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Dynamic stiffness based computation of response for framed machine foundations

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Abstract. The paper deals with the applications of spectral finite element method to the dynamic analysis of framed foundations supporting high speed machines. Comparative performance of approximate dynamic stiffness methods formulated using static stiffness and lumped or consistent or average mass matrices with the exact spectral finite element for a three dimensional Euler-Bernoulli beam element is presented. The convergence of response computed using mode superposition method with the appropriate dynamic stiffness method as the number of modes increase is illustrated. Frequency proportional discretisation level required for mode superposition and approximate dynamic stiffness methods is outlined. It is reiterated that the results of exact dynamic stiffness method are invariant with reference to the discretisation level. The Eigen-frequencies of the system are evaluated using William-Wittrick algorithm and Sturm number generation in the LDL^T decomposition of the real part of the dynamic stiffness matrix, as they cannot be explicitly evaluated. Major's method for dynamic analysis of machine supporting structures is modified and the plane frames are replaced with springs of exact dynamic stiffness and dynamically flexible longitudinal frames. Results of the analysis are compared with exact values. The possible simplifications that could be introduced for a typical machine induced excitation on a framed structure are illustrated and the developed program is modified to account for dynamic constraint equations with a master slave degree of freedom (DOF) option.

Keywords: machine foundations; dynamic stiffness method; spectral finite element; wittrick-william's algorithm; sturm number; major's method; dynamic constraint problem.

1. Introduction

The handbook on machine foundations (Srinivasulu and Vaidyanathan 1985) is a popular design guide for machine foundation in the Indian sub-continent. Dynamic stiffness approach to response of machine foundations is however not covered by them. Initial works on dynamic stiffness matrix are due to Kolusek (1973), Richard and Leung (1977) and Howson *et al.* (1983). The exact dynamic stiffness matrix for Timoshenko beam element, linearly tapering Euler-Bernoulli element and two dimensional beam-column joints are given by Gopalakrishnan *et al.* (1992) and Gopalakrishnan and Doyle (1994). A dynamic stiffness for the analysis of non-uniform Timoshenko beams in the

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presence of axial loads for various boundary conditions is proposed by Leung and Zhou (1995). The symbolic computing package REDUCE has been used to generate an analytical expression for each of the dynamic stiffness terms of an axially loaded uniform Timoshenko beam element (Banerjee and Williams 1994). A new solution approach termed as the spectral transfer matrix method (STMM) is introduced and a beam with periodic supports and a plane lattice structure with several beam-like periodic lattice substructures are analyzed (Lee 2000). Experimentally computed frequency response function and dynamic stiffness is used to compute the damages in an RC structure (Maeck *et al.* 2000, Lee and Shin 2002).

Recently, Zhu and Leung (2009) have presented a formulation based on hierarchical finite element to solve a geometrically non-linear free and forced vibration problem for a non-uniform beam on elastic foundation. Arc-length based iteration method is used to solve the geometrically non-linear problem. Another interesting recent paper is due to Leung (2008), wherein the author derives for the first time, an exact dynamic stiffness formulation for solving axial-torsional buckling of framed members.

2. Mode superposition and dynamic stiffness methods

Dynamic response of an elastic structure is a linear combination of its eigenmodes. The advantage of the modal superposition method is that response of a multi-degree of freedom structure in the Cartesian domain is synthesized from a few fundamental uncoupled degrees of freedom in the modal domain. Mode superposition method, comprising of either modal displacement method or modal acceleration method is time consuming only up to extraction of modes.

It is worthwhile to examine the expression for the steady state dynamic displacement profile of a structure, when subjected to the operating frequency (ω),

$$\{d\} = [\phi]_{nxm}\{z\}_{mx1} \quad \text{or} \quad \{d\} = \sum_{i=1}^{i=m} (cf)_i \cdot \{\phi\}_i \tag{1}$$

$$cf_i = \frac{\{\phi\}_i^T\{f\}}{m_i(\omega_{ni}^2 - \omega^2 + i2\xi\omega\omega_{ni})} \tag{2}$$

and

$$\{\phi\}_i^T[K]_s\{\phi\}_i = \omega_i^2$$
$$\{\phi\}_i^T[M]\{\phi\}_i = 1 \tag{3}$$

Two major drawbacks of the mode superposition method are the uncertainty associated with the computation of Eigen pairs using inadequate discretisation and the inability to judge *a priori*, the number of modes that may participate in the response. A guideline for the band of modes participating in response of a framed structure subjected to high frequency of excitation is given by Lakshmanan *et al.* (2004). Recently, Lakshmanan and Gopalakrishnan (2007) have proposed a new design approach for framed foundations with un-certainties in machine speed and material properties. There are situations where only the steady state component of response corresponding to each frequency of excitation is required and this is accomplished by solving the dynamic equilibrium equation directly. The property of a linear structure is such that, the structure also responds in the harmonic excitation frequency. Using this property, the equation involving spatial and temporal variables are simplified to a system having only spatial variables, with solution to be sought at each frequency. Evaluation of linear dynamic response under steady state harmonic loading conditions and random dynamic loadings and transient loads using dynamic stiffness approach are viable alternative to mode superposition method. The dynamic stiffness method is independent of the number of modes participating in the response is not a matter of concern for as the response predicted by the mode superposition method converges to the response predicted by dynamic stiffness method (considering that both the methods use the same mass formulation and same equivalent damping) as the number of modes increase. However, the uncertainty due to discretisation remains and can only be solved by the exact dynamic stiffness formulation. In dynamic stiffness method, the dynamic equilibrium equation, for a multi-degree of freedom system, under steady state conditions is solved as a static problem, replacing static stiffness with the dynamic stiffness matrix. Frequency domain solution of the dynamic equilibrium problem using dynamic stiffness formulation has many advantages, which include circumventing eigenvalue analysis, ease in incorporation of frequency dependent soil spring stiffness, incorporation of visco-elastic material properties, implementation of frequency dependent fluid film stiffness in the case of a rotor and so on. It may be noted that exact dynamic stiffness can be derived for only a few structural elements like Euler-Bernoulli and Timoshenko beams, thin walled beams and rectangular plates. The shape functions for the exact dynamic stiffness formulation are derived by solving the appropriate fourth order differential equation, reduced to spatial co-ordinates after eliminating the temporal variation with Fourier frequency components. The exact dynamic stiffness formulation is also termed as spectral finite element or a continuum element. For other types of finite elements, where an exact dynamic stiffness is not possible, the approximate dynamic stiffness could be formulated using static stiffness and lumped or consistent or average mass. The performance of approximate dynamic stiffness methods are adequate under low frequencies of excitation but shall result in unacceptable levels of errors under high frequencies. The applications of high frequency of excitation include machine induced excitation where the super harmonic components of the basic operating frequency can be 100 Hz-150 Hz. The other applications of high excitation frequencies are transient dynamic loads such as impact and blast loads and sonic excitation of structures. High frequency response of a structure is dependent on the fineness of the discretisation and hence the result of an approximate dynamic stiffness formulation is not equally valid at higher frequency ranges.

3. Approximate dynamic stiffness formulations

3.1 Lumped mass model

$$\begin{aligned}\{F\} &= [-\omega^2[M]_{L1} - \omega^2[M]_{L2} + i\omega[C] + (1+2i\zeta)[K]_S]\{d\} \\ \{F\} &= [K]_{DL}\{d\}\end{aligned} \tag{4}$$

Lumped mass matrix is due to the lumping of mass from structural members ($L1$) and also due to non-structural members such as machinery ($L2$). The latter component, $L2$, is always present in all the formulations, irrespective of whether a lumped or a consistent or an average mass formulation is adopted. Imaginary part of the dynamic stiffness comprises of contribution from physical dampers present in the system and due to the structural damping factor. The term due to the structural damping should always be present to prevent instability at resonances for structures, where physical dampers are not used.

3.2 Consistent mass model:

$$\{F\} = [-\omega^2[M]_C - \omega^2[M]_{L2} + i\omega[C] + (1+2i\zeta)[K]_S]\{d\}$$
$$\{F\} = [K]_{DC}\{d\} \tag{5}$$

3.3 Average mass model

$$\{F\} = [-0.5\omega^2([M]_C + [M]_{L1}) - \omega^2[M]_{L2} + i\omega[C] + (1+2i\zeta)[K]_S]\{d\}$$
$$\{F\} = [K]_{DA}\{d\} \tag{6}$$

This model uses an average of the consistent and lumped mass matrices. Lumped mass matrix formulation generally under-estimates the frequencies, whereas a consistent mass formulation over-estimates the frequencies. Hence in average mass matrix, it is assumed that the under-estimation of frequencies by the lumped mass model is neutralized by the over-estimation using consistent mass, to a large extent.

4. Exact dynamic stiffness formulation

Flexural vibration of a thin beam follows the governing equation as,

$$EI\frac{\partial^4 y}{\partial x^4} + \rho A\frac{\partial^2 y}{\partial t^2} = p(x,t) = \Sigma p_\omega(x,\omega)$$
$$y(x,t) = \Sigma Y(x)A(\omega)e^{i.\omega.t} \tag{7}$$
$$\lambda^4 = \frac{\rho A\omega^2}{EI}$$

The equation of motion has the solution of the form,

$$Y(x) = A\sin\lambda x + B\cos\lambda x + C\sinh\lambda x + D\cosh\lambda x \tag{8}$$

$$s = \sin\lambda L \ldots\ldots c = \cos\lambda L \tag{9}$$
$$S = \sinh\lambda L \ldots\ldots C = \cosh\lambda L$$

Shape Functions can be derived as (Leung 1993)

$$[N] = \begin{Bmatrix} \cos\lambda x \\ \sin\lambda x \\ \cosh\lambda x \\ \sinh\lambda x \end{Bmatrix}^T \begin{bmatrix} \frac{1}{2} - \frac{F_4}{2\lambda^2 l^2} & \frac{F_2}{2\lambda^2 l} & -\frac{F_3}{2\lambda^2 l^2} & \frac{F_1}{2\lambda^2 l} \\ -\frac{F_6}{2\lambda^3 l^3} & \frac{1}{2\lambda} + \frac{F_4}{2\lambda^3 l^2} & -\frac{F_5}{2\lambda^3 l^3} & -\frac{F_3}{2\lambda^3 l^2} \\ \frac{1}{2} + \frac{F_4}{2\lambda^2 l^2} & -\frac{F_2}{2\lambda^2 l} & \frac{F_3}{2\lambda^2 l^2} & -\frac{F_1}{2\lambda^2 l} \\ \frac{F_6}{2\lambda^3 l^3} & \frac{1}{2\lambda} - \frac{F_4}{2\lambda^3 l^2} & \frac{F_5}{2\lambda^3 l^3} & \frac{F_3}{2\lambda^3 l^2} \end{bmatrix} \tag{10}$$

$$F_1 = \frac{\lambda l(S-s)}{\delta}; \quad F_2 = \frac{\lambda l(Cs-cS)}{\delta}; \quad F_3 = \frac{\lambda^2 l^2(C-c)}{\delta}$$

$$F_4 = -\frac{\lambda^2 l^2 Ss}{\delta}; \quad F_5 = -\frac{\lambda^3 l^3(S+s)}{\delta}; \quad F_3 = -\frac{\lambda^3 l^3(Cs+cS)}{\delta} \tag{11}$$

$$\delta = 1 - Cc$$

and the stiffness matrix is,

$$\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix} = B \begin{bmatrix} -\lambda^2(cS+sC) & & Symmetric & \\ \lambda sS & sC-cS & & \\ -\lambda^2(s+S) & \lambda(c-C) & \lambda^2(cS+sC) & -\lambda^3 S \\ \lambda(C-c) & S-s & -\lambda sS & sC-cS \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \theta_1 \\ \delta_2 \\ \theta_2 \end{Bmatrix} \tag{12}$$

$$\text{where} \quad B = \frac{\lambda EI}{1-cC} \tag{13}$$

Similar expressions for axial force - deformation and torque-twist relationships can be derived. In the global domain, equilibrium equation is stated as,

$$\begin{aligned} \{F\} &= [-\omega^2[M]_{L2} + i\omega[C] + 2i\zeta[K]_S + [K]_{DD}]\{d\} \\ \{F\} &= [K]_{DE}\{d\} \end{aligned} \tag{14}$$

5. Description of a typical high speed machine foundation

A structure supporting a high speed machine like turbo-generator has a series of vertically symmetric transverse frames interconnected by two stiff longitudinal beams. The transverse frames support typical turbine units such as, high pressure (HPT), intermediate (IPT) and low pressure turbines (LPT), turbo-generator and an excitation unit for the field coil. Each of these machines span between a pair of plane frames and the contact point for the rotating element is a bearing of a typical fluid film type. The bearings are located on the top mid span of the transverse frames. A rotor cannot be balanced to zero eccentricity and depending on the balancing grade of the rotor a residual unbalance is left over. Typical excitation forces are due to the imbalance, which cause a rotating vector with a magnitude of $me\omega^2$. This results in sinusoidal forces in both the vertical and transverse horizontal directions each having a phase difference of $\pi/2$ radians. The rotor can be misaligned which may result in dynamic longitudinal force and clamping moment on the bearing. This machine runs at a speed of 50 Hz and due to imperfections and friction in the bearings, the sinusoidal disturbing force is distorted. The distorted force is periodic but generates super harmonic components like 2N, 3N etc, where 'N' is the primary RPM of the machine. This gives rise to high frequency components which can be of the order of about 150 Hz. The other cause of a high frequency excitation is the short-circuiting force of the exciter, whose primary frequency is 100 Hz.

6. TGDYN program

The "Turbo-Generator Dynamic Analysis" (TGDYN) is a program developed by the authors for the linear dynamic analysis of machine foundation. The typical modules are for computation of steady state response due to normal and abnormal eccentricity of rotor. Amplitude and strength checks are carried out for the normal and abnormal imbalances respectively. The steady state response computation can be processed either following mode superposition route or dynamic stiffness route. Mode superposition has lumped and consistent mass options for generating the Eigen pairs, using sub-space iteration. The dynamic stiffness has approximate formulations involving lumped, consistent and average mass. The exact dynamic stiffness option is also implemented for an Euler-Bernoulli beam element and is being extended to Timoshenko beam element. Soil-structure, single pile-structure, group pile - structure interaction effects can be handled through any of the frequency domain approaches. Frequency dependent stiffness of the pile tip is generated through another module based on Lakshmanan and Minai's Model (1981). The frequency information for the dynamic stiffness approach, in the absence of an eigen value analysis is obtained through the Sturm number generation employing Wittrick-William algorithm (Gopalakrishnan *et al.* 2001). There are other modules which cater to dynamic load cases using seismic response spectrum analysis and time history analysis approaches for impulsive short-circuiting forces. Critical static load cases include thermal loads, which may involve uniform elongation of members or bowing of members subjected to heat on one face only. The program, though developed for machine foundations, is general and can be used for other structures as well. The element library consists of three dimensional beams, truss, generalized six-DOF springs and truss type dampers. The storage of stiffness and mass matrices are in the form of a vector sky-line form. The 'COLSOL' routine is suitably converted and modified to account for the complex number arithmetic. The program validation is done through NISA and ANSYS 5.6 software.

7. Convergence of the mode superposition solution to appropriate dynamic stiffness method

As the number of participating modes increase, steady state response due to mode superposition, using a particular mass formulation, asymptotically reaches the response predicted using the dynamic stiffness method of the same mass formulation. Towards proving this statement, a one bay and one storied frame (Fig. 1) is analyzed for steady state dynamic displacements both under vertical as well as horizontal harmonic loads. The frequencies of excitation considered are 25, 50, 75 and 100 Hz. For each frequency, mode superposition and dynamic stiffness analysis is carried out. Eigen frequencies and vectors are extracted with lumped mass formulation using sub-space iteration technique. Response is also predicted using the dynamic stiffness method using a combination of lumped mass and static stiffness. Mode superposition uses a percentage of critical damping as viscous damping (ξ), whereas the dynamic stiffness method uses structural damping factor (2ζ). The structural damping factor is similar to the loss factor of a visco-elastic material and the product of the structural damping and the stiffness is assumed to be the imaginary portion of the complex dynamic stiffness matrix. The equivalence of damping in both the cases is arrived at by equating the maximum values of steady state displacements at resonance.

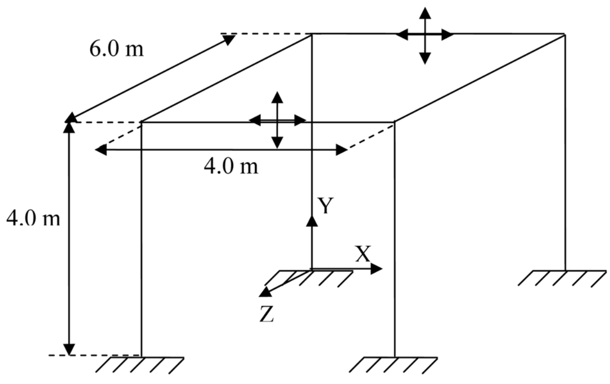

Fig. 1 Typical three dimensional machine foundation used in modal convergence study and for comparison of various dynamic stiffness formulations

$$d_{max} = \frac{F}{2\xi.K_S} \tag{15}$$

$$F = (-\omega^2 M + (1 + 2i\zeta)K_S)d, \text{ at resonance } K_S = \omega^2 M \tag{16}$$

$$d_{max} = \frac{F}{2\zeta.K_S}, \text{ therefore } \zeta = \xi \tag{17}$$

In the mode superposition method, number of modes participating in the response is cumulatively increased and displacement values are plotted. The results of dynamic stiffness method are also shown on the same graph. Figs. 2 and 3 (with lumped mass and static stiffness formulation) show the variation of steady-state dynamic displacements predicted by both the methods in the horizontal and vertical directions respectively. The curve is invariant of the modes in the dynamic stiffness method whereas the response due to mode superposition cumulatively increases and converges to a constant value as given by the dynamic stiffness method. Those modes, which do not have a participation in the total response, do not give rise to a change in displacement and the response

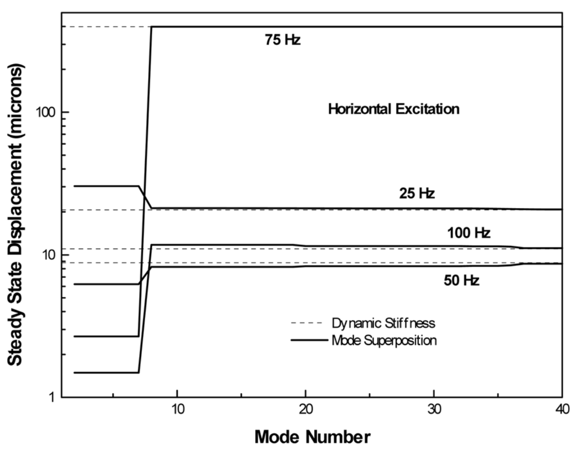

Fig. 2 Convergence of mode superposition results to lumped mass based approximate dynamic stiffness method (transverse excitation)

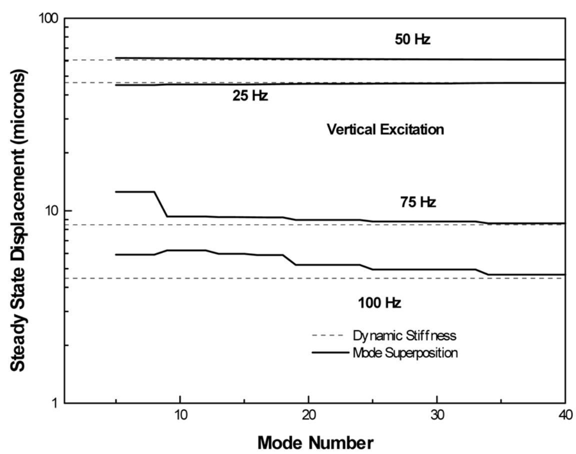

Fig. 3 Convergence of mode superposition results to lumped mass based approximate dynamic stiffness method (vertical excitation)

gets major contributions only from a few selective modes and the curve shows major changes only in these modes. It is also seen that number of modes, which participate in the response, are relatively less for lower frequency of excitation and beyond certain modes, the contribution due to additional modes are negligible. Convergence of response under higher frequencies of excitation is relatively slower and requires more number of modes to participate in the response. In the proximity of a global resonance, the response is dictated by the nearby mode and is least influenced by other modes.

8. Performance of various dynamic stiffness formulations

Results of three approximate dynamic stiffness methods, formulated using lumped mass, consistent

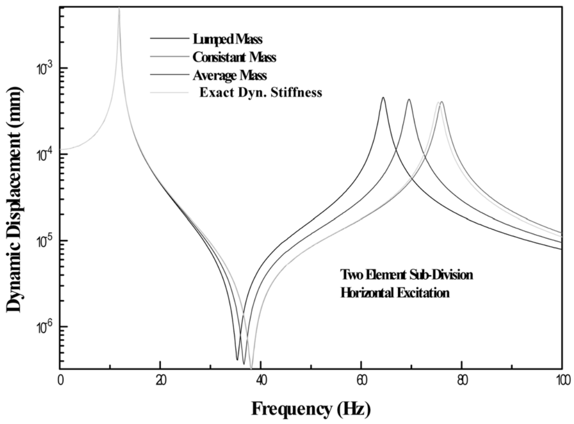

Fig. 4 Horizontal displacement versus excitation frequency for various dynamic stiffness formulations (two sub-elements, horizontal excitation)

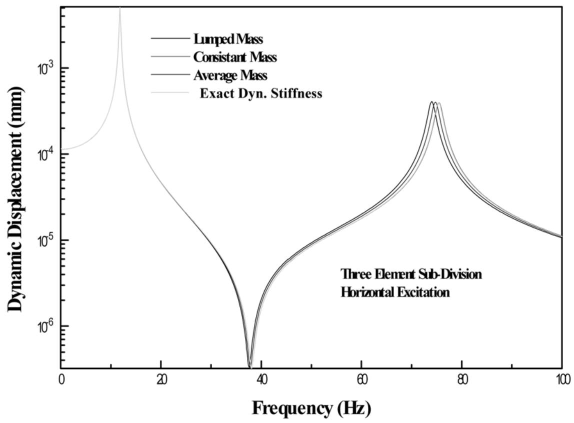

Fig. 5 Horizontal displacement versus excitation frequency for various dynamic stiffness formulations (three sub-elements, horizontal excitation)

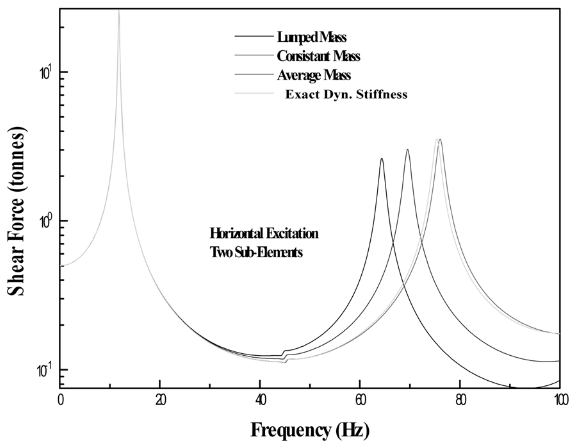

Fig. 6 Column-shear versus excitation frequency for various dynamic stiffness formulations (two sub-elements, horizontal excitation)

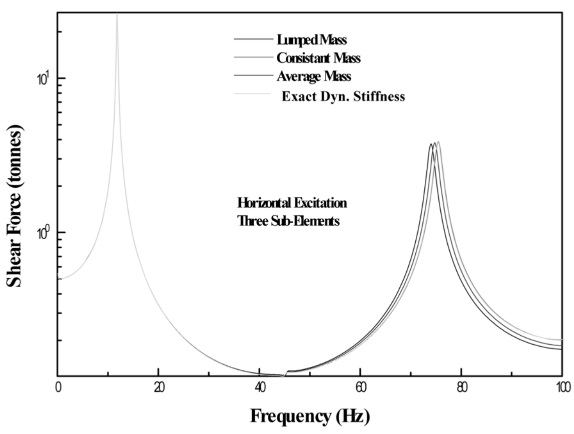

Fig. 7 Column-shear versus excitation frequency for various dynamic stiffness formulations (three sub-elements, horizontal excitation)

mass and average mass are presented. Comparison of performance of the various models is carried out through steady state dynamic analysis of a one bay and one storied frame (Fig. 1) subjected to both vertical and transverse-horizontal harmonic loads. The dynamic forces are applied at bearing locations, at the centre of transverse beam. The forces are assumed to be in-phase. The sub-element discretisation is progressively increased and the displacement and stress resultants are studied. Figs. 4 and 5 show the variation of driving point displacement under transverse excitation for two and three sub-element discretisation. Same first lateral frequency and response is predicted by all methods. Second resonant frequency and the corresponding displacement, predicted by consistent mass is closer to the exact dynamic stiffness results whereas lumped mass results are the farthest. Results of average mass formulation are between the results predicted by lumped and consistent mass. As discretisation level is increased, each of the approximate formulation converges to a constant value. Results of exact dynamic stiffness method are invariant of the discretisation level.

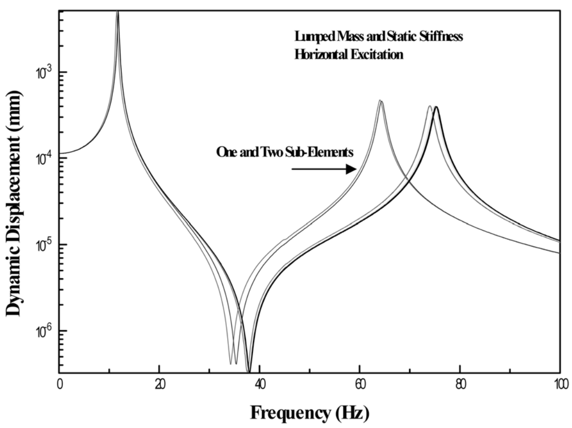

Fig. 8 Convergence of horizontal displacement for various discretisations in lumped mass formulation (horizontal excitation)

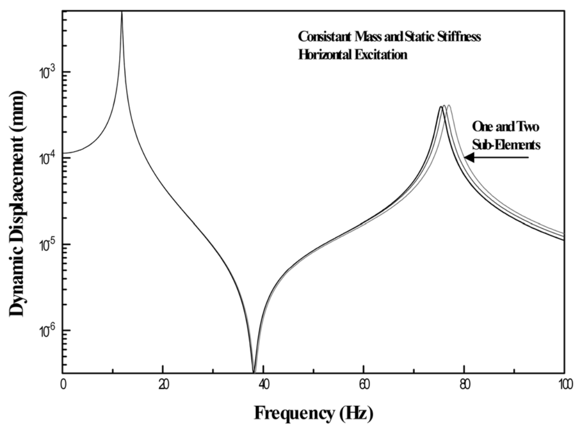

Fig. 9 Convergence of horizontal displacement for various discretisations in consistent mass formulation (horizontal excitation)

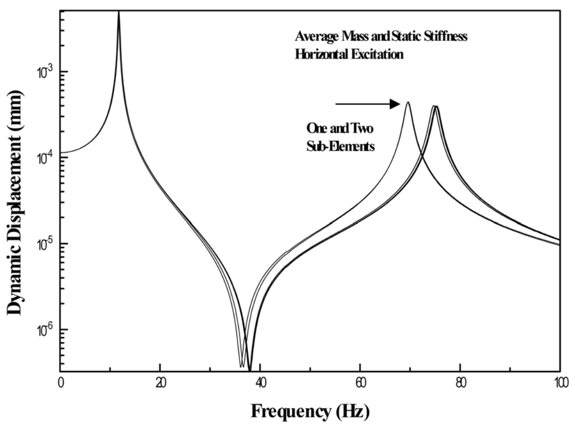

Fig. 10 Convergence of horizontal displacement for various discretisations in average mass formulation (horizontal excitation)

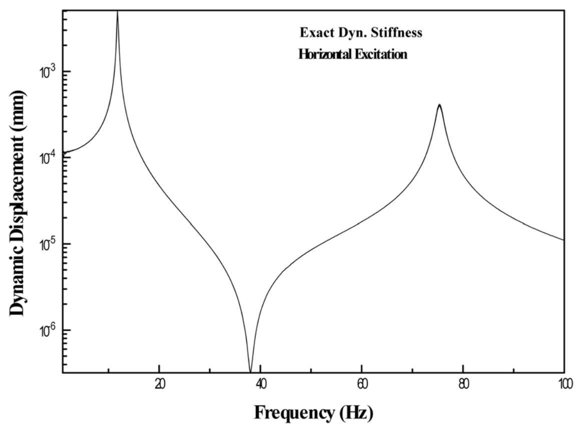

Fig. 11 Horizontal displacement for various discretisations in exact dynamic stiffness formulation (horizontal excitation)

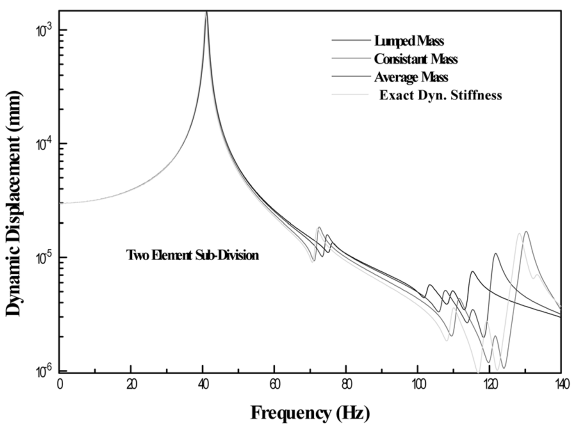

Fig. 12 Vertical displacement versus excitation frequency for various dynamic stiffness formulations (two sub-elements, vertical excitation)

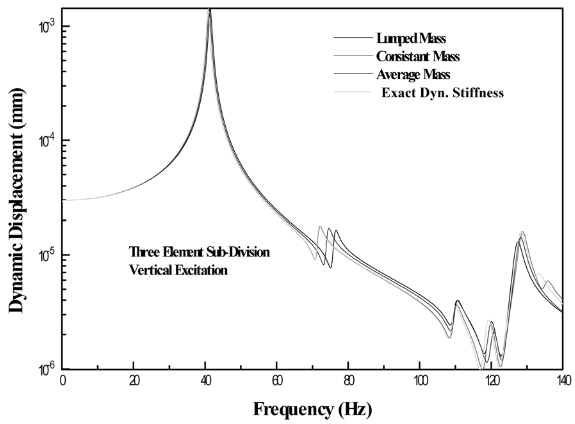

Fig. 13 Vertical displacement versus excitation frequency for various dynamic stiffness formulations (three sub-elements, vertical excitation)

Figs. 6 and 7 show the variation of column shear under transverse excitation for two and three sub-element discretisation. Shear force is a third derivative of displacement and will need more elements to converge. Figs. 8 to 11 show the performance of each of the mass formulation as the fineness increases. In this study, it is seen that lumped mass method requires at least four sub-elements to converge to actual results whereas consistent mass needs only three sub-elements to converge. Figs. 12 to 13 show the variation of driving point displacement under vertical excitation for two and three sub-element discretisation. Same first vertical frequency and response is predicted by all methods. Difference between the approximate and exact dynamic stiffness method is more at weakly excited modes and less at strongly excited modes. Strongly excited modes are those which have a large amplitude at resonance and weakly excited modes have comparatively less amplitude at resonance. Second pre-dominant resonant frequency, predicted by consistent mass is closer to the exact

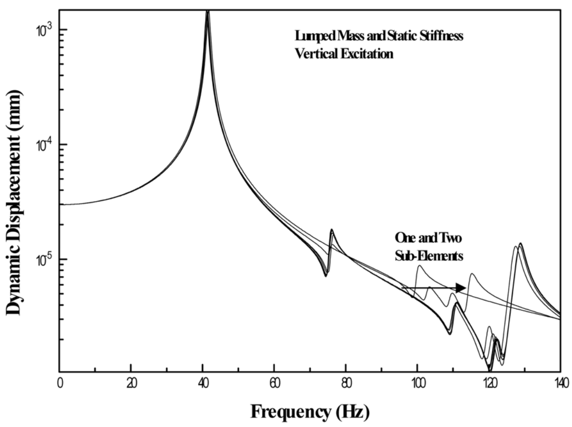

Fig. 14 Convergence of vertical displacement for various discretisations in lumped mass formulation (vertical excitation)

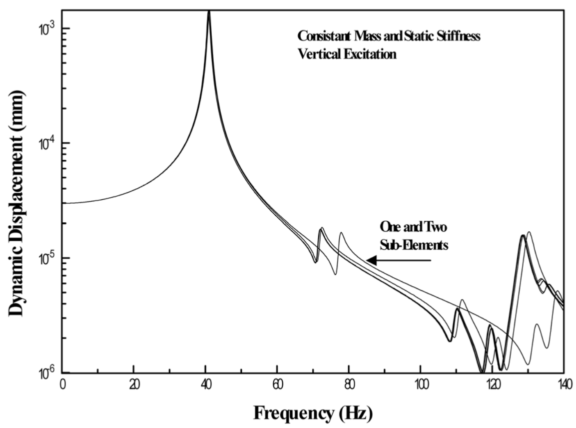

Fig. 15 Convergence of vertical displacement for various discretisations in consistent mass formulation (vertical excitation)

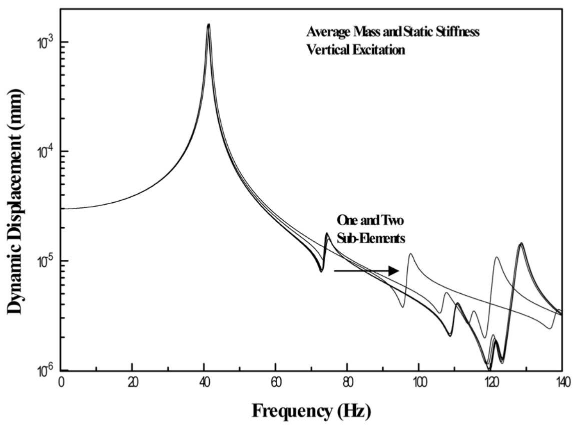

Fig. 16 Convergence of vertical displacement for various discretisations in average mass formulation (vertical excitation)

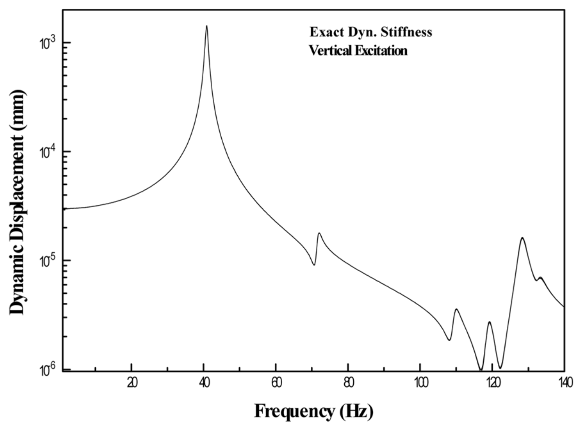

Fig. 17 Vertical displacement for various discretisations in exact dynamic stiffness formulation (vertical excitation)

dynamic stiffness results, whereas results obtained by lumped mass approach are the farthest. Results of exact dynamic stiffness method are invariant of the discretisation level. Figs. 14 to 17 show the performance of each of the mass formulation, under vertical excitation, as the fineness increases. In this study, at least five sub-elements are required to converge to actual results in the case of lumped mass approach, whereas consistent mass approach requires four sub-elements to converge. Out of all the approximate methods, response predicted by consistent mass formulation is closer to the exact one. Though, generally, lumped mass method under-estimates and consistent mass over-estimates the natural frequencies, results of consistent mass formulation are closer to the exact values and the results of average mass formulation do not improve the results substantially.

9. Sturm's theorem and wittrick-william algorithm

Generally, for a small dynamic problem involving only three or four unknowns, $\det[K-\omega^2 M]$ is expanded into a polynomial form and the roots are extracted. The matrices [**K**] and [**M**] are invariant with the frequency of excitation. The polynomial coefficients are of alternate signs and differ in magnitude of an order or two between successive terms. Hence the equation is susceptible to large errors due to ill-conditioning. To improve its efficiency and reliability, the determinant search method of eigen value solution can be modified, to use the Sturm sequence property of the eigen value solution. The congruence transformation of a symmetric matrix involves writing a typical matrix in the form, (Leung 1993):

$$[A] = [S][B][S]^T \tag{18}$$

where $[S]$ is a non-singular matrix. The congruent transformation is different from a similarity transformation like $[A] = [P]^{-1}[B][P]$. In a similarity transformation, similar matrices have the same determinant and eigen pairs. However, the Sylvester's law of inertia states that for a pair of congruent matrices, the total number of positive, negative and zero eigen values are same.

The generalized equation of an eigenvalue problem can be stated as,

$$[K]_s\{x\} = \omega^2[M]\{x\}$$
$$([K]_s - \omega^2[M])\{x\} = \{0\} \tag{19}$$

Assuming [Φ] as a matrix, whose columns are represented by ortho-normalised eigenmodes of the above problem and by pre and post multiplying by [Φ], the following expression is obtained,

$$[\Phi]^T([K]_s - \omega^2[M])[\Phi] = [\Phi]^T[K]_s[\Phi] - \omega^2[\Phi]^T[M][\Phi] = [\Lambda] - \omega^2[I] \tag{20}$$

As [K] and [M] are symmetric and positive definite matrices, they can be decomposed in the form,

$$([K]_s - \omega^2[M]) = [L_\omega][D_\omega][L_\omega]^T$$
$$[D_\omega] = [L_\omega]^{-1}([K]_s - \omega^2[M])[L_\omega]^{-T} \tag{21}$$

As stated earlier, the matrices in the previous two expressions are thus obtained using congruence transformation of the $[K_s - \omega^2 M]$ matrix and therefore should have the same number of positive, negative and zero eigen values. The eigen values of the diagonal matrix $[D_\omega]$ are its diagonal entries (D_{ii}). Also for the matrix, $[\Lambda] - \omega^2[I]$, the eigen values are $\omega_i^2 - \omega^2$, where ω is the frequency up to which the Eigen values are sought. This proves the Sturm's theorem which states that the number of eigen values less than a particular frequency value of ω is the simple count of negative diagonal entries in the LDL^T decomposition of the $[K_s - \omega^2 M]$ matrix.

The Sturm's theorem for discrete finite element system is extended to continuum models of dynamic stiffness method using William - Wittrick algorithm. This states that the number of frequencies less than a particular trial frequency ω is stated as

$$\Sigma J_m + s\{D(\omega)\}$$
$$s\{D(\omega)\} = s(\mathrm{Re}(K_{DE})) \tag{22}$$

where ΣJ_m is the number of fixed-fixed frequencies less than ω, summed up for all elements and $s\{D(\omega)\}$ is the Sturm's count of the frequency dependent dynamic stiffness matrix.

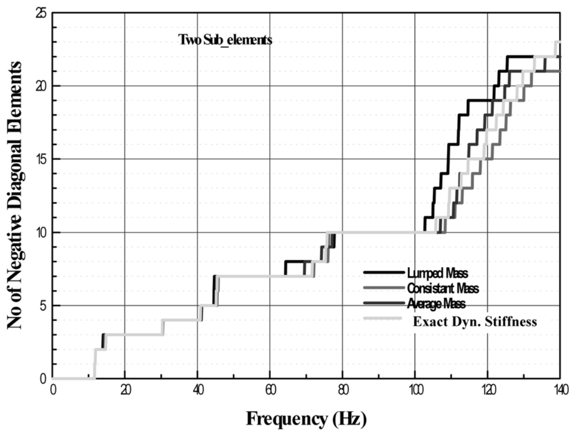

Fig. 18 Sturm number versus excitation frequency for various dynamic stiffness formulations (two sub-elements)

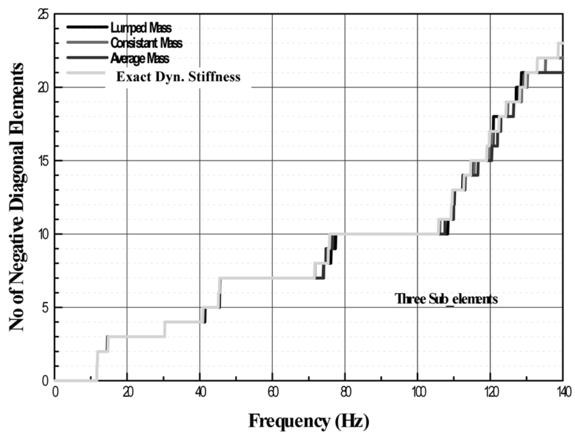

Fig. 19 Sturm number versus excitation frequency for various dynamic stiffness formulations (three sub-elements)

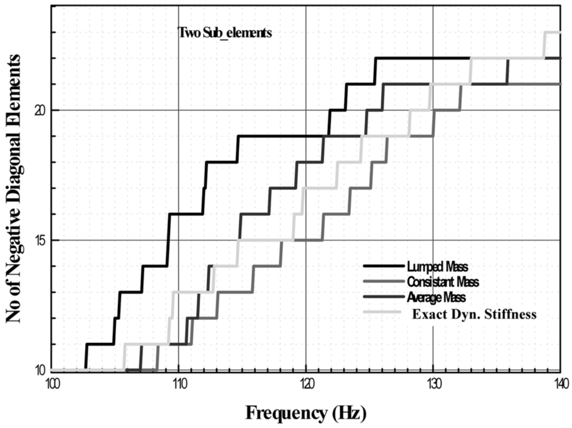

Fig. 20 Sturm number versus excitation frequency for various dynamic stiffness formulations (two sub-elements)

The program TGDYN has facility to extract the Sturm number at each frequency of excitation and hence eigen values are in directly obtained. The accuracy of eigen values depend on the frequency resolution with which Sturm sequence check is performed. Figs. 18 and 19 show the Sturm number for two and three sub-elements. The performance is similar to the transverse and vertical responses, but the plots reveal information on all frequencies and not necessarily on those pre-dominantly excited ones. Fig. 20 shows the zoomed up view for a two sub-element level, indicating over-estimation of eigen frequencies by consistent mass, under estimation by lumped mass. However, the deviation is more for the lumped mass while average mass under estimates the frequencies, in this study. Few of the commercial software successfully use average mass owing to the fact that static stiffness formulation for an element other than beams are generally stiff and the combination of static stiffness and average mass may balance the errors induced.

10. Minimum discretisation for mode superposition and approximate dynamic stiffness methods

A reasonable guideline should be available to estimate the minimum number of sub elements to which a main element has to be sub-divided in order to estimate the deformation and the stress resultants of the element to a sufficient accuracy. The simplest guideline for calculating the number of nodes or elements within a main structural element spanning between two beam-column joint is based on the anticipated deformed shape of the element. The upper bound mode number at which an element vibrates for the particular operating frequency can be estimated using the frequency expressions for the simply supported or cantilever beam elements. Assuming a structural element to be simply supported between the two beam column joints, if it had been found to be vibrating between the (n-1) and n^{th} modes for that operating frequency, then the minimum number of nodes to be provided across the element is suggested as, 2n-1. (Excluding the intersection nodes). One can similarly find the number of nodes, if the element is assumed to be a cantilever between the beam-column joints. This logic is based on the assumption that the maximum mode that influences the dynamic behavior of a structure at its element level could be the n^{th} mode.

The natural frequency of a simply supported beam is given by the expression,

$$\omega_n = n^2\pi^2\sqrt{\frac{EI}{ml^4}} \tag{23}$$

With 'ω_{op}' being known, the value of 'n' is derived as

$$n = \frac{l.\sqrt{\omega_{op}}}{\pi}\left(\frac{m}{EI}\right)^{1/4} \tag{24}$$

For non-resonant conditions, 'n' is a fraction. Similarly, for a cantilever structure, the expression for frequency can be approximated as,

For $n = 1$,

$$\omega = 1.875^2\sqrt{\frac{EI}{ml^4}} \tag{25}$$

and for $n > 1$,

$$\omega = (n-0.5)^2\pi^2\sqrt{\frac{EI}{ml^4}} \tag{26}$$

Hence the value of 'n' is derived as,

$$n = 0.5 + \frac{l.\sqrt{\omega_{op}}}{\pi}\left(\frac{m}{EI}\right)^{1/4} \tag{27}$$

For an axially loaded, simply supported member, 'n' is derived as

$$n = 0.5 + \frac{l.\omega_{op}}{\pi}\sqrt{\frac{\rho}{E}} \tag{28}$$

Based on the above expressions and depending on the cross sections and the mass density, the upper-bound frequency of excitation is computed for each of the member. The actual convergence under lateral excitation for excitation frequency less than 100 Hz is reached for 4 sub-elements when using lumped mass approach and 3 sub-elements using consistent mass approach. Similarly, under vertical excitation for excitation frequency less than 100 Hz, convergence is reached with 5 sub-elements when lumped mass approach is used and 4 sub-elements using consistent mass approach.

The table suggests that at least four elements are required for frequencies less than 80 Hz and 5

Table 1 Upper bound frequency (Hz) of excitation for each discretisation, predicted by different expressions

Expression Used	Structural Element								
	Beam1			Beam2			Column1		
	2 sub Elements	4 Sub Elements	6 Sub Elements	2 sub Elements	4 Sub Elements	6 Sub Elements	2 sub Elements	4 Sub Elements	6 Sub Elements
Simply supported Beam	78.5	314.0	706.5	34.9	139.6	314.1	58.9	235.6	530.1
Cantilever Beam	28.0	176.7	490.7	12.4	78.5	218.1	21.0	132.5	368.2
Axial Rod	216.5	650.0	1082.0	144.4	433.2	721.8	216.5	650.0	1082.0

elements may be required for frequencies less than 100 Hz. Any beam element spanning between two beam-column joint is neither simply supported nor a cantilever. The arbitrary rotational spring stiffness provided by the joint, however shall give rise to higher values of element-level natural frequencies only. However to take care of these uncertainties, and as a measure of caution, the values predicted by the expression could be enhanced by 25% for consistent mass and 50% for lumped mass.

11. Exact dynamic stiffness for a symmetrical plane frame

The first step in the implementation of the exact dynamic stiffness method for a three dimensional case is to develop simplified expressions for a plane frame. This is done for a symmetrical plane frame, typical of a transverse frame of a turbo-generator supporting structure with a loading at the mid-span of the cross beam. There are nine DOF, including the load application point. By a dynamically consistent load lumping procedure, the load at the centre of the beam could be transferred to its ends. This is similar to the transfer of element loads to nodal loads but the shape functions are the exact ones.

The nodal load is given as, for a distributed load, $P_1 = \int p(x)N_1(x).dx$ and for a concentrated load, $P_1 = P_0N_1(x_0)$. This reduces the total degrees of freedom to six. Due to known positions of loads and symmetry of the structure, further simplifications can be made.

For a transverse loading case, $\delta_{2x} = \delta_{1x}$; $\delta_{2y} = -\delta_{1y}$; $\theta_{2z} = \theta_{1z}$ (29)

For a vertical loading case, $\delta_{2x} = -\delta_{1x}$; $\delta_{2y} = \delta_{1y}$; $\theta_{2z} = -\theta_{1z}$ (30)

Further, vertical and transverse loading cases are applied successively such that when transverse load is applied there are no vertical loads. Using these conditions, for a case of a plane frame (Fig. 21), dynamic stiffnesses are developed for vertical and transverse load cases. Figs. 22 and 23 show the variation of steady state amplitudes when the transverse and vertical loads are applied on the top of the column, as given by the expression and also through TGDYN. The results show an interesting feature. For the vertically loaded frame, with the load application point on the top of the column, there are no lateral displacement and rotations for low frequencies. But these values become substantial at high frequencies, reinforcing the fact that the same simplifications and

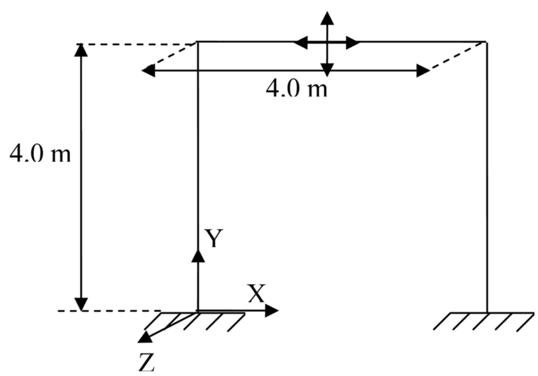

Fig. 21 Structure for validation of expressions derived using exact dynamic stiffness method for plane frames

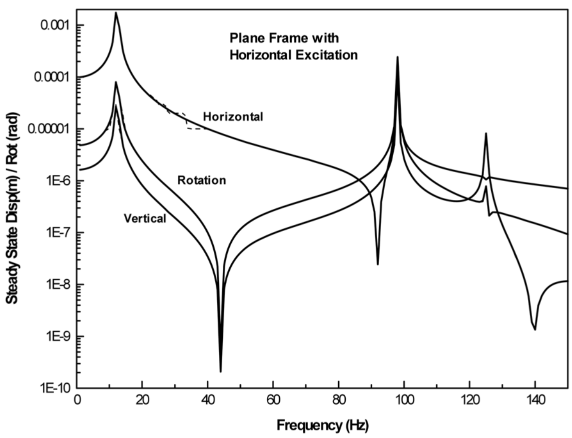

Fig. 22 Comparison of response obtained using developed expressions for a plane frame (transverse excitation)

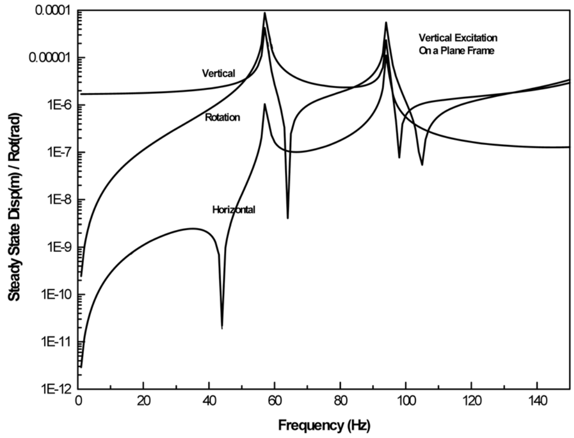

Fig. 23 Comparison of response obtained using developed expressions for a plane frame (vertical excitation)

approximations as adopted for static and low frequency cases cannot be carried forward at high frequencies. The other fact is that the total response at the bearings is a vectorial summation of three components, namely, (a) response of the top of the columns under dynamically equivalent nodal loads, (b) response at the centre of the beam, due to a uniform base excitation of amplitude as computed in the first stage and (c) the response of the mid-span point of a fixed-fixed beam with a dynamic load. This observation is also different from a static and low frequency case, wherein the response of the column top is also the response of the centre of the transverse beam.

12. Modification of Major's method with dynamic spring stiffness

The preliminary design of a machine foundation is done by the method suggested by Major (1980). Each of the transverse frames is converted to an equivalent spring having the same lateral stiffness as that of the frame (Fig. 24). Longitudinal beams are assumed to be rigid in the horizontal plane. This result in a coupled translation and rotation and both the translational and rotational spring stiffnesses are provided by the transverse frames. The resulting two DOF system is solved

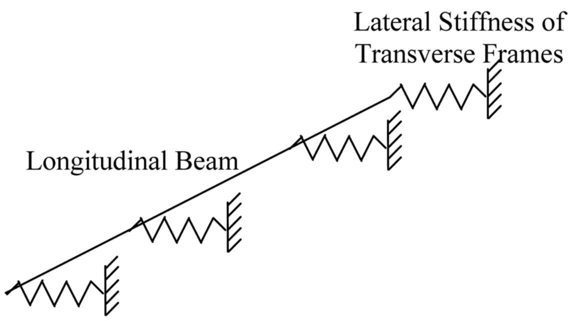

Fig. 24 Major's approximations in typical machine supporting structure

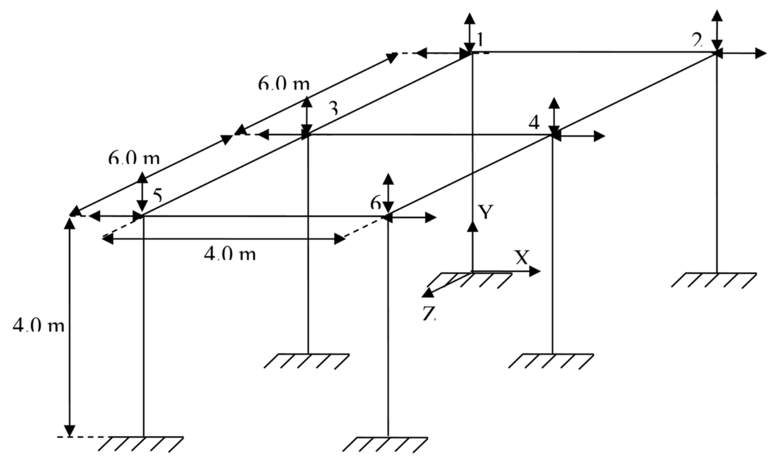

Fig. 25 Typical machine foundation for comparison of modified major's method and dynamic constraint problem

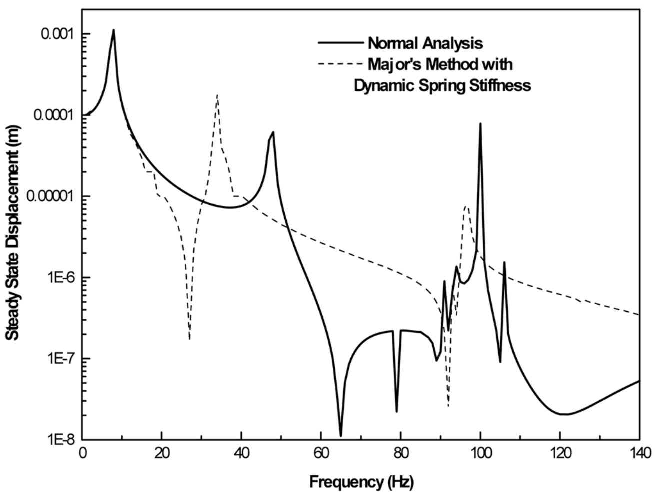

Fig. 26 Performance of modified major's approximation with reference to exact response (transverse excitation)

and the amplitude at resonance is computed assuming a large damping in the system. In the vertical direction, Major's method assumes that frames are generally unconnected and the contribution from the longitudinal beams is assumed to be absent. The lacuna in the procedure is that, using fundamental frequencies and mode shapes, it may not be able to determine the amplitudes at 50 or 100 Hz and to compensate for this, resonance amplitude at the fundamental lateral frequency is computed, with a slightly larger damping. Hence an attempt is made to replace the static spring stiffness of the lateral frames with the exact dynamic spring stiffness. The longitudinal members are also assumed to be dynamically flexible with two DOF at each of the frame connecting point (translation, parallel to the transverse frames and the associated rotation). The two longitudinal beams are combined as a single beam with additive area and moments of inertia about the vertical axis. The above system is analyzed for horizontal harmonic forces applied on the frames. The steady state translational amplitudes are compared with the results of the original three dimensional frame (Fig. 25). The results are plotted in Fig. 26. The figure shows that beyond 14 Hz, the deviation between the amplitudes predicted by the modified Major's procedure and the exact dynamic stiffness method is beyond acceptable level. When subjected to horizontal transverse forces, each frame moves at an equal extent along the same direction on the top of the columns, but to accommodate the rotation of the longitudinal beams about the vertical axis, these frames have to move longitudinally at equal and opposite magnitudes. Neglecting the longitudinal movement of the frame, even though the applied force is only along the transverse direction, is not a justifiable approximation, particularly at high frequencies. The compatible rotation of the frames about the vertical axis is also not ensured. This could be the cause of deviation from the modified Major's procedure to the exact dynamic stiffness method.

13. Introduction of dynamic constraint equations in the TGDYN program

As in the case of a static analysis, neglecting a particular DOF as close to zero has to be done with care. The response shows a totally different pattern at high frequencies as different from a low frequency excitation. Typical illustration is the vertical excitation of the plane frame with identical loads on the top of the column. The best possible simplification could be the proper identification of constraint equations, suitable at higher frequencies of excitation and their proper implementation in the program. Introduction of constraint equation for the steady state dynamic response computation is equivalent to a static analysis problem and all the simplifications that could be carried out in a

Table 2 List of constraints for a machine foundation subjected to vertical and horizontal loads (similar constraints exist for node pairs 3-4 and 5-6 etc in Fig. 25)

DOF	Transverse Loading	Vertical Loading
X/Y/Z Translation	$\delta_{2x} = \delta_{1x}$; $\delta_{2y} = -\delta_{1y}$; $\delta_{2z} = -\delta_{1z}$	$\delta_{2x} = -\delta_{1x}$; $\delta_{2y} = \delta_{1y}$; $\delta_{2z} = \delta_{1z}$
X/Y/Z Rotation	$\theta_{2x} = -\theta_{1x}$; $\theta_{2y} = \theta_{1y}$; $\theta_{2z} = \theta_{1z}$	$\theta_{2x} = \theta_{1x}$; $\theta_{2y} = -\theta_{1y}$; $\theta_{2z} = -\theta_{1z}$

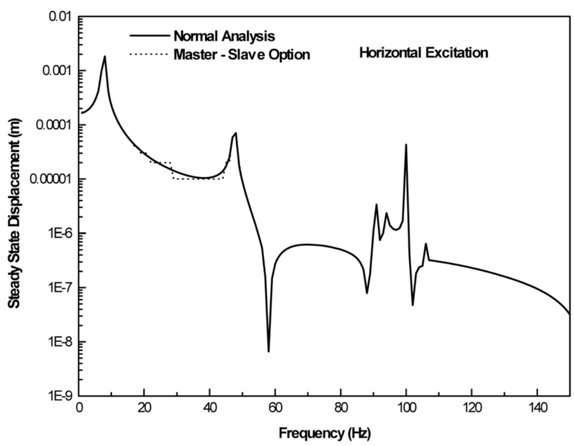

Fig. 27 Response computed with dynamic constraints as against the normal method (transverse excitation)

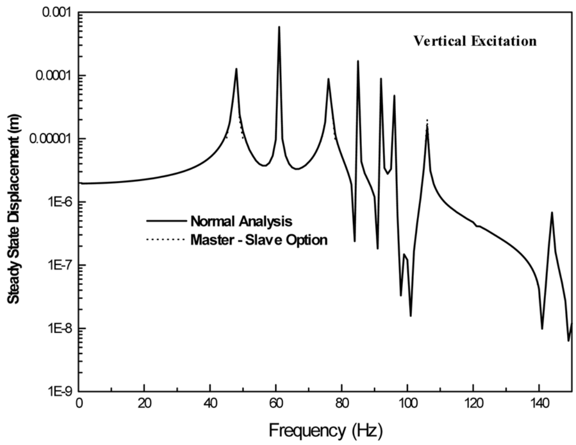

Fig. 28 Response computed with dynamic constraints as against the normal method (vertical excitation)

static problem could be introduced in the dynamic problem also. If two unknown DOFs are connected together by a simple relation, like one displacement is equal or negative or a factor of the other displacement, suitable modifications can be made in the stiffness matrix assembly and also in the applied load vector. The simplification is possible strictly under certain conditions like symmetry of geometry, loading, material and cross section properties. Another version of the dynamic analysis program is introduced with facility to define constraint equations and this has also been verified with the analysis results such that no constraints are explicitly stated. The available constraints are to be identified and after that these can be explicitly stated in the input. Table 2 states the constraints that are normally available for a machine supporting framed structure for both the vertical and transverse loadings.

After introduction of the facility of constraint dynamic equation in the program, a dynamic analysis is carried out both under transverse and vertical loads for the frame shown in Fig. 25. The results are identical and are shown in Figs. 27 and 28 for the transverse and vertical directions respectively. The constraint facility is not problem specific and any other possible constraint, available to a particular class of problem can also be introduced in the program input.

14. Conclusions

Computation of steady state dynamic response by mode superposition method can be erroneous at higher frequencies due to the non-exact nature of the higher eigenvalues, arising from coarse discretisation and the poor judgment of the range of frequencies participating in a dynamic response. Frequency domain methods using direct dynamic stiffness solve the problems involved in the missing of modes but still have approximations due to poor sub-element discretisation. The flaws of the mode superposition method and the various approximate dynamic stiffness methods show up only at high frequencies of excitation, a typical example being a high speed machine foundation. Local and semi global modes manifest at higher frequencies and a finer discretisation has to be ensured to prevent filtering out of any of the higher modes. The paper outlines the relative performance of the various dynamic stiffness formulations, obtained through lumped, consistent and average mass schemes with reference to the exact dynamic stiffness method. All the above formulations, in addition to mode superposition route are implemented in the TGDYN program. The convergence of response obtained through mode superposition method, as the number of modes increase with the same formulation of the dynamic stiffness method is established. The sub-element discretisation is progressively increased and the displacement and stress resultants are studied. As discretisation level is increased, each of the approximate formulation converges. Results of exact dynamic stiffness method are invariant of the discretisation level. Out of all the approximate methods, response predicted by consistent mass formulation is closer to the exact one. Generally, lumped mass method underestimates the frequencies and consistent mass over-estimates them. However, results of consistent mass formulation are closer to the exact values and results of average mass formulation does not substantially improve the over all performance. Simplified expressions are developed for the stiffness of symmetrical plane frame subjected to machine-bearing-type excitation in both the vertical and transverse directions and are compared with the results of the program. The Sturm number generation on the real part of the complex dynamic stiffness matrix is also implemented such that natural frequency information is made available even when the response is computed through direct frequency domain approaches. In cases, where commercial software is to be used, a simplified guideline is developed such that, depending on the frequency of excitation, a minimum sub-element discretisation can be ensured. Simplifications, similar to those suggested by Major are attempted on a typical machine foundation, in which the transverse frames are replaced by equivalent springs. Unlike Major's method, dynamic spring stiffness is used and the longitudinal beams are also made dynamically flexible. However, such a model also does not yield good results beyond 14 Hz and the absence of the longitudinal DOF of the frame brings in erroneous results. The best possible simplification procedure is through implementation of dynamic constraint equations in the program and this is also implemented. For a typical framed machine supporting structure, suitable constraint equations are identified and the results of the modified program are compared and found to be the same as the one in which the constraint equations are over-looked.

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Notations

N : No of degrees of freedom
M : No of modes extracted
$[\phi]$: Mode shape matrix of size $N \times M$
cf_i : A complex scalar of contribution factor for ith mode,
$\{z\}$: A vector of cf_i, of size, $M \times 1$
$[M]_{L1}$: Lumped mass matrix approximated from distributed mass
$[M]_{L2}$: Lumped mass matrix from non-structural members
$[C]$: Damping matrix due to physical dampers
$[K]_S$: Static stiffness matrix of the system
$[K]_{DD}$: Distributed exact real part of dynamic stiffness matrix
$[K]_{DL}$: Approximate complex dynamic stiffness matrix due to mass lumping
$[K]_{DC}$: Approximate complex dynamic stiffness matrix due to consistent mass
$[K]_{DA}$: Approximate complex dynamic stiffness matrix due to average of lumped and consistent mass
$[K]_{DE}$: Exact. complex dynamic stiffness matrix
$\{F\}$: Complex force vector due to harmonic motion
$\{d\}$: Complex displacement vector due to harmonic motion
ω or ω_{op} : Forcing frequency in radians/sec
ω_{ni} : Natural frequency of nth mode in radians/sec
ζ : Structural damping factor
EI : Flexural rigidity of the beam section
ρA : Mass per unit length of beam

ξ : Viscous damping percentage
$[A],[S],[B],[P]$: Example matrices to illustrate congruent and similarity transformations
$[L_\omega],[D_\omega],[L_\omega]^T$: Lower triangular, diagonal and upper triangular matrices of the decomposed $[K]-\omega^2[M]$ matrix at every ω
ΣJ_m : is the number of fixed-fixed frequencies less than ω summed up for all elements
$s\{D(\omega)\}$: Sturm count of dynamic stiffness matrix
i : $\sqrt{-1}$