic wave propagation in pre-stressed solids has a long history (Tolstoy 1982). A definite theory explaining the wave propagation of a material under initial stress was developed by Biot (Biot and Drucker 1965), which clearly showed that the effect of initial stress on wave propagation cannot be represented by a change in elastic coefficients.

During the last fifty years, based on Biot's works, many scholars have been explored the theory of wave propagation in pre-stressed elastic solids, these results demonstrated that the initial stress affect the wave propagation. Recently, using theoretical analysis, Ogden and Singh (2014) derived the governing amplitude motions in a rotating transversely isotropic initially stressed elastic solid, which indicated the initially stressed changed the governing equation. And, the initially stressed inhomogeneous isotropic layer can affect the wave propagation also been verified in recent studies (Dhua and Chattopadhyay 2015). Mahmoud and co-workers (Mahmoud *et al.* 2014) solved the equations of dynamics for transversely isotropic material under the effect of initial stress and magnetic field. Kumar (Kumar and Kumar 2013) studied wave propagation at the boundary

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surface of elastic and initial stressed material with voids media.

Additionally, in laboratory, Ji *et al*. (2007, 2013a, b) presented a lot of data on seismic velocities and anisotropy in minerals and rocks. Mohammad (Asef and Najibi 2013), Zhao (Zhao *et al*. 2011), Wang (Wang *et al*. 2009), and others (Kakar and Kakar 2014, Li and Tao 2015, Shen and Karakus 2015) also conducted many tests on wave propagation with initial stress. The results indicated that velocity of seismic waves and elastic coefficients are much influenced by initial stress, and thus it is imperative to deal with the problem of wave propagation in a material with initial stress. However, these studies mainly pay attention that the initial stress is homogeneous.

In practice, due to the gravity variation and external force inhomogeneous, the initial stress may not be homogeneous, i.e., initial stress gradient exists. The seismic wave is a typical example of stress wave propagation in a medium with gravity gradient. Thus, many scholars studied the influence of gravity gradient change for wave propagation. An early study examined the gravity effect on surface wave by Bromwich (1898). Vinh, Munish and other scholars investigated the influence of gravity on surface waves (Sethi *et al*. 2014, Vinh and Seriani 2010), the results explored that the gravity change the governing equation of surface wave propagation. Vinh and Seriani (Vinh and Linh 2013, Vinh and Seriani 2010) investigated the propagation of Stoneley and Rayleigh waves in an incompressible isotropic elastic half-space under the effect of gravity. These results have considerably increased our understanding of wave propagation in materials with initial stress, indicating the overburden pressure may significantly affect propagation velocities. However, initial stress gradient is commonly exist in nature, the previous research less mentioned the different initial stress gradient on wave propagation.

The purpose of this study is to investigate how the different initial stress gradient affects wave propagation. The governing equation of wave propagation in one dimension of elastic continuum materials is derived by taking the influence of the inhomogeneous initial stress into account. Also, by using mathematical software, displacement-time relationships of wave propagation in a material under different initial stress gradient conditions are illustrated. Alternatively, the spall process in earth surface is simulated to verify the influence of wave propagation.

2. Theoretical formulations of elastic wave propagation

The stress-strain relationship of wave propagation in a rod was descripted as

$$\sigma_x = E\frac{\partial u}{\partial x} \tag{1}$$

where x is the space coordinate measured along the rod, σ_x is the stress, u is the displacement, and E is Young's modulus. The classical governing equation of dynamics is

$$\frac{\partial}{\partial x}\left(E\frac{\partial u}{\partial x}\right) = \rho\frac{\partial^2 u}{\partial t^2} \tag{2}$$

In addition, if the bar has an initial static stress and the stress changes along with the locations. Assuming the stress and location follow the relationship as

$$\sigma_s = \sigma(x) \tag{3}$$

Aassuming the rod is an incompressible elastic material, and then Eq. (2) can be written as

$$\frac{\partial \sigma_s(x)}{\partial x} + E\frac{\partial u^2}{\partial x^2} = \rho \frac{\partial u^2}{\partial t^2} \tag{4}$$

Herein, initial stress gradient Γ is defined as

$$\Gamma = \frac{\partial \sigma_s(x)}{\partial x} \tag{5}$$

Eq. (4) indicates that the governing equation of wave propagation is different from the classical form when the inhomogeneous initial stress is taken into account. Due to the existence of initial stress gradient, a term is added.

Therefore, in order to verify the stress gradient that can affect stress wave propagation, a simple rod model is used, the right end of the rod is fixed and the left end is conducted a dynamic loading *f* (*t*), and stress gradient exists along the *x*-direction, shown in Fig. 1.

In this condition, the boundary conditions are

$$\begin{cases} \left.\dfrac{\partial u}{\partial x}\right|_{x=0} = \dfrac{f(t)}{E} \\ u|_{x=l} = 0 \end{cases} \tag{6}$$

The initial conditions are

$$\begin{cases} u|_{t=0} = 0 \\ \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = \dfrac{f(t)}{E} \end{cases} \tag{7}$$

The propagation of stress wave can be calculated by mathematical physics method. The physic process of stress wave propagation in the rod is in turn as the follow mathematical equations

$$\begin{cases} \dfrac{\partial \sigma_s(x)}{\partial x} + \dfrac{E\partial u^2}{\partial x^2} = \rho \dfrac{\partial^2 u}{\partial t^2} \\ \left.\dfrac{\partial u}{\partial x}\right|_{x=0} = \dfrac{f(t)}{E}, u|_{x=l} = 0 \\ u|_{t=0} = 0, \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = 0 \end{cases} \tag{8}$$

This is a mathematic physic equation, and the analytical solution can be solved by the principle of separation when the initial stress function is verified, but it is not straightforward. Meanwhile, the result of Eq. (8) can be illustrated by mathematical software, such as MATLAB. Thus, the mathematical software is used to illustrate the displacement-time relationships of wave

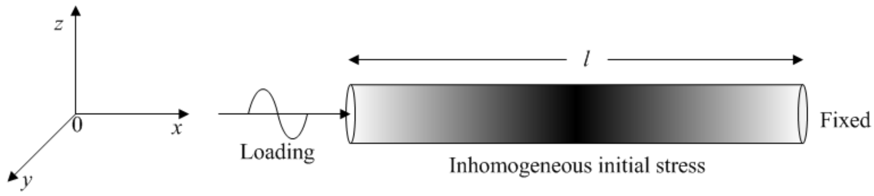


Fig. 1 Dynamic loading model

propagation process in the following sections.

3. Examples of wave propagation in elastic materials

When the initial stress function is verified, the result of displacement-time relationship of wave propagation can be investigated. For example, to consider two typical examples, one is the initial stress is linear change with the location, the other is non-linear change. Thus, when the relationship of initial stress with location is descripted as follows

$$\sigma_s(x) = kx \tag{9}$$

Where k is a constant coefficient, i.e., the initial stress is linear relationship with the location, the Eq. (8) can be written as

$$\begin{cases} k + \dfrac{E\partial u^2}{\partial x^2} = \rho\dfrac{\partial^2 u}{\partial t^2} \\ \left.\dfrac{\partial u}{\partial x}\right|_{x=0} = \dfrac{f(t)}{E}, u|_{x=l} = 0 \\ u|_{t=0} = 0, \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = 0 \end{cases} \tag{10}$$

And, if the initial stress is non-linear change with the location as follows

$$\sigma_s(x) = kx^2 \tag{11}$$

The definite problem Eq. (8) can be written as

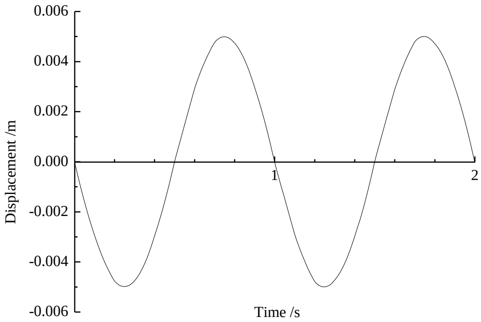
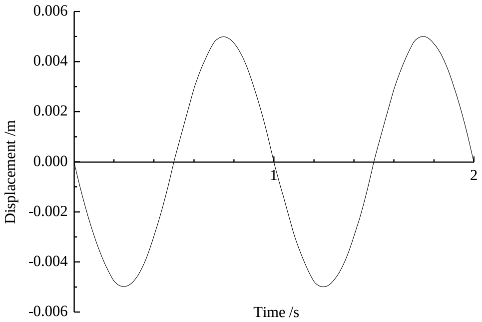
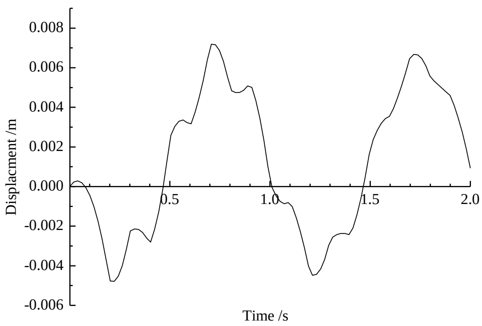
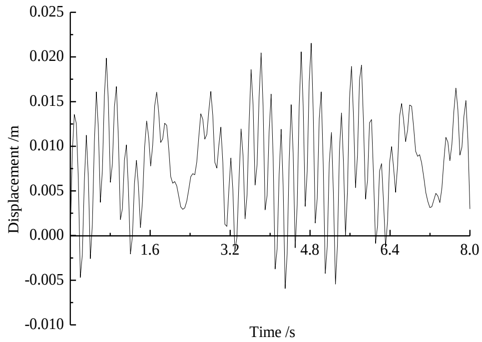
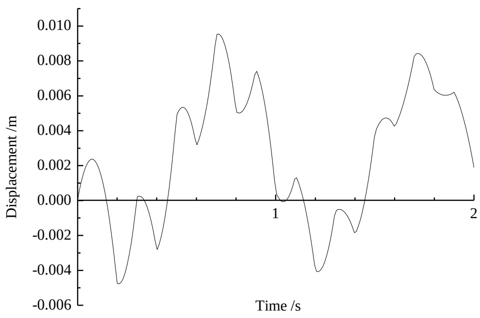
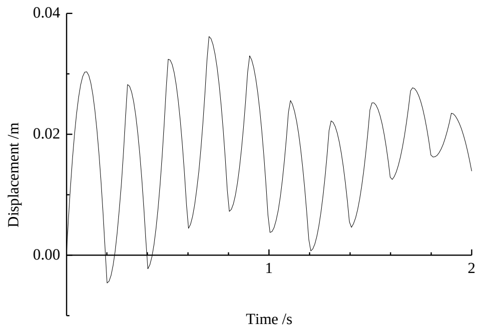
$$\begin{cases} 2kx + \dfrac{E\partial u^2}{\partial x^2} = \rho\dfrac{\partial^2 u}{\partial t^2} \\ \left.\dfrac{\partial u}{\partial x}\right|_{x=0} = \dfrac{f(t)}{E}, u|_{x=l} = 0 \\ u|_{t=0} = 0, \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = 0 \end{cases} \tag{12}$$

When k is equal to zero corresponding with the material is un-stressed. Thus, for a certain known dynamic loading, $f(t)$, when k is known, the displacement-time curve can be illustrated by mathematical software. Additionally, the physical properties of rock are used for numerical simulation as: E = 40 GPa, ρ = 2700 kg/m^3, the rod length l =10 m. The loading wave function is described as

$$f(t) = 2\times10^7 \sin\left(\frac{\pi}{0.5}t\right) \tag{13}$$

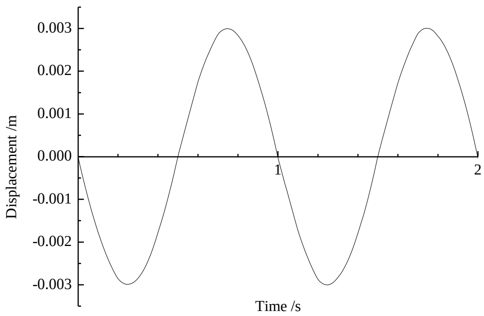
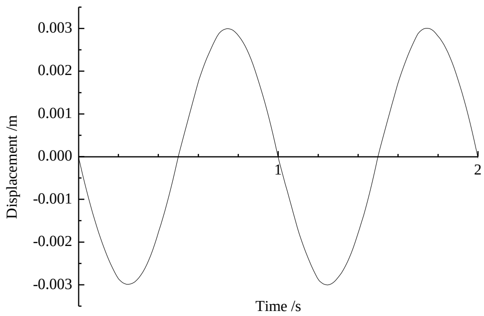
When k is defined different value, the results of displacement-time curve at x = 0, 4, and 8 can be illustrated in Tables 1, 2 and 3.

Table 1 Comparison of displacement-time curves under different initial stress states (x=0)

$\sigma_s(x)$ / k values	$\sigma_s(x) = kx$	$\sigma_s(x) = kx^2$
$k = 0$		
$k = 1 \times 10^6$		
$k = 2 \times 10^6$		

The results indicated that when the material is un-stressed, i.e., k is equal to zero, the form of displacement-time relationship is the same as the original loading wave. However, when stress gradient is not equal to zero, the waveform changes with the initial stress gradient. When the initial stress is linear relationship with the location, the magnitude of waveform change of $\sigma_s(x) = 1 \times 10^6 x$ less than the $\sigma_s(x) = 2 \times 10^6 x$. And, when k is the same, the magnitude of waveform change of $\sigma_s(x) = 1 \times 10^6 x^2$ higher than the $\sigma_s(x) = 1 \times 10^6 x$. The higher initial stress gradient leads to the waveform of the wave propagation in the material more different from the original loading wave, and the period and the value of the displacement are also significantly changed. Thus, the initial stress gradient changed the wave propagation, causing different response. The following section further investigates the initial stress gradient on wave propagation.

Table 2 Comparison of displacement-time curves under different initial stress states (x = 4)

<table>
<tr><th>$\sigma_s(x)$ / k values</th><th>$\sigma_s(x) = kx$</th><th>$\sigma_s(x) = kx^2$</th></tr>
<tr><td>$k = 0$</td><td>
</td><td>
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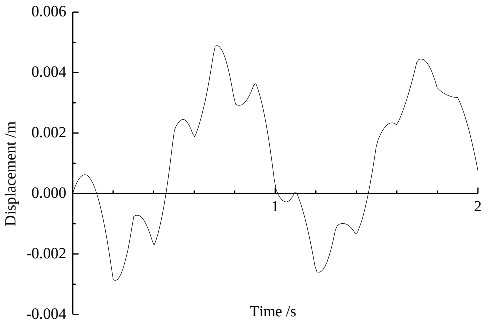
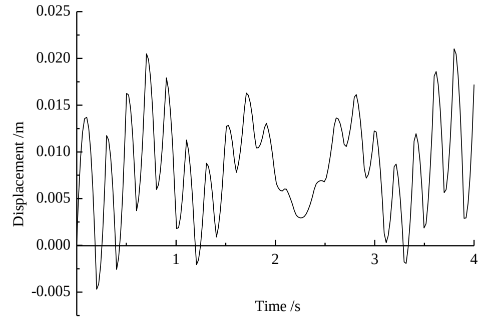
<tr><td>$k = 1 \times 10^6$</td><td>
</td><td>
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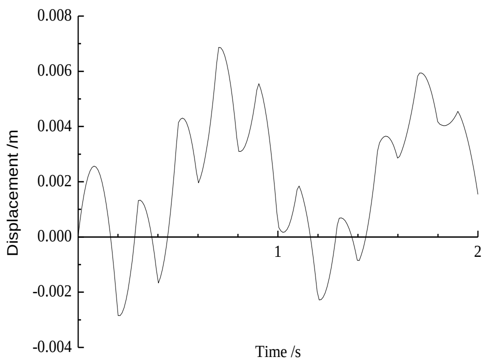
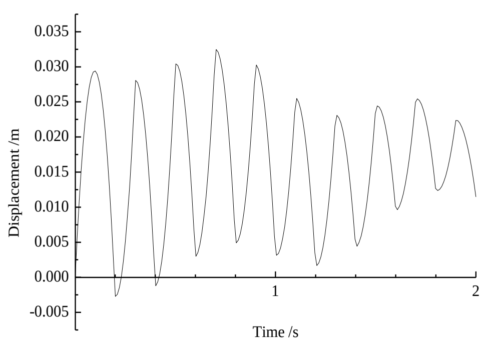
<tr><td>$k = 2 \times 10^6$</td><td>
</td><td>
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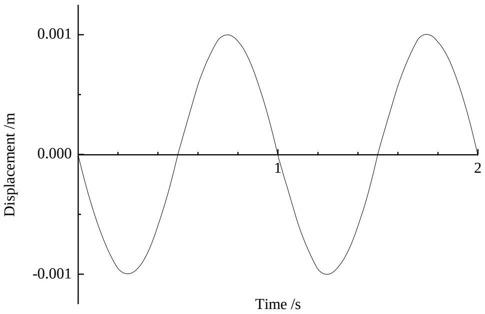
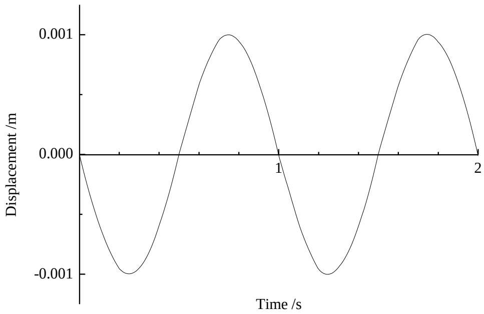
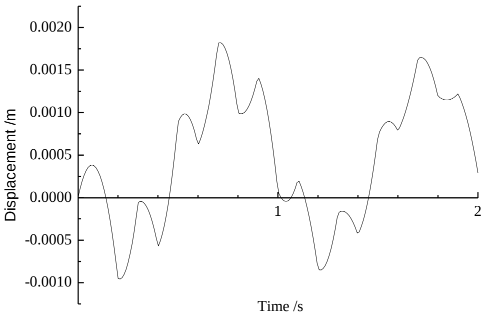
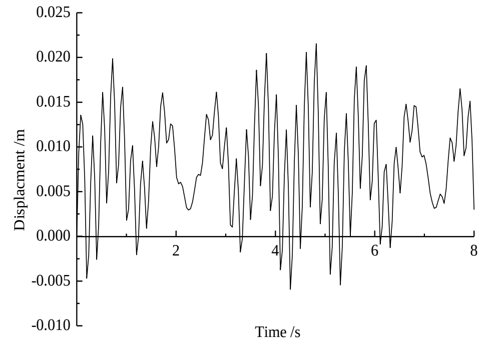
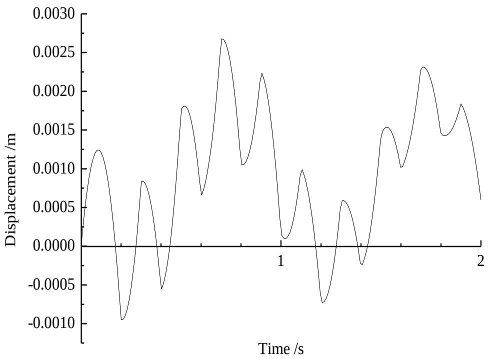
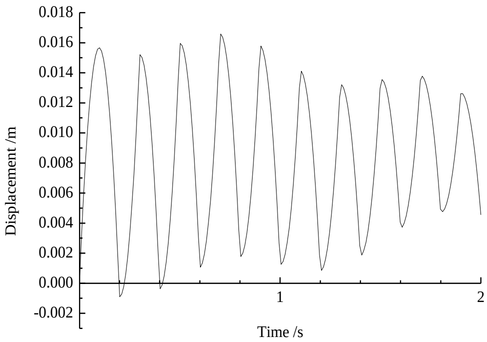
Table 3 Comparison of displacement-time curves under different initial stress states (x = 8)

$\sigma_s(x)$ / k values	$\sigma_s(x) = kx$	$\sigma_s(x) = kx^2$
$k = 0$		
$k = 1 \times 10^6$		
$k = 2 \times 10^6$		

4. Numerical simulations of the rock spall process under different initial stress gradient

The initial stress gradient changes the wave propagation and it will cause different failure response. Thus, numerical simulation is used to verify the influence of initial stress gradient on material failure response. In practical, there are many conditions that stress wave propagation in a medium with initial stress gradient. A typical example is stress wave propagation from underground to earth surface, and in this process, the stress wave propagation in a medium with initial stress gradually decreases. Additionally, explosion generated compression stress wave

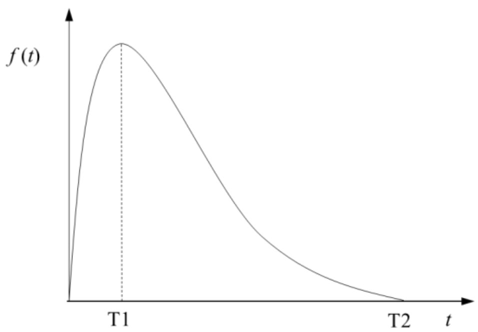


Fig. 2 Dynamic loading curve

becoming tension due to reflection at the free surface, and spall is induced when the tensile strength of the near surface geologic material is exceeded. Thus, spall of the earth's surface by underground explosions is simulated to explore the influence of initial stress gradient on rock failure.

Typically, an explosion induced stress wave has a short rise time and long decay time, which can be approximately combined as an exponential function and sine function, shown in Fig. 2.

Assuming the dynamic loading can be closely approximated by the expression

$$f(t) = e^{-\alpha t} \sin \beta t \tag{14}$$

where, $f(t)=0$ for $t \le 0$ or $t \ge T2$, set $\beta = \pi / T2$, then $f(T2) = 0$, and the parameter a is determined by the condition that $f(T1)$ is maximum, thus

$$\alpha = \beta \cot \beta T_1 = (\pi / T2) \cot \pi T_1 / T_2 \tag{15}$$

The numerical process contains two steps; one is the stress initialization of material, because of the gravity, initial stress is inhomogeneous, i.e., having gradient; the following step is the dynamic stress wave propagation in the pre-stressed material from underground to surface. In this study, the finite element program LS-DYNA was employed to simulate the process of stress wave propagation in rock material with initial stress gradient. The solver of LS-DYNA is based on explicit and implicit solver, static initial initialization and stress wave propagation process can be performed in sequence in the software package, respectively.

To simplify the problem, assuming the earth is only composited by rock, and the continuous surface cap model (CSCM) was employed to model the rock in this study, which was proved suitable for use with rock in earlier studies by Tao *et al*. (2012, 2013). The uni-axial compression strength is 153 MPa, and uni-axial tensile strength is 9.3 MPa. The complete descriptions of the model can be found in the LS-DYNA keyword user's manual (Hallquist 2007).

The finite element models were constructed and solved by using the LS-DYNA program. The geometric parameters of the numerical model are described in Fig. 3.

The initial geo-stress is conducted in *y*-direction, and a stress wave loaded in the up boundary of the cylindrical bore. The coordinate origin is the central of the cylindrical bore, the distance from the origin to –*x*-direction is set as 400 m, to +*x*-direction also set as 400 m, to –*y*-direction is

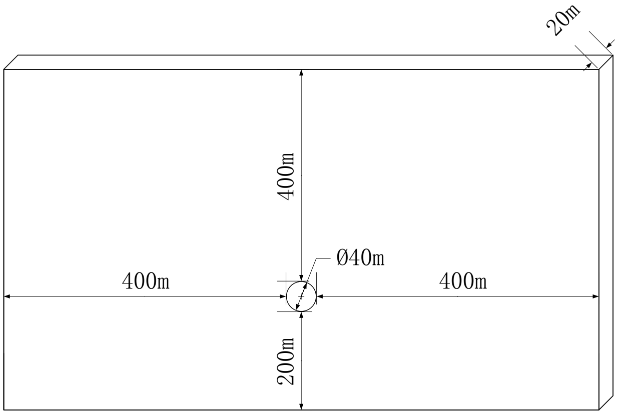


Fig. 3 The geometry size of the model

200 m, and the thickness of the model is 20 m, in *z*-direction. The –*x* and +*x* directions are set as non-reflection boundary, the top surface is set as free boundary, and the bottom boundary in *y*-direction is fixed.

Eight-node solid elements were used for the model, and convergence tests were then conducted to determine how many elements would be needed to achieve a reliable estimation by decreasing the size of the elements until the difference of the results between two consecutive element sizes is less than 5%. The convergence tests resulted in the section of the element number that was employed in the simulation (Tao *et al*. 2013).

To obtain the steady state of initial stress required to solve this problem, *dynamic relaxation*, a method of applying a pre-stressed provided by LS-DYNA, was used to perform the stress gradient initialization, such as gravity, and a function was applied to conduct the initial stress, different functions led to different initial stress and stress gradient. After this step, a database that updates the geometry and the stress history of the rock allows the values of the deformed shape, the pre-stress and strain in the rock to be reformulated. In addition, if no initial stress is considered in this problem, the explosion loading can directly be conducted in the up boundary of the cylinder hole. For example, when the gravity acceleration is 9.8 m/s^2, the results of initialization is shown in Fig. 4.

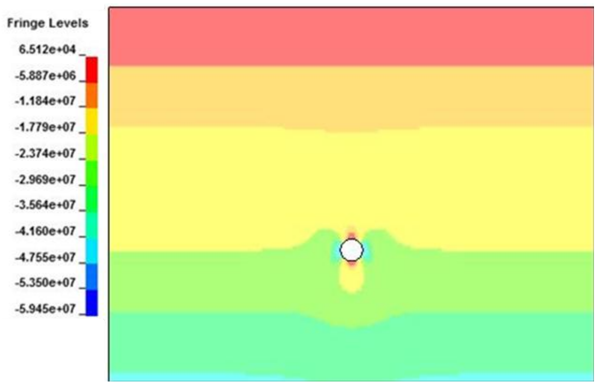
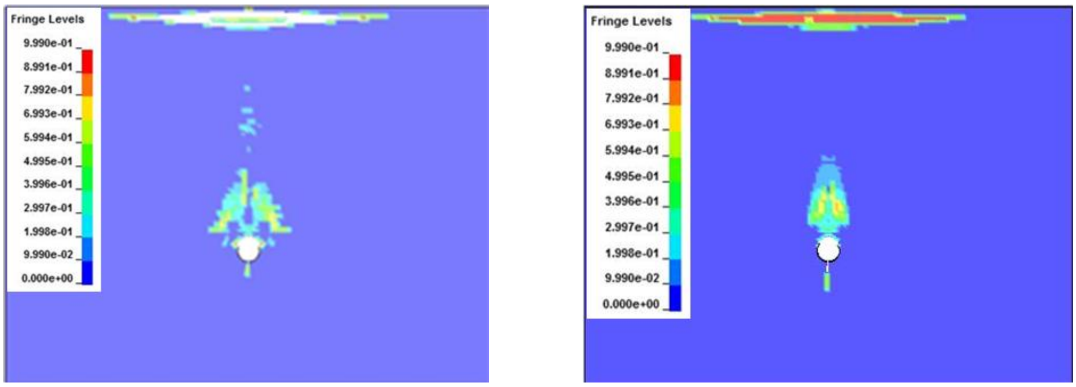
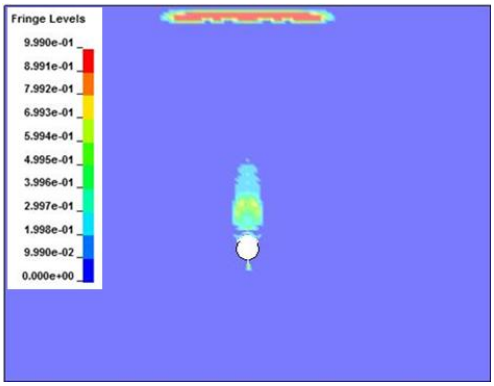


Fig. 4 Initial stress state in y direction



(a) Spalling results in gravity acceleration is zero (b) Spalling results in gravity acceleration is 9.8 m/s^2



(c) Spalling results in gravity acceleration 19.6 m/s^2

Fig. 5 Spalling results in different gravity acceleration

Fig. 4 indicated that initial stress is gradually increased from surface to deep depth. After the stress initialization, the explicit code was written for dynamic loading processes. An equivalent explosion loading curve and a command for calling the results of the initialization were added. Additional loading curves represent different stress waves. To compare how the initial stress affects the wave propagation, different gravity accelerations are considered corresponding with different initial stress gradient. Gravity acceleration is defined as 0, 9.8 m/s^2 and 19.6 m/s^2, respectively, the results shown in Fig. 5.

The result illustrated that when a larger blasting carried out in underground, it first caused damage around the blast source, then, after the stress wave reached the ground surface, it caused typical spalling damage in the vicinity surface. However, when the gravity is consider, different gravity acceleration leads to different saplling response, and both the magnitude damage for blast source and spalling are less than the gravity is not considered. Furthermore, as increase the gravity acceleration, the higher gradient coefficient leads to less rock failure. As the previous section analyzed that the initial stress gradient changed the waveform, period of the wave stress. Although the loading stress wave is the same, the initial stress gradient changed the wave stress, then, it causes different spalling results around the earth surface. Therefore, the numerical simulation further verified that the initial stress gradient have obvious influence on wave propagation.

5. Conclusions

The governing equation for wave propagating in a generally material with inhomogeneous initial stress is derived. By using a sample model, the definite problem of wave propagating in a rod with initial stress gradient is investigated, the calculation examples demonstrated that the initial stress gradient affects the wave propagation, and it will change the waveform and period. The earth surface spalling process is simulated to further explore the influence of initial stress gradient on wave propagating. The results derived in this work may be used to compute the exact effect of initial stress and initial stress gradient on wave propagation. For example, when a real data is known, the mathematical model can be used for a variety of geophysical problems, such as earthquake wave propagation, deep underground mining et al.

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