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Rayleigh wave in an anisotropic heterogeneous crustal layer lying over a gravitational sandy substratum

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Abstract. The purpose of this paper is to study the propagation of Rayleigh waves in an anisotropic heterogeneous crustal layer over a gravitational semi-infinite sandy substratum. It is assumed that the heterogeneity in the crustal layer arises due to exponential variation in elastic coefficients and density whereas the semi-infinite sandy substratum has homogeneous sandiness parameters. The coupled effects of heterogeneity, anisotropy, sandiness parameters and gravity on Rayleigh waves are discussed analytically as well as numerically. The dispersion relation is obtained in determinant form. The proposed model is solved to obtain the different dispersion relations for the Rayleigh wave in the elastic medium of different properties. The results presented in this study may be attractive and useful for mathematicians, seismologists and geologists.

Keywords: rayleigh waves; heterogeneity; elasticity; sandy substratum; anisotropy

1. Introduction

The knowledge of wave propagation in elastic and viscoelastic layered medium with different regions or boundaries helps geophysicists and seismologists to interpret the seismic pattern of earth at different edge or border. Material anisotropy and in-homogeneities of the earth affect the wave propagation. Several studies have been carried out to explain the nature of Rayleigh waves in isotropic homogeneous and heterogeneous media but less literature is available to show the effect of anisotropy of the medium on the Rayleigh wave propagation. So, in order to understand the accurate seismic pattern it becomes important and necessary to consider anisotropy and different heterogeneities of the elastic material (Wilson 1942). The elastic wave problems in elastic semi-infinite substratum are subject of many investigators. Moreover, the static and dynamic problems of the earth can be understood with the help of gravity parameter. In eighteenth century, Bromwich (1898) suggested gravity parameter to explain surface wave propagation in an elastic solid medium. In extension of work proposed by Bromwich, Love (1911) has shown the influence of gravity parameter on Rayleigh wave velocity. Biot (1965) investigated the influence of gravity parameter and initial hydrostatic stress on Rayleigh wave velocity. Vishwakarma and Gupta (2014) discussed Love waves under the effect of the rigid boundary. Gupta (2013) discussed the Love waves in prestressed layer over prestressed half-space. Addy and Chakraborty (2015) has studied a problem

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in which they have shown the effect of temperature and initial stress on Rayleigh waves in a viscoelastic medium but they did not show the effect of anisotropy and in-homogeneities of the medium. Vashishth and Sharma (2008) analyzed elasticity anisotropy and poroviscoelastic anisotropy of rocks. Singh and Bala (2013) have discussed a problem on Rayleigh wave in temperature field using theory of thermoelasticity. Vinh (2009) gave a complete solution of Rayleigh waves in elastic media under the effect of gravity and initial stress. Abd-Alla *et al*. (2011) investigated propagation of Rayleigh waves in granular medium under various parameters. Kumar and Singh (2011) discussed the effect of mountain on propagation of Rayleigh waves. Abd-Alla *et al*. (2010) studied Influences of rotation, magnetic field, initial stress, and gravity on Rayleigh waves in a homogeneous orthotropic elastic half-space. Sethi *et al*. (2012) discussed the effect of gravity and stress on Rayleigh waves propagating in inhomogeneous orthotropic elastic media. Acharya and Monda (2002) investigated Rayleigh surface waves in nonlocal viscoelastic solids with small wavelengths. Abd-Alla and Ahmed (1997) studied the effect of gravity, temperature and stress on Rayleigh waves in orthotropic elastic medium. Recently, Ghatuary and Chakraborty (2015) studied the effect of temperature, magnetic field and stress on Rayleigh waves propagating in isotropic homogeneous elastic half-space. Extensive literature has been devoted to the Rayleigh wave in anisotropic configurations by Favretto-Cristini *et al*. (2011). Similarly Vinh and Seriani (2009) have discussed the propagation of Rayleigh waves by introducing the concept of gravity. Pal *et al*. (2014) discussed the propagation of Rayleigh waves in anisotropic layer overlying a semi-infinite sandy medium in the absence of gravitational field.

In this study, a sincere attempt is made to explain the behaviour of Rayleigh wave in anisotropy heterogeneous crustal layer over a sandy semi-infinite substratum. It is believed that the upper boundary plane is free. Biot's equations are modified in context of heterogeneity of the medium with gravity. The heterogeneity in the layer is assumed to vary exponentially. The dispersion relation in 6×6 determinant form is obtained. The effects of heterogeneity and depth on the phase velocity are also shown in corresponding figures. In special cases, Rayleigh waves in sandy semi-infinite substratum under gravity, Rayleigh waves in isotropic semi-infinite substratum under gravity, Rayleigh waves in sandy semi-infinite substratum, Rayleigh waves in isotropic semi-infinite substratum, Rayleigh waves in a heterogeneous orthotropic layer lying over sandy semi-infinite substratum under gravity and Rayleigh waves in a heterogeneous isotropy layer lying over sandy semi-infinite substratum under gravity are derived. These said cases are further studied by plotting curves between real and imaginary phase velocity 'c' and wave number 'k' for different values of heterogeneity parameter, thickness of layer and sandy parameter for a particular model.

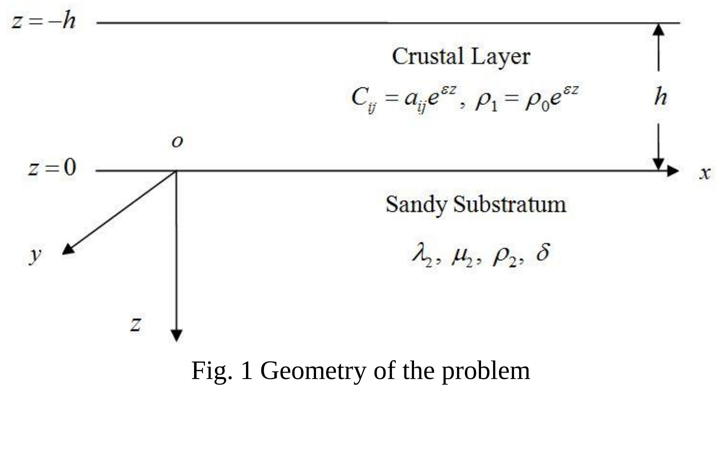

Fig. 1 Geometry of the problem

2. Formulation of the problem

We assume an anisotropic and heterogeneous elastic layer of finite thickness h lying over a sandy semi-infinite substratum under the effect of gravity. It is considered that the interface of these two media is at $z = 0$, whereas upper free surface is at $z = -h$. Further, the x-axis is along the direction of wave propagation and z-axis vertically downwards (Fig. 1). Let (u, v, w) denote the displacement components at any point $P(x, y, z)$ of the medium then $v = 0$ and u, w are functions of (x, z) and t only, because the displacement of Rayleigh wave does not depend on y-axis.

3. Solution of the problem

3.1 Solution for the crustal layer

It is assumed that the equilibrium conditions for the initial stresses are due to gravitational field, given by

$$\frac{\partial \tau}{\partial x} = 0, \frac{\partial \tau}{\partial z} + \rho g = 0, \tag{1a}$$

$$\tau_{xx} = \tau_{zz} = \tau,\ \tau_{xz} = 0. \tag{1b}$$

The dynamical equations of motion governing the propagation of three dimensional waves under the effect of gravity are given by Biot (1965)

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \rho_1 g \frac{\partial w_1}{\partial x} = \rho_1 \frac{\partial^2 u_1}{\partial t^2}, \tag{2}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \rho_1 g \frac{\partial w_1}{\partial y} = \rho_1 \frac{\partial^2 v_1}{\partial t^2}, \tag{3}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} - \rho_1 g \left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} \right) = \rho_1 \frac{\partial^2 w_1}{\partial t^2}, \tag{4}$$

where ρ_1 is the density of the layer.

The stress-strain relations for a heterogeneous anisotropic layer are of the form

$$\tau_{xx} = e^{\varepsilon z} \left\{ a_{11} e_{xx} + a_{12} e_{yy} + a_{13} e_{zz} + a_{14} e_{yz} + a_{15} e_{xz} + a_{16} e_{xy} \right\}, \tag{5a}$$

$$\tau_{yy} = e^{\varepsilon z} \left\{ a_{12} e_{xx} + a_{22} e_{yy} + a_{23} e_{zz} + a_{24} e_{yz} + a_{25} e_{xz} + a_{26} e_{xy} \right\}, \tag{5b}$$

$$\tau_{zz} = e^{\varepsilon z} \left\{ a_{13} e_{xx} + a_{23} e_{yy} + a_{33} e_{zz} + a_{34} e_{yz} + a_{35} e_{xz} + a_{36} e_{xy} \right\}, \tag{5c}$$

$$\tau_{yz} = e^{\varepsilon z} \left\{ a_{14} e_{xx} + a_{24} e_{yy} + a_{34} e_{zz} + a_{44} e_{yz} + a_{45} e_{xz} + a_{46} e_{xy} \right\}, \tag{5d}$$

$$\tau_{xz} = e^{\varepsilon z}\left\{a_{15}e_{xx} + a_{25}e_{yy} + a_{35}e_{zz} + a_{45}e_{yz} + a_{55}e_{xz} + a_{56}e_{xy}\right\}, \tag{5e}$$

$$\tau_{xy} = e^{\varepsilon z}\left\{a_{16}e_{xx} + a_{26}e_{yy} + a_{36}e_{zz} + a_{46}e_{yz} + a_{56}e_{xz} + a_{66}e_{xy}\right\}, \tag{5f}$$

and the density is taken as $\rho_1 = \rho_0 e^{\varepsilon z}$.

Now, the equations of motion for the propagation of Rayleigh waves in a heterogeneous anisotropic medium obeying Eqs. (2)-(5), become

$$\begin{aligned} a_{11}\frac{\partial^2 u_1}{\partial x^2} + a_{15}\frac{\partial^2 w_1}{\partial x^2} + a_{55}\frac{\partial^2 u_1}{\partial z^2} + a_{35}\frac{\partial^2 w_1}{\partial z^2} + 2a_{15}\frac{\partial^2 u_1}{\partial x \partial z} + (a_{13}+a_{55})\frac{\partial^2 w_1}{\partial x \partial z} \\ + \varepsilon a_{15}\frac{\partial u_1}{\partial x} + \varepsilon a_{55}\frac{\partial w_1}{\partial x} + \varepsilon a_{55}\frac{\partial u_1}{\partial z} + \varepsilon a_{35}\frac{\partial w_1}{\partial z} + \rho_0 g\frac{\partial w_1}{\partial x} = \rho_0\frac{\partial^2 u_1}{\partial t^2}, \end{aligned} \tag{6}$$

$$\begin{aligned} a_{15}\frac{\partial^2 u_1}{\partial x^2} + a_{55}\frac{\partial^2 w_1}{\partial x^2} + a_{35}\frac{\partial^2 u_1}{\partial z^2} + a_{33}\frac{\partial^2 w_1}{\partial z^2} + (a_{13}+a_{55})\frac{\partial^2 u_1}{\partial x \partial z} + 2a_{35}\frac{\partial^2 w_1}{\partial x \partial z} \\ + \varepsilon a_{13}\frac{\partial u_1}{\partial x} + \varepsilon a_{35}\frac{\partial w_1}{\partial x} + \varepsilon a_{35}\frac{\partial u_1}{\partial z} + \varepsilon a_{33}\frac{\partial w_1}{\partial z} - \rho_0 g\frac{\partial u_1}{\partial x} = \rho_0\frac{\partial^2 w_1}{\partial t^2}. \end{aligned} \tag{7}$$

On seeking solution for the above equations in the form $u_1(x, z, t) = A(z)e^{ik(x-ct)}$ and $w_1(x, z, t) = B(z)e^{ik(x-ct)}$, we have

$$\begin{aligned} &\left[a_{55}D^2 + (2ika_{15} + \varepsilon a_{55})D + (\rho_0 k^2 c^2 - a_{11}k^2 + ik\varepsilon a_{15})\right]A \\ &+ \left[a_{35}D^2 + \left\{ik(a_{13}+a_{55}) + \varepsilon a_{35}\right\}D + (-a_{15}k^2 + ik\varepsilon a_{55} + ik\rho_0 g)\right]B = 0, \end{aligned} \tag{8}$$

$$\begin{aligned} &\left[a_{35}D^2 + \left\{ik(a_{13}+a_{55}) + \varepsilon a_{35}\right\}D + (-a_{15}k^2 + ik\varepsilon a_{13} - ik\rho_0 g)\right]A \\ &+ \left[a_{33}D^2 + (2ika_{35} + \varepsilon a_{33})D + (\rho_0 k^2 c^2 - a_{55}k^2 + ik\varepsilon a_{35})\right]B = 0. \end{aligned} \tag{9}$$

Following the standard method for solving simultaneous linear algebraic equations with constant coefficients, we write $A(z) = \phi e^{-k\zeta z}$, $B(z) = \psi e^{-k\zeta z}$ and by using Eqs. (8) and (9), we have

$$\begin{aligned} &\left[a_{55}\zeta^2 - \left(2ia_{15} + \frac{\varepsilon a_{55}}{k}\right)\zeta + \left(\rho_0 c^2 - a_{11} + \frac{i\varepsilon a_{15}}{k}\right)\right]\phi \\ &+ \left[a_{35}\zeta^2 - \left\{i\left(a_{13}+a_{55}\right) + \frac{\varepsilon a_{35}}{k}\right\}\zeta + \left(-a_{15} + \frac{i\varepsilon a_{55}}{k} + \frac{i\rho_0 g}{k}\right)\right]\psi = 0, \end{aligned} \tag{10}$$

$$\begin{aligned} &\left[a_{35}\zeta^2 - \left\{i\left(a_{13}+a_{55}\right) + \frac{\varepsilon a_{35}}{k}\right\}\zeta + \left(-a_{15} + \frac{i\varepsilon a_{33}}{k} - \frac{i\rho_0 g}{k}\right)\right]\phi \\ &+ \left[a_{33}\zeta^2 - \left(2ia_{35} + \frac{\varepsilon a_{33}}{k}\right)\zeta + \left(\rho_0 c^2 - a_{55} + \frac{i\varepsilon a_{35}}{k}\right)\right]\psi = 0. \end{aligned} \tag{11}$$

In order to obtain a non trivial solution of Eqs. (10) and (11), the following condition should be fulfilled

$$\alpha_0\zeta^4+\alpha_1\zeta^3+\alpha_2\zeta^2+\alpha_3\zeta+\alpha_4=0, \tag{12}$$

where, α_0, α_1, α_2, α_3 and α_4 are given in the Appendix.

If by ζ_j (j = 1,..., 4) we denote the roots of Eq. (12), the ratio of the displacement components $\frac{A_j}{B_j}$ from Eq. (10) corresponding to $\zeta=\zeta_j$ is

$$\frac{B_j}{A_j}=\frac{\psi_j}{\phi_j}=\frac{-\left[a_{55}\zeta_j^2-\left(2ia_{15}+\frac{\varepsilon a_{55}}{k}\right)\zeta_j+\left(\rho_0c^2-a_{11}+\frac{i\varepsilon a_{15}}{k}\right)\right]}{\left[a_{35}\zeta_j^2-\left\{i\left(a_{13}+a_{55}\right)+\frac{\varepsilon a_{35}}{k}\right\}\zeta_j+\left(-a_{15}+\frac{i\varepsilon a_{55}}{k}+\frac{i\rho_0 g}{k}\right)\right]}\equiv p_j. \tag{13}$$

Thus, the solution of Eqs. (6) and (7) can be written as

$$u_1=\left(\phi_1e^{-k\zeta_1z}+\phi_2e^{-k\zeta_2z}+\phi_3e^{-k\zeta_3z}+\phi_4e^{-k\zeta_4z}\right)e^{ik(x-ct)}, \tag{14}$$

$$w_1=\left(p_1\phi_1e^{-k\zeta_1z}+p_2\phi_2e^{-k\zeta_2z}+p_3\phi_3e^{-k\zeta_3z}+p_4\phi_4e^{-k\zeta_4z}\right)e^{ik(x-ct)}. \tag{15}$$

where k is the wave number and c is the phase velocity.

3.2 Solution for the sandy substratum

The dynamical equations of motion for the propagation of Rayleigh waves under the effect of gravity are given by Biot (1965)

$$\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{xz}}{\partial z}+\rho_2 g\frac{\partial w_2}{\partial x}=\rho_2\frac{\partial^2u_2}{\partial t^2}, \tag{16}$$

$$\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{zz}}{\partial z}-\rho_2 g\frac{\partial u_2}{\partial x}=\rho_2\frac{\partial^2w_2}{\partial t^2}, \tag{17}$$

where ρ_2 is the density of the sandy medium. The stress-strain relations for a sandy medium are given by the relations

$$\tau_{xx}=\delta\left[\left(\lambda_2+2\mu_2\right)\frac{\partial u_2}{\partial x}+\lambda_2\frac{\partial w_2}{\partial z}\right], \tag{18a}$$

$$\tau_{zz}=\delta\left[\lambda_2\frac{\partial u_2}{\partial x}+\left(\lambda_2+2\mu_2\right)\frac{\partial w_2}{\partial z}\right], \tag{18b}$$

$$\tau_{xz}=\delta\lambda_2\left(\frac{\partial w_2}{\partial x}+\frac{\partial u_2}{\partial z}\right), \tag{18c}$$

where ρ_2 is the density of the sandy medium. The stress-strain relations for a sandy medium are given by the relations

$$\delta\left(\lambda_2+2\mu_2\right)\frac{\partial^2 u_2}{\partial x^2}+\delta\left(\lambda_2+\mu_2\right)\frac{\partial^2 w_2}{\partial x\partial z}+\delta\mu_2\frac{\partial^2 u_2}{\partial z^2}+\rho_2 g\frac{\partial w_2}{\partial x}=\rho_2\frac{\partial^2 u_2}{\partial t^2}, \tag{19}$$

$$\delta\mu_2\frac{\partial^2 w_2}{\partial x^2}+\delta\left(\lambda_2+\mu_2\right)\frac{\partial^2 u_2}{\partial x\partial z}+\delta\left(\lambda_2+2\mu_2\right)\frac{\partial^2 w_2}{\partial z^2}-\rho_2 g\frac{\partial u_2}{\partial x}=\rho_2\frac{\partial^2 w_2}{\partial t^2}. \tag{20}$$

Assuming that the solution of above equations is of the form $u_2(x, z, t) = [Fe^{-k\gamma z} + Ge^{k\gamma z}]e^{-ik(x-ct)}$ and $w_2(x, z, t) = [Pe^{-k\gamma z} + Qe^{k\gamma z}]e^{-ik(x-ct)}$ and substituting in Eqs. (19) and (20), we have

$$\left\{\rho_2 c^2-\delta\left(\lambda_2+\mu_2\right)+\mu_2\delta\gamma^2\right\}F-\left\{i\delta\left(\lambda_2+\mu_2\right)\gamma+\frac{i\rho_2 g}{k}\right\}G=0, \tag{21}$$

$$\left\{\rho_2 c^2-\delta\left(\lambda_2+\mu_2\right)+\mu_2\delta\gamma^2\right\}G+\left\{i\delta\left(\lambda_2+\mu_2\right)\gamma+\frac{i\rho_2 g}{k}\right\}I=0, \tag{22}$$

$$\left\{\rho_2 c^2-\delta\left(\lambda_2+\mu_2\right)+\mu_2\delta\gamma^2\right\}P-\left\{i\delta\left(\lambda_2+\mu_2\right)\gamma+\frac{i\rho_2 g}{k}\right\}G=0, \tag{23}$$

$$\left\{\rho_2 c^2-\delta\left(\lambda_2+\mu_2\right)+\mu_2\delta\gamma^2\right\}Q-\left\{i\delta\left(\lambda_2+\mu_2\right)\gamma+\frac{i\rho_2 g}{k}\right\}G=0, \tag{24}$$

where γ is the parameter to be determined, k is the wave number and c is the phase velocity.

Eliminating F, G, P and Q from Eqs. (21)-(24), we get the following biquadratic algebraic equation in γ

$$\beta_0\gamma^4+\beta_1\gamma^2+\beta_2=0, \tag{25}$$

where, β_0, β_1 and β_2 are defined in the Appendix . Let $\pm\chi_1$, $\pm\chi_2$ be the roots of Eq. (25), then

$$\chi_1^2=\frac{-\beta_1-\sqrt{\beta_1^2-4\beta_0\beta_2}}{2\beta_0} \quad \text{and} \quad \chi_2^2=\frac{-\beta_1-\sqrt{\beta_1^2-4\beta_0\beta_2}}{2\beta_0}.$$

In view of Eq. (25), the displacements in the gravitational sandy layer are given by

$$u_2(x,z,t)=\left[F_1e^{-k\gamma_1 z}+F_2e^{-k\gamma_2 z}+G_1e^{k\gamma_2 z}+G_2e^{k\gamma_2 z}\right]e^{ik(x-ct)}, \tag{26}$$

$$w_2(x,z,t)=\left[m_1F_1e^{-k\gamma_1 z}+m_2F_2e^{-k\gamma_2 z}-m_1G_1e^{k\gamma_1 z}-m_2G_2e^{k\gamma_2 z}\right]e^{ik(x-ct)}, \tag{27}$$

where $P_j = m_jF_j$ and $Q_j = -m_jG_j$, the ratio of the displacement components, is

$$m_j=\frac{\dfrac{c^2}{c_1^2}-\delta+\delta\dfrac{c_2^2}{c_1^2}\gamma_j^2}{\left\{i\delta\left(1-\dfrac{c_2^2}{c_1^2}\right)\gamma_j-\dfrac{i\rho_2 g}{k}\right\}}\,(j=1,2). \tag{28}$$

The approximate solution for Eqs. (26) and (27) is given by

$$u_2(x,z,t)=\left[F_1e^{-k\gamma_1 z}+F_2e^{-k\gamma_2 z}\right]e^{ik(x-ct)}, \tag{29}$$

$$w_2(x,z,t)=\left[m_1F_1e^{-k\gamma_1 z}+m_2F_2e^{-k\gamma_2 z}\right]e^{ik(x-ct)}. \tag{30}$$

4. Boundary conditions and dispersion relation

(i) At the interface, $z=0$, the continuity of the displacement along the x direction requires that $u_1 = u_2$ and $w_1 = w_2$, where u_1 and w_1 are the displacement component in the layer along the x and $z$directions respectively.
(ii) At the interface, $z=0$, the continuity of the stress requires that $(\tau_{xz})_{medium1} = (\tau_{xz})_{medium2}$ and $(\tau_{zz})_{medium1} = (\tau_{zz})_{medium2}$, where τ_{xz} and τ_{zz} are the relevant stress components.
(iii) At the upper boundary plane (free surface) i.e., at $z = -h$, the stresses vanish i.e., $(\tau_{xz})_{medium1} = 0$ and $(\tau_{zz})_{medium1} = 0$.

Using the boundary conditions (i), (ii), (iii) and Eqs. (14), (15), (29) and (30) we have, respectively

$$\phi_1+\phi_2+\phi_3+\phi_4-F_1-F_2=0, \tag{31}$$

$$p_1\phi_1+p_2\phi_2+p_3\phi_3+p_4\phi_4-m_1F_1-m_2F_2=0, \tag{32}$$

$$T_1\phi_1+T_2\phi_2+T_3\phi_3+T_4\phi_4-\delta\mu_2(im_1-\gamma_1)F_1-\delta\mu_2(im_2-\gamma_2)F_2=0, \tag{33}$$

$$T_5\phi_1+T_6\phi_2+T_7\phi_3+T_8\phi_4-\delta\left\{i\lambda_2-\left(\lambda_2+2\mu_2\right)\gamma_1m_1\right\}F_1-\delta\left\{i\lambda_2-\left(\lambda_2+2\mu_2\right)_3\gamma_2m_2\right\}F_2=0, \tag{34}$$

$$T_1\phi_1e^{k\zeta_1h}+T_2\phi_2e^{k\zeta_2h}+T_3\phi_3e^{k\zeta_3h}+T_4\phi_4e^{k\zeta_4h}=0, \tag{35}$$

$$T_5\phi_1e^{k\zeta_1h}+T_6\phi_2e^{k\zeta_2h}+T_7\phi_3e^{k\zeta_3h}+T_8\phi_4e^{k\zeta_4h}=0. \tag{36}$$

Eliminating ϕ_1, ϕ_2, ϕ_3, ϕ_4, F_1 and F_2 from Eqs. (29) to (34), we have

$$\begin{vmatrix} 1 & 1 & 1 & 1 & -1 & -1 \\ p_1 & p_2 & p_3 & p_4 & -m_1 & -m_2 \\ T_1 & T_2 & T_3 & T_4 & -\delta\mu_2(im_1-\gamma_1) & -\delta\mu_2(im_2-\gamma_2) \\ T_5 & T_6 & T_7 & T_8 & -\delta\left\{i\lambda_2-\left(\lambda_2+2\mu_2\right)\gamma_1m_1\right\} & -\delta\left\{i\lambda_2-\left(\lambda_2+2\mu_2\right)\gamma_2m_2\right\} \\ T_1e^{k\zeta_1h} & T_2e^{k\zeta_2h} & T_3e^{k\zeta_3h} & T_4e^{k\zeta_4h} & 0 & 0 \\ T_5e^{k\zeta_1h} & T_6e^{k\zeta_2h} & T_7e^{k\zeta_3h} & T_8e^{k\zeta_4h} & 0 & 0 \end{vmatrix}=0. \tag{37}$$

Eq. (37) is the dispersion equation for the Rayleigh waves in a heterogeneous anisotropic layer lying over a sandy substratum under gravity. eq. (37) is the dispersion equation and its solution can be expressed as Real(c) + i Imaginary (c).The real part of the dispersion relation gives the phase velocity of Rayleigh waves and the imaginary part of it represents its dispersive nature. It can be noted that both the frequency and the layer thickness *h* affect the solution c, not only frequency through *k*.

5. Special cases

Case 1

If $h \to 0$, Eq. (37) reduced to

$$\begin{vmatrix} \delta\mu_2(im_1-\gamma_1) & \delta\mu_2(im_2-\gamma_2) \\ \delta\left\{i\lambda_2-\left(\lambda_2+2\mu_2\right)\gamma_1 m_1\right\} & \delta\left\{i\lambda_2-\left(\lambda_2+2\mu_2\right)\gamma_2 m_2\right\} \end{vmatrix}=0 \tag{38}$$

Eq. (38) is the dispersion equation for Rayleigh waves in a sandy semi-infinite substratum under gravity.

Case 2

If $h \to 0$, $\delta \to 1$, Eq. (37) reduced to

$$\begin{vmatrix} \mu_2(im_1'-\gamma_1') & \mu_2(im_2'-\gamma_2') \\ \left\{i\lambda_2-\left(\lambda_2+2\mu_2\right)\gamma_1' m_1'\right\} & \left\{i\lambda_2-\left(\lambda_2+2\mu_2\right)\gamma_2' m_2'\right\} \end{vmatrix}=0 \tag{39}$$

where root corresponding to Eq. (25) are given by $m_j' = \dfrac{\dfrac{c^2}{c_1^2}-1+\dfrac{c_2^2}{c_1^2}\gamma_j'^2}{\left\{i\left(1-\dfrac{c_2^2}{c_1^2}\right)\right\}\gamma_j' - \dfrac{i\rho_2 g}{k}}\,(j=1,\ 2).$

Eq. (39) is the dispersion equation for Rayleigh waves in isotropic semi-infinite substratum under gravity.

Case 3

If $h \to 0$, $g \to 0$, Eq. (37) reduces to

$$\begin{vmatrix} \delta\mu_2(im_1''-\gamma_1'') & \delta\mu_2(im_2''-\gamma_2'') \\ \delta\left\{i\lambda_2-\left(\lambda_2+2\mu_2\right)\gamma_1'' m_1''\right\} & \delta\left\{i\lambda_2-\left(\lambda_2+2\mu_2\right)\gamma_2'' m_2''\right\} \end{vmatrix}=0 \tag{40}$$

where the root corresponding to Eq. (25) are given by $m_j'' = \dfrac{\dfrac{c^2}{c_1^2}-1+\dfrac{c_2^2}{c_1^2}\gamma_j''^2}{\left\{i\left(1-\dfrac{c_2^2}{c_1^2}\right)\gamma_j''\right\}}\,(j=1,\ 2).$

Eq. (40) is the dispersion equation for Rayleigh waves in sandy semi-infinite substratum which is in agreement with the result given by Pal *et al*. (2014).

Case 4

If we take $h \to 0$, $g \to 0$, $\delta \to 1$, Eq. (37) reduces to

$$\begin{vmatrix} \mu_2 (im_1''' - \gamma_1''') & \mu_2 (im_2''' - \gamma_2''') \\ \left\{ i\lambda_2 - (\lambda_2 + 2\mu_2)\gamma_1''' m_1''' \right\} & \left\{ i\lambda_2 - (\lambda_2 + 2\mu_2)\gamma_2''' m_2''' \right\} \end{vmatrix} = 0 \tag{41}$$

where the root corresponding to Eq. (25) are given by $m_j' = \dfrac{\dfrac{c^2}{c_1^2} - 1 + \dfrac{c_2^2}{c_1^2}\gamma_j'^2}{\left\{ i\left(1 - \dfrac{c_2^2}{c_1^2}\right)\gamma_j' \right\}} (j = 1,\ 2)$.

Eq. (41) is the dispersion equation for Rayleigh waves in an isotropic semi-infinite substratum.

Case 5

If we take $a_{15} = a_{35} = 0$. Eq. (37) reduces dispersion equation for Rayleigh waves in a heterogeneous orthotropic layer lying over sandy semi-infinite substratum under gravity.

Case 6

If we take $a_{15} = a_{35} = 0$, $a_{11} = a_{33} = \lambda_1 + 2\mu_1$, $a_{13} = \lambda_1$, $a_{55} = \mu_1$, Eq. (37) reduces dispersion equation for Rayleigh waves in a heterogeneous isotropy layer lying over sandy semi-infinite substratum under gravity.

6. Numerical results and discussion

In order to show the effect of non-homogeneity and phase velocity dependence on the wave number, we have taken data in Table 1 for layer and sandy substratum. Analytical curves of the wave number calculated with MATLAB software are plotted versus the phase velocity in Fig. 2 to fig 15. For each case, the real and imaginary parts are shown separately. In Fig. 2, the relevant graph is plotted for the real part of phase velocity real(c) against the wave number (k) for different values of the heterogeneity parameter ε (0.0011, 0.0013, 0.0015 and 0.0017) at constant depth h = 5 km and sandiness parameter δ = 1.5. The curves show that the phase velocity decreases for increasing wave number. However as we increase the heterogeneity parameter ε, the magnitude of the phase velocity increases for all k's. The behavior of the curves is same but the curves are getting closer and closer with increasing values of wave number. It is clearly seen that the effect of heterogeneity on phase velocity is more significant for small values of the wave number, and as the wave number increases, the effect of heterogeneity decreases. In Fig. 3, the relevant graph is plotted for the imaginary part of phase velocity against the wave number k for different values of heterogeneity parameter ε (0.0011, 0.0013, 0.0015 and 0.0017) and constant depth at constant depth h = 5 km and sandiness parameter δ = 1.5. The figure shows that the phase velocity decreases for increasing wave numbers, but as we increase ε, the phase velocity increases for all k's as before. In Fig. 4, the relevant graph is plotted for the real part of phase velocity against the wave number k for ε = 0.0015 and different depths h (0, 5, 10 and 15) km at δ = 1.5. It can be seen from the graph that the phase velocity decreases for increasing wave number. As we increase the

Table 1 Data for elastic and sandy medium

Symbol	Numerical value	Units
a_{11}	17.77	Gpa
a_{12}	3.78	Gpa
a_{13}	3.76	Gpa
a_{14}	0.24	Gpa
a_{15}	-0.28	Gpa
a_{16}	0.03	Gpa
a_{22}	19.45	Gpa
a_{23}	4.13	Gpa
a_{24}	-0.41	Gpa
a_{25}	0.07	Gpa
a_{26}	1.13	Gpa
a_{33}	21.79	Gpa
a_{34}	-0.12	Gpa
a_{35}	-0.01	Gpa
a_{36}	0.38	Gpa
a_{44}	8.30	Gpa
a_{45}	0.66	Gpa
a_{46}	0.06	Gpa
a_{55}	7.62	Gpa
a_{56}	0.52	Gpa
a_{56}	7.77	Gpa
ρ_0	2216	kg/m^3
λ_2	1.82	Gpa
μ_2	3.52	Gpa
ρ_2	3380	kg/m^3

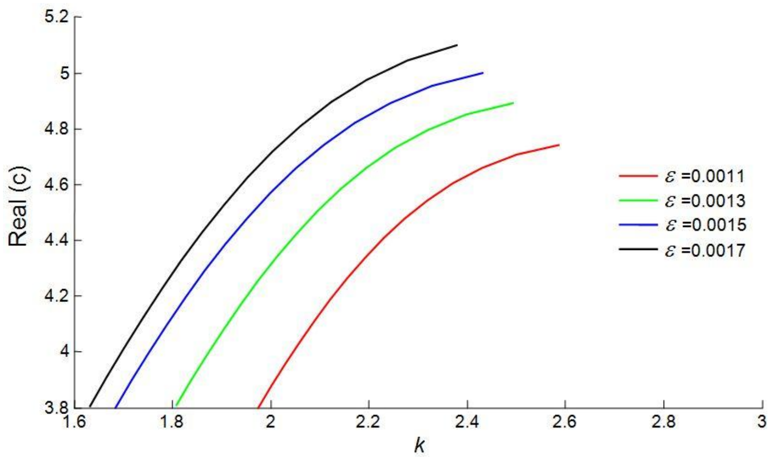

Fig. 2 Variation of phase velocity Real(c) against wave number k for different ε at constant h = 5 km, δ = 1.5 when layer is anisotropy in nature

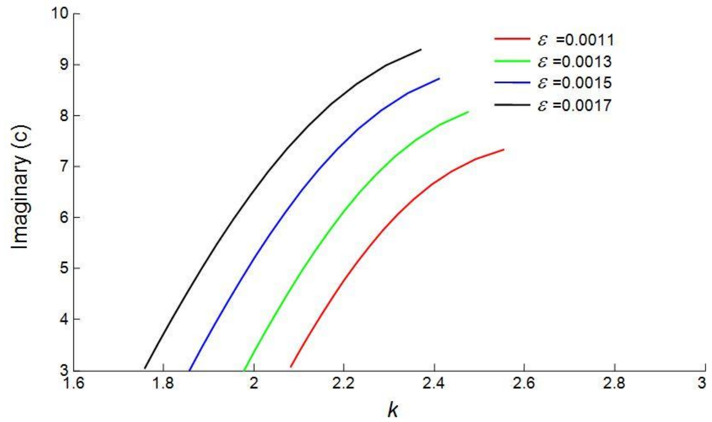

Fig. 3 Variation of phase velocity Imaginary(c) against wave number k for different ε at constant h = 5 km, δ = 1.5 when layer is anisotropy in nature

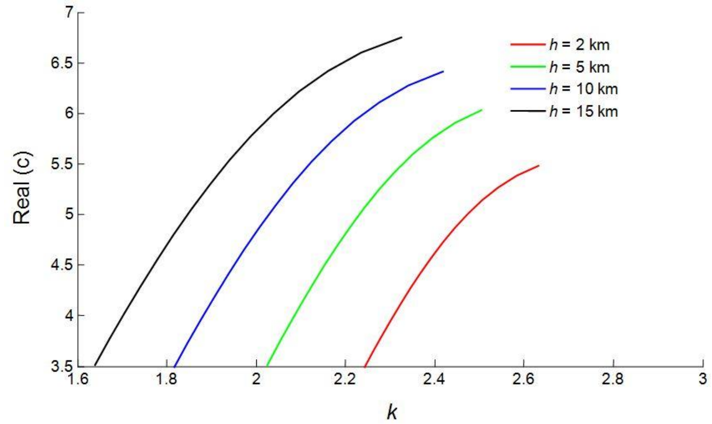

Fig. 4 Variation of phase velocity Real(c) against wave number k for different h at constant ε = 0.0015, δ = 1.5 when layer is anisotropy in nature

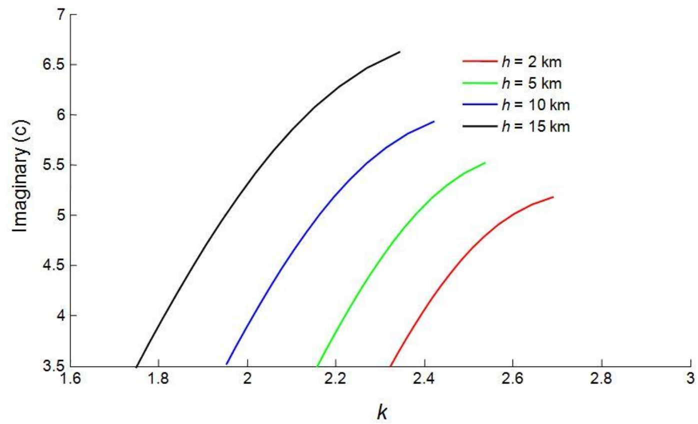

Fig. 5 Variation of phase velocity Imaginary(c) against wave number k for different h at constantε = 0.0015, δ = 1.5 when layer is anisotropy in nature

thickness of the layer, the phase velocity increases for some wave number k's, but all curves for different thicknesses come closer as k increases. It is shown that the thickness of the layer has significant effects on the phase velocity. In Fig. 5, the relevant graph is plotted for the imaginary part of the phase velocity against the wave number k for $\varepsilon = 0.0015$ and different depths h (0, 5, 10 and 15) km at $\delta = 1.5$. The figure suggests that as we increase the thickness of the layer, the phase velocity increases for all k's, but the behaviour of the curve remains same. In Figs. 6 and 7 the relevant graph is plotted for phase velocity Real(c) and Imaginary(c) against wave number k for different δ at constant h = 4 km, $\varepsilon = 0.0015$ when layer is anisotropy in nature. It shows as sandiness parameter δ increases Real(c) increases but Imaginary(c) decreases with increase of δ. However both Real(c) and Imaginary(c) increases with increasing wave number k's. In Fig. 8 and Fig. 9 the relevant graph is plotted for phase velocity Real(c) and Imaginary(c) against wave number k for different h at constant $\varepsilon = 0.0015$, $\delta = 1.5$ when layer is assumed to be orthotropic in nature. In Fig. 8, Real(c) increases as h increases, this trend is reciprocal in Fig. 9 for*m* Imaginary(c). Figs. 10 and 11shows the variation of phase velocity Real(c) and Imaginary(c)

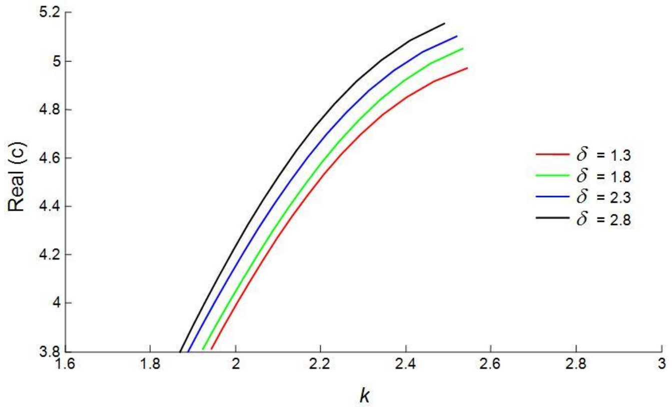

Fig. 6 Variation of phase velocity Real(c) against wave number k for different δ at constant h = 5 km, $\varepsilon = 0.0015$ when layer is anisotropy in nature

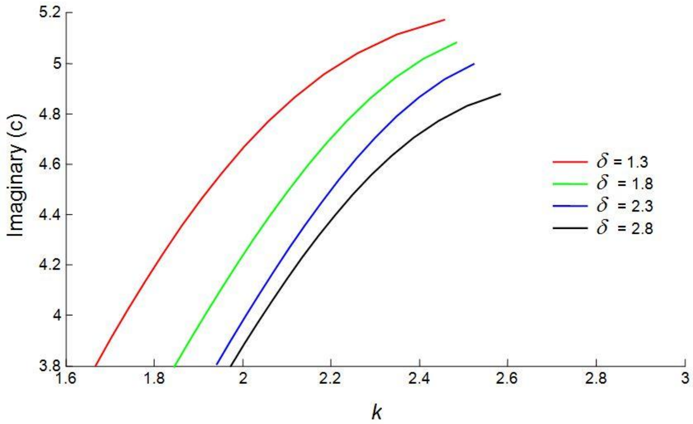

Fig. 7 Variation of phase velocity Imaginary(c) against wave number k for different δ at constant h = 5 km, $\varepsilon = 0.0015$ when layer is anisotropy in nature

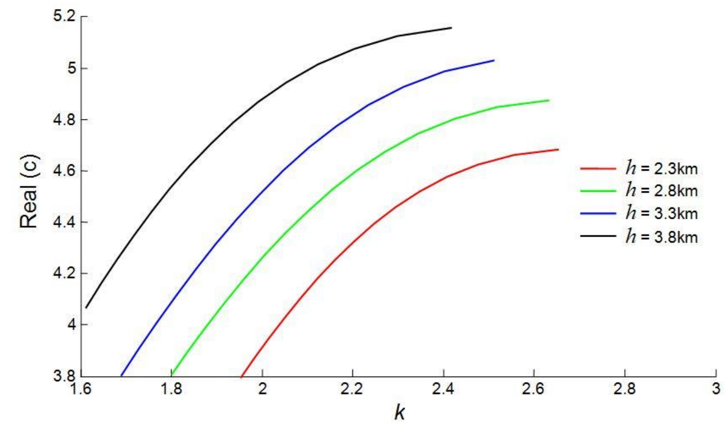

Fig. 8 Variation of phase velocity Real(c) against wave number k for different h at constant ε = 0.0015, δ = 1.5 when layer is orthotropic in nature

against wave number k for different δ at constant h = 4 km, ε = 0.0015 when layer is orthotropic in nature. In Fig. 10 Real(c) increases as δ increases while Imaginary(c) decreases in Fig. 11 as δ increases. In Fig. 12, the relevant graph is plotted for the real part of phase velocity real(c) against the wave number (k) for different values of h at constant ε = 0.0015, δ = 1.5 when layer is isotropic in nature. In Fig. 13, the relevant graph is plotted for the imaginary part of phase velocity Imaginary(c) against the wave number (k) for different values of h at constant ε = 0.0015, δ = 1.5 when layer is isotropic in nature. In Fig. 12 Real(c) increases as δ increases while Imaginary(c) decreases in Fig. 13 as δ increases. Figs. 14-15 shows the variation of phase velocity Real(c) and Imaginary(c) against wave number k for different δ at constant h = 4 km, ε = 0.0015 when layer is isotropic in nature. In Fig. 14, Real(c) increases as δ increases, this trend is reciprocal in Fig.15 for Imaginary(c).

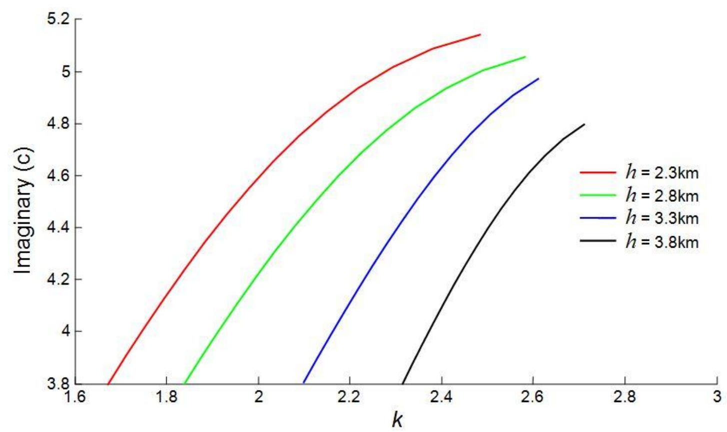

Fig. 9 Variation of phase velocity Imaginary(c) against wave number k for different h at constantε = 0.0015, δ = 1.5 when layer is orthotropic in nature

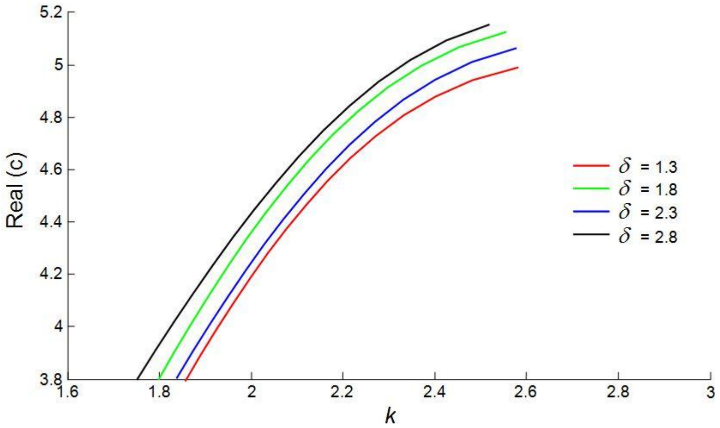

Fig. 10 Variation of phase velocity Real(c) against wave number k for different δ at constant h = 5 km, ε = 0.0015 when layer is orthotropic in nature

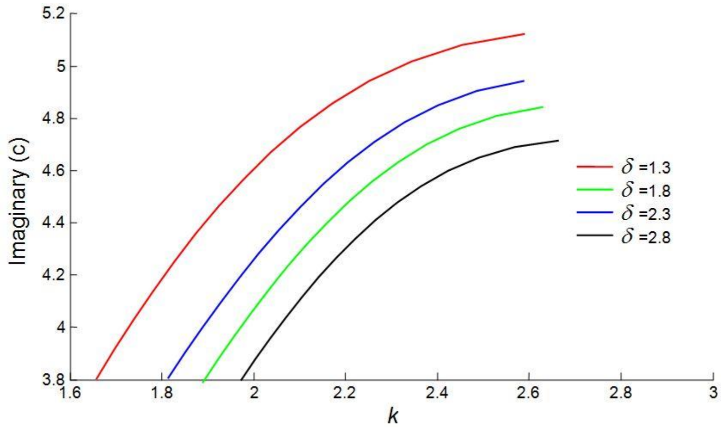

Fig. 11 Variation of phase velocity Imaginary(c) against wave number k for different δ at constant h = 5 km, ε = 0.0015 when layer is orthotropic in nature

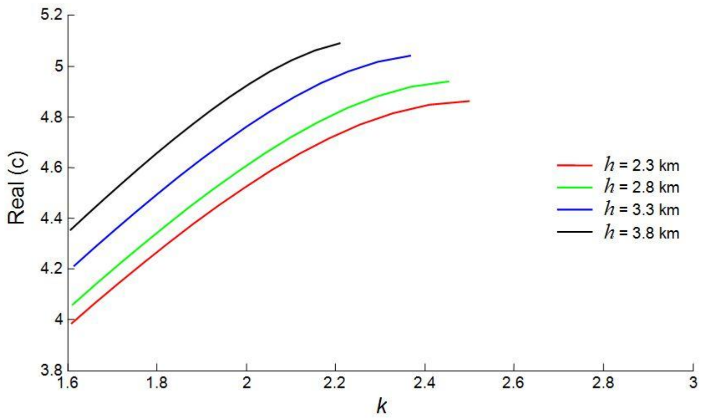

Fig. 12 Variation of phase velocity Real(c) against wave number k for different h at constant ε = 0.0015, δ = 1.5 when layer is isotropic in nature

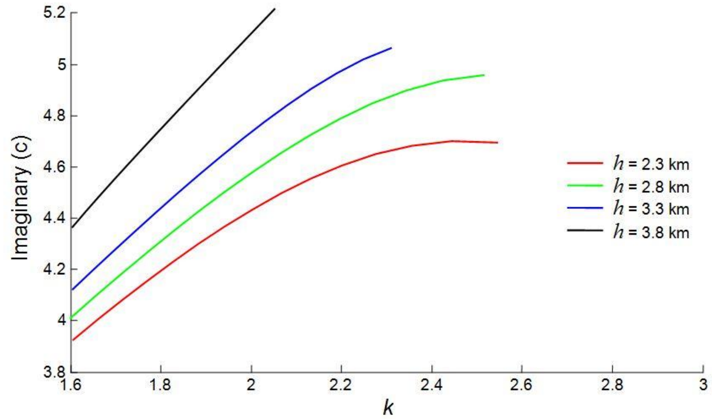

Fig. 13 Variation of phase velocity Imaginary(c) against wave number k for different h at constantε = 0.0015, δ = 1.5when layer is isotropic in nature

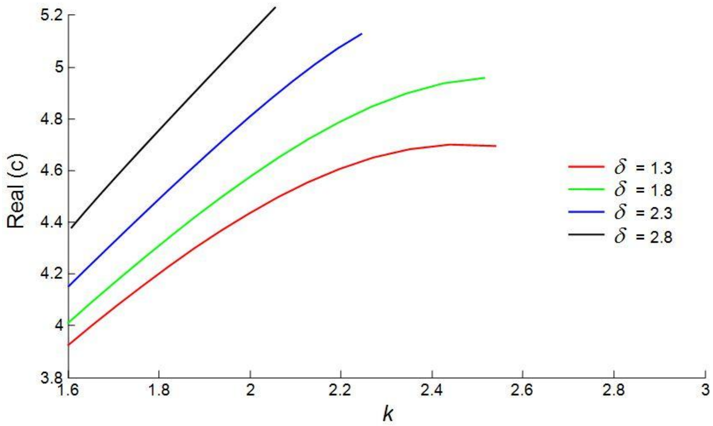

Fig. 14 Variation of phase velocity Real(c) against wave number k for different δ at constant h = 5 km, ε = 0.0015 when layer is isotropic in nature

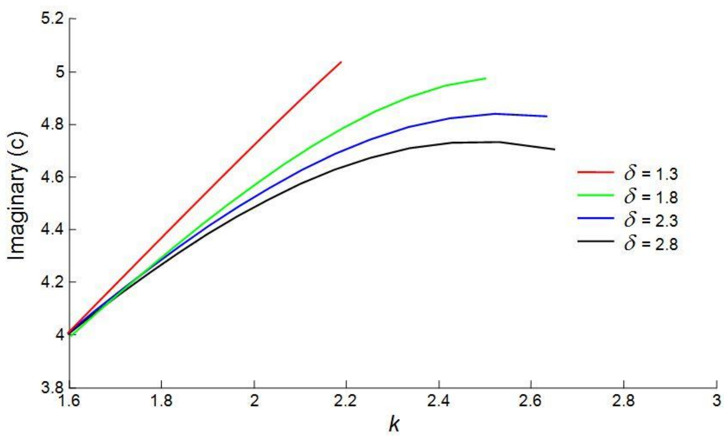

Fig. 15 Variation of phase velocity Imaginary(c) against wave number k for different δ at constant h = 5 km, ε = 0.0015 when layer is isotropic in nature

7. Conclusions

An analytical and numerical approach is used to study the propagation of Rayleigh wave in a heterogeneous anisotropy crustal layer over a gravitational sandy semi-infinite substratum under the assumption of free upper boundary plane. From above numerical analysis, it may be conclude that:

(a) In entire figures, the magnitude of both real and imaginary phase velocity of Rayleigh waves increases with wave number for increase in the thickness of layer except for orthotropic layer.
(b) Phase velocity of Rayleigh wave shows remarkable change with heterogeneity and sandiness parameters of the material.
(c) The real phase velocity of Rayleigh wave increases for increase of sandiness parameter but this trend is reversed for imaginary phase velocity.
(d) The material anisotropy affects the Rayleigh wave phase velocity remarkably.
(e) From the above discussion it can be concluded that heterogeneous elastic properties and heterogeneous density of materials affect the Rayleigh wave.
(f) Also, it is seen that the Rayleigh waves are affected by the direction of wave propagation.
(g) The heterogeneity parameter ε of the layer, sandiness parameter δ of semi-infinite substratum and anisotropy of the layer play a significant role in the Rayleigh wave propagation.
(h) It can be concluded from diagrams that the thickness of the layer have remarkable effect on the phase velocity.
(i) The gravity parameter has no major role on phase velocity due to its low involvement in the final dispersion relation.

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Appendix

$$\alpha_0 = a_{33}a_{35} - a_{35}{}^2, \alpha_1 = -2\left\{ i\left(a_{15}a_{33} - a_{13}a_{35}\right) + \frac{\varepsilon\left(a_{33}a_{55} - a_{35}{}^2\right)}{k} \right\}$$

$$\alpha_2 = \left(a_{33} + a_{55}\right)\rho_0 c^2 - 4a_{15}a_{35} - a_{11}a_{33} + a_{13}{}^2 + 2a_{13}a_{15} + 2a_{15}a_{35} + \frac{3i\varepsilon\left(a_{15}a_{33} - a_{13}a_{35}\right)}{k} + \frac{\varepsilon^2\left(a_{33}a_{55} - a_{35}{}^2\right)}{k^2}$$

$$\alpha_3 = -\left\{ \begin{array}{l} \left(2ia_{15} + 2ia_{35} + \frac{\varepsilon a_{35}}{k} + \frac{\varepsilon a_{33}}{k}\right)\rho_0 c^2 - 2ia_{11}a_{35} + 2ia_{13}a_{15} + \frac{2\varepsilon a_{13}a_{55}}{k} - \frac{2\varepsilon a_{15}a_{35}}{k} \\ \qquad - \frac{\varepsilon a_{11}a_{33}}{k} + \frac{\varepsilon a_{13}{}^2}{k} - \frac{i\varepsilon^2 a_{13}a_{35}}{k} - \frac{i\varepsilon^2 a_{35}a_{55}}{k} + \frac{i\varepsilon^2 a_{35}a_{55}}{k^2} + \frac{i\varepsilon^2 a_{33}a_{15}}{k^2} \end{array} \right\}$$

$$\alpha_4 = \rho_0{}^2 c^4 - \left(a_{55} + a_{11} + \frac{i\varepsilon a_{35}}{k}\right)\rho_0 c^2 + a_{11}a_{55} - a_{15}{}^2 - \frac{i\varepsilon a_{11}a_{35}}{k} + \frac{i\varepsilon a_{13}a_{15}}{k} - \frac{i\rho_0 g a_{15}}{k} + \frac{i\varepsilon a_{15}a_{55}}{\rho} + \frac{\varepsilon^2 a_{13}a_{55}}{k^2} - \frac{\varepsilon\rho_0 g a_{55}}{k^2}$$

$$\beta_0 = \delta^2 \frac{c_2^2}{c_1^2}, \quad \beta_1 = \delta^2\left(1 - \frac{c_2^2}{c_1^2}\right) + \delta\left(\frac{c^2}{c_1^2} - \delta\right) + \delta\left(\frac{c^2}{c_2^2} - \delta\right), \quad \beta_2 = \frac{c_2^2}{c_1^2}\left(\frac{c^2}{c_1^2} - \delta\right)\left(\frac{c^2}{c_2^2} - \delta\right) - \frac{\rho_2^2 g^2}{k^2}$$

$$c_1^2 = \frac{\lambda_2 + 2\mu_2}{\rho_2}, \quad c_2^2 = \frac{\mu_2}{\rho_2}$$

$$T_1 = ia_{15} - p_1\zeta_1 a_{35} + \left(ip_1 - \zeta_1\right)a_{55},\; T_2 = ia_{15} - p_2\zeta_2 a_{35} + \left(ip_2 - \zeta_2\right)a_{55}$$

$$T_3 = ia_{15} - p_3\zeta_3 a_{35} + \left(ip_3 - \zeta_3\right)a_{55}, T_4 = ia_{15} - p_4\zeta_4 a_{35} + \left(ip_4 - \zeta_4\right)a_{55}$$

$$T_5 = ia_{13} - p_1\zeta_1 a_{33} + \left(ip_1 - \zeta_1\right)a_{35},\; T_6 = ia_{13} - p_2\zeta_2 a_{33} + \left(ip_2 - \zeta_2\right)a_{35}$$

$$T_7 = ia_{13} - p_3\zeta_3 a_{33} + \left(ip_3 - \zeta_3\right)a_{35}, T_8 = ia_{13} - p_4\zeta_4 a_{33} + \left(ip_4 - \zeta_4\right)a_{35}$$